

Graduate Texts in Mathematics

Jayce R. Getz · Heekyoung Hahn

An Introduction to Automorphic Representations

With a view toward trace formulae



Springer

Graduate Texts in Mathematics

Series Editors:

Patricia Hersh, *University of Oregon*

Ravi Vakil, *Stanford University*

Jared Wunsch, *Northwestern University*

Advisory Boards:

Alexei Borodin, *Massachusetts Institute of Technology*

Richard D. Canary, *University of Michigan*

David Eisenbud, *University of California, Berkeley & SLMath*

Brian C. Hall, *University of Notre Dame*

June Huh, *Princeton University*

Eugenia Malinnikova, *Stanford University*

Akhil Mathew, *University of Chicago*

Peter J. Olver, *University of Minnesota*

John Pardon, *State University of New York*

Jeremy Quastel, *University of Toronto*

Wilhelm Schlag, *Yale University*

Barry Simon, *California Institute of Technology*

Melanie Matchett Wood, *Harvard University*

Yufei Zhao, *Massachusetts Institute of Technology*

Graduate Texts in Mathematics bridge the gap between passive study and creative understanding, offering graduate-level introductions to advanced topics in mathematics. The volumes are carefully written as teaching aids and highlight characteristic features of the theory. Although these books are frequently used as textbooks in graduate courses, they are also suitable for individual study.

Jayce R. Getz · Heekyoung Hahn

An Introduction to Automorphic Representations

With a view toward trace formulae

Jayce R. Getz
Department of Mathematics
Duke University
Durham, NC, USA

Heekyoung Hahn
Department of Mathematics
Duke University
Durham, NC, USA

ISSN 0072-5285

ISSN 2197-5612 (electronic)

Graduate Texts in Mathematics

ISBN 978-3-031-41151-9

ISBN 978-3-031-41153-3 (eBook)

<https://doi.org/10.1007/978-3-031-41153-3>

Mathematics Subject Classification: 11F23, 11F66, 11F67, 11F70, 11F72, 11F75, 11F80, 11R34, 11R39, 11R56, 11S25, 11S37, 11S40, 14G12, 14G35, 14L15, 14L17, 14L24, 14L30, 22D10, 22D12, 22D30, 22E30, 22E35, 22E40, 22E45, 22E46, 22E47, 22E50, 22E55

© Springer Nature Switzerland AG 2024

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors, and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Paper in this product is recyclable.

To Angela and Adsila, with love

Preface

The instructor of an introductory course on automorphic representations, the student beginning their foray into the subject, and the experienced mathematical researcher in another discipline trying to obtain conversational knowledge in the field are all faced with a formidable task. The prerequisites are vast, as the theory of automorphic representations draws on, and makes substantial contributions to, much of algebraic geometry, harmonic analysis, number theory, and representation theory. Moreover, it is a subject in which one can spend a career moving in any direction one chooses.

The student should not be overly discouraged, however, because there is no person currently alive who has the necessary prerequisites to really understand the entire subject. This is meant as no disrespect to the luminaries in the field that, either through their work or personal contact, have made a profound impact on the mathematical community and the authors of this book. It is simply that broad and deep a subject of research. Extremely few important results were proved by one individual (or even by a group working on a single project). Most of the important results in the field have been obtained by individuals or small teams with very different specialties working on various pieces of a larger whole, sometimes over the course of decades. We say this to encourage the reader; one need not know everything in order to get started and make important contributions.

In the authors' experience, after scrounging around the literature one learns enough to begin discovering mathematics connected to their particular point of view. The goal of this book is to make this process easier. The more particular aim is to develop enough of the language and machinery of the subject that after reading this book one can start reading original research papers. We have also tried to be concise enough that the reader does not get mired in so many details that they are unable to see the overarching themes and goals of the subject. This means that we have not made any systematic attempt to prove every result that we use in the book. Our preference has been to omit proofs that require substantial digressions that interrupt the flow of ideas. However, we have included such proofs if they are not easily

accessible in the literature. When proofs are not provided we have made an effort to provide precise references where the reader can find more details.

This book is written so that it can be used as a text for a semester or year-long course, a resource for self-study of more advanced topics, and as a reference for researchers. We hope that our efforts to provide definitions, proofs, or precise references for proofs will be helpful to the last group. We treat general reductive groups over arbitrary global fields.

Prerequisites for the book include introductory graduate-level courses in algebra and analysis together with undergraduate topology. Graduate-level classes in algebraic number theory, algebraic geometry, representation theory, and algebraic topology are helpful at various places. However the book is written so that the reader who does not have this knowledge can still proceed, supplementing the material in the book from other resources when necessary.

We now describe several outlines for possible courses. The sections we have not included have valuable information that may be substituted or added as desired. Here is an outline for a semester-long introduction to automorphic representation theory:

Affine Algebraic Groups §1.2, §1.3, §1.4, §1.5, §1.6, §1.7, §1.8, §1.9

Adeles §2.1, §2.2, §2.3, §2.4, §2.5, §2.6, §2.7

Discrete Automorphic Representations §3.1, §3.2, §3.3, §3.4, §3.5, §3.6, §3.7

Archimedean Representation Theory §4.2, §4.3, §4.4, §4.5, §4.6, §4.7

Representations of Totally Disconnected Groups §5.1, §5.2, §5.3, §5.4, §5.5, §5.6, §5.7

Automorphic Forms §6.1, §6.2, §6.3, §6.4, §6.5, §6.6

Unramified Representations §7.1, §7.2, §7.3, §7.4, §7.5, §7.6, §7.7

It is useful to say in words what these chapters cover. Given a global field F and a reductive algebraic group G over F , a discrete automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ is an irreducible unitary representation $A_G \backslash G(\mathbb{A}_F)$ which is isomorphic to a subrepresentation of $L^2(A_G G(F) \backslash G(\mathbb{A}_F))$. The first goal of the course outline above is to make sense of this definition. The second goal, realized in Chapter 6, is to give a refinement of it that is technically very useful and make the connection with classical automorphic forms transparent. The precise relationship between the two definitions is explained in §6.6. Finally in Chapter 7 one finds a rough form of the Langlands functoriality conjecture.

For a second semester or an advanced class one has more flexibility. Chapter 8 covers deeper topics in non-Archimedean representation theory, including the crucial related notions of Jacquet modules and supercuspidal representations. It also includes a statement of the Bernstein-Zelevinsky classification (see §8.4). These topics, though important, come up only in passing later in the book, and could be omitted. Chapter 9 on the cuspidal spectrum gives a complete proof of the fact that the cuspidal spectrum decomposes discretely. It is a basic fact that is hard to extract from the literature, but it could easily be treated as a black box if desired. Chapter 10 outlines the theory of Eisenstein series. This foundational theory lies at the heart of many

constructions (and analytic difficulties) in the theory of automorphic representations. However, the book is written so that Chapter 10 can be skipped without much loss of continuity.

With these comments in mind, a second course that does not cover simple relative trace formulae might have an outline as follows:

Rankin-Selberg L -functions §11.2, §11.3, §11.4, §11.5, §11.6, §11.7, §11.9
 Langlands Functoriality §12.1, §12.2, §12.3, §12.4, §12.5, §12.6, §12.7, §12.8
 Known Cases of Global Langlands Functoriality §13.3, §13.4, §13.5, §13.6, §13.7, §13.8
 Distinction and Period Integrals §14.2, §14.3, §14.4, §14.5, §14.6, §14.7
 The Cohomology of Locally Symmetric Spaces §15.2, §15.3, §15.4, §15.5, §15.6, §15.8

The next three chapters of the book together give a proof of a simple version of the (relative) trace formula. This implies the trace formula, twisted trace formula, etc. in simple cases. In a course these three chapters should be covered together, and they rely in particular on §14.3. If covering the simple relative trace formula is a priority for the instructor, they may consider dropping Chapter 15 or postponing it to the end of the semester.

The spectral side of the trace formula is considered in §16.1 and §16.2. The geometric side is the focus of Chapter 17. Chapter 9 on the discreteness of the cuspidal spectrum and Chapter 17 are easily the most technical chapters of the book. The first requires some analysis and the second some algebraic geometry. In some sense they are also the most important, because they give complete proofs of results that are very difficult to extract from the literature, if they exist at all. That said, just as much of Chapter 9 can be taken as a black box, the same is true of Chapter 17. In a course it might make sense to treat §17.2 and §17.4 only superficially and focus on the definition of the local relative orbital integrals in §17.5 and the definition of global relative orbital integrals in §17.8. The main result of the chapter is Theorem 17.8.4.

After the preparatory work described in the previous paragraph the general simple relative trace formula in §18.2 follows easily. Sections 18.3–18.5 provide specializations of the simple relative trace formula that are important for illustrating what the formula means. Readers interested in analytic number theory should peruse §18.7, §18.8, §18.9 to obtain a complete derivation of the famous Kuznetsov formula in a form suitable for applications to analytic number theory. After all the hard work required to derive (even simple) relative trace formulae, it would be remiss not to discuss some applications. These are contained in Chapter 19. The most important sections are §19.2 and §19.3, but to really understand the theory in §19.3 the examples and clarifications in the remaining sections of Chapter 19 are invaluable. Some suggestions for further work are contained in §19.6.

Most of the material in this book has appeared in one form or another in the literature. We have made an honest attempt to give accurate attributions for both results and expository treatments. We would appreciate

being informed of any serious errors we have made either in mathematics or in attributing results. As much as possible we treat arbitrary reductive groups over arbitrary global fields, including the case of positive characteristic. We have also refrained from assuming that the affine algebraic groups under consideration are reductive until necessary. This both emphasizes the results that really depend on the fact that G is reductive and develops the preliminaries necessary to understand, for example, unitary representations of nonreductive groups. This is not vacuous, as the theory of Jacobi forms already plays a role in automorphic forms [BS98].

The results on orbital integrals in Chapter 17 and the simple relative trace formula in Chapter 18 are based partially on work of the second author [Hah09] and both authors [GH15]. However they have not appeared previously in the generality in which we prove them here. In particular we treat arbitrary reductive groups over arbitrary global fields. Thus we allow positive characteristic.

In the interest of full disclosure, we should say a word about foundations. We define affine schemes (and in particular algebraic groups) as certain functors on k -algebras, and in §1.10 and §17.6 we make use of some more serious algebraic geometry involving sheaves of sets on a site. This puts us up against the usual set-theoretic difficulties. To handle these difficulties we assume the universe axiom; fix an uncountable universe; and implicitly assume that the categories of k -algebras, sets, etc. with which we are working are elements of that universe. This is a standard convention in algebraic geometry. For more details we refer to [Poo17, Appendix A] and [SGA72, §0]. It is likely that the reliance on the universe axiom could be removed at the cost of complicating our definitions.

Durham, NC, USA
July 2023

Jayce R. Getz
Heekyoung Hahn

Acknowledgements

The authors thank Francesc Castella, Andrew Fiori, and Cameron Franc for typesetting the first draft of the lecture notes that formed the germ of this book, and thank Pam Gu, Dick Hain, Chun-Hsien Hsu, Tasho Kaletha, Taeyeoup Kang, Minhyong Kim, Chung-Ru Lee, Seewoo Lee, Spencer Leslie, Huajie Li, Siyan Daniel Li-Huerta, Yi Luo, Jason Polak, Timo Richarz, Leslie Saper, Chad Schoen, and Zixuan Zhang for many useful corrections and comments. Many people have been generous in answering questions as this book was written, including James Arthur, Avner Ash, David Ginzburg, Nadya Gurevich, Michael Harris, Rahul Krishna, Erez Lapid, Joseph Rabinoff, Freydoon Shahidi, Yihang Zhu, Michał Zydor, and any others who we may have forgotten. The authors give special thanks to Brian Conrad, who pointed out inaccuracies in the first four chapters; Solomon Friedberg, who taught a course out of a draft of this book; Alex Youcis for his very careful reading of the first twelve chapters; and to the anonymous referees. Their comments led to many clarifications and corrections. The authors also thank Ken Ono for suggesting this book project and Elizabeth Loew at Springer for her publishing expertise. The first author wishes to thank the National Science Foundation for support (DMS-1901883 and DMS-1405708) at various times during the preparation of this book. Part of this book was written while both authors were members at the Institute for Advanced Study in the spring of 2018 supported in part by C. Simonyi Endowment, and the final draft was completed while both authors were on sabbatical at Postech in South Korea in the 2020-2021 academic year and jointly supported in part by Postech Mathematical Institute (NRF-2017R1A2B2001807) and by YoungJu Choie (NRF-2018R1A4A1023590). We thank these institutions for their hospitality and providing us with excellent working conditions. We also thank the students of Hahn's course at Postech for their questions which led to various improvements. Finally, we truly appreciate the encouragement of Mahdi Asgari, Peter Sarnak, and many graduate students.

Contents

1	Affine Algebraic Groups	1
1.1	Introduction	1
1.2	Affine schemes	2
1.3	Affine group schemes	7
1.4	Extension and restriction of scalars	12
1.5	Reductive groups	13
1.6	Lie algebras	16
1.7	Tori	19
1.8	Root data	25
1.9	Parabolic subgroups	32
1.10	Homogeneous spaces	37
	Exercises	42
2	Adeles	45
2.1	Adeles	45
2.2	Adelic points of affine schemes	51
2.3	Relationship with restricted direct products	55
2.4	Hyperspecial subgroups and models	57
2.5	Approximation in affine algebraic groups	64
2.6	The adelic quotient	67
2.7	Reduction theory	71
	Exercises	73
3	Discrete Automorphic Representations	77
3.1	Representations of locally compact groups	77
3.2	Haar measures on locally compact groups	81
3.3	Gelfand-Pettis integrals	83
3.4	Convolution algebras of test functions	85

3.5	Haar measures on local fields	86
3.6	Haar measures on the points of affine algebraic groups	87
3.7	Discrete automorphic representations	91
3.8	Decomposition of representations	92
3.9	The Fell topology	95
3.10	Type I groups	97
3.11	Why affine groups?	98
	Exercises	99
4	Archimedean Representation Theory	103
4.1	The passage between analysis and algebra	103
4.2	Smooth vectors	104
4.3	Restriction to compact subgroups	110
4.4	$(\mathfrak{g}, K)$ -modules	114
4.5	Hecke algebras with K -types	121
4.6	Infinitesimal characters	123
4.7	Classification of $(\mathfrak{g}, K)$ -modules for $\mathrm{GL}_{2\mathbb{R}}$	126
4.8	Matrix coefficients	127
4.9	The Langlands classification	129
	Exercises	132
5	Representations of Totally Disconnected Groups	135
5.1	Totally disconnected groups	135
5.2	Smooth functions on td-groups	136
5.3	Smooth and admissible representations	139
5.4	Contragredients	143
5.5	The unramified Hecke algebra	146
5.6	Restricted tensor products of modules	148
5.7	Flath's theorem	151
	Exercises	154
6	Automorphic Forms	157
6.1	Smooth functions	157
6.2	Classical automorphic forms	158
6.3	Adelic automorphic forms over number fields	161
6.4	Adelic automorphic forms over function fields	166
6.5	The cuspidal subspace	167
6.6	Discrete automorphic representations revisited	170
6.7	From modular forms to automorphic forms	173
6.8	Ramanujan's Δ -function	178
	Exercises	179

7	Unramified Representations	183
7.1	Unramified representations	183
7.2	The Satake isomorphism for split groups	185
7.3	The Langlands dual group	187
7.4	Parabolic subgroups of L -groups	193
7.5	The Satake isomorphism for unramified groups	196
7.6	The principal series	203
7.7	Weak global L -packets	208
	Exercises	211
8	Non-Archimedean Representation Theory	213
8.1	Introduction	213
8.2	Parabolic induction	215
8.3	Jacquet modules	218
8.4	The Bernstein-Zelevinsky classification	222
8.5	Traces, characters, and coefficients	225
8.6	Parabolic descent of representations	229
8.7	Parabolic descent of orbital integrals	232
	Exercises	236
9	The Cuspidal Spectrum	239
9.1	Introduction	239
9.2	The cuspidal subspace	242
9.3	Deduction of the discreteness of the spectrum	246
9.4	The basic estimate	249
9.5	The function field case	259
9.6	Rapidly decreasing functions	263
9.7	Cuspidal automorphic forms	264
	Exercises	267
10	Eisenstein Series	269
10.1	Induced representations	269
10.2	Intertwining operators	271
10.3	Eisenstein series	275
10.4	Constant terms	277
10.5	Decomposition of the spectrum	281
10.6	Local preparation for isobaric representations	283
10.7	Isobaric representations	286
10.8	A theorem of Mœglin and Waldspurger	289
	Exercises	291
11	Rankin-Selberg L-functions	293
11.1	Paths to the construction of automorphic L -functions	293
11.2	Generic characters	294
11.3	Generic representations	295
11.4	Formulae for Whittaker functions	299

11.5	Local Rankin-Selberg L -functions	303
11.6	Unramified Rankin-Selberg L -functions	307
11.7	Global Rankin-Selberg L -functions	308
11.8	The nongeneric case	311
11.9	The converse theorem	312
	Exercises	314
12	Langlands Functoriality	315
12.1	The Weil group	315
12.2	The Weil-Deligne group and L -parameters	319
12.3	The Archimedean Langlands correspondence	325
12.4	The local Langlands correspondence for GL_n	327
12.5	The local Langlands conjecture	328
12.6	Global Langlands functoriality	333
12.7	Langlands L -functions	339
12.8	Algebraic representations	340
	Exercises	344
13	Known Cases of Global Langlands Functoriality	347
13.1	Introduction	347
13.2	Parabolic induction	348
13.3	L -maps into general linear groups	348
13.4	Base change	349
13.5	The strong Artin conjecture	352
13.6	The Langlands-Shahidi method	354
13.7	Functoriality for the classical groups	356
13.8	Endoscopic classification of representations	362
13.9	The function field case	368
	Exercises	369
14	Distinction and Period Integrals	371
14.1	Introduction	371
14.2	Distinction in the local setting	372
14.3	Global distinction and period integrals	374
14.4	Spherical varieties	378
14.5	Symmetric subgroups	381
14.6	Relationship with the endoscopic classification	385
14.7	Period integrals in the Gan-Gross-Prasad setting	387
14.8	Necessary conditions for distinction	392
	Exercises	393
15	The Cohomology of Locally Symmetric Spaces	395
15.1	Introduction	395
15.2	Locally symmetric spaces	396
15.3	Local systems	399
15.4	$(\mathfrak{g}, K_\infty)$ -cohomology	402

15.5	The cohomology of Shimura manifolds	403
15.6	The relation to distinction	406
15.7	More on $(\mathfrak{g}, K_\infty)$ -cohomology	408
15.8	Shimura varieties	410
	Exercises	417
16	Spectral Sides of Trace Formulae	419
16.1	The automorphic kernel function	419
16.2	Relative traces	423
16.3	The full expansion of the automorphic kernel	429
16.4	Functions with cuspidal image	429
	Exercises	431
17	Orbital Integrals	433
17.1	A refined study of orbits	433
17.2	Group actions, orbits, and stabilizers	434
17.3	Classes and quotients	439
17.4	Local geometric classes	446
17.5	Local relative orbital integrals	450
17.6	Torsors	451
17.7	Adelic classes	455
17.8	Global relative orbital integrals	462
	Exercises	469
18	Simple Trace Formulae	473
18.1	A brief history of trace formulae	473
18.2	A general simple relative trace formula	475
18.3	Products of subgroups	483
18.4	The simple trace formula	485
18.5	The simple twisted trace formula	486
18.6	A variant	488
18.7	The Petersson-Bruggeman-Kuznetsov formula	492
18.8	Kloosterman integrals	494
18.9	Sums of Whittaker coefficients	500
	Exercises	504
19	Applications of Trace Formulae	507
19.1	Existence and comparison	507
19.2	The Weyl law	508
19.3	The comparison strategy	509
19.4	Jacquet-Langlands transfer and base change	513
19.5	Twisted endoscopy	517
19.6	The interplay of distinction and twisted endoscopy	518
	Exercises	522

A	Groups Attached to Involutions of Algebras	525
A.1	Algebras with involution	525
A.2	Split simple algebras	530
A.3	Classification of forms	532
A.4	Parabolic subgroups	533
B	The Iwasawa Decomposition	541
B.1	Introduction	541
B.2	Some group schemes	542
B.3	The dynamic method	544
B.4	Proof of Theorem B.1.1	547
B.5	Addenda in the hyperspecial case	548
C	Poisson Summation	551
C.1	The standard additive characters	551
C.2	Local Schwartz spaces and Fourier transforms	553
C.3	Global Schwartz spaces and Poisson summation	553
	Exercises	555
D	Alternate Conventions Related to Adelic Quotients	557
D.1	The quotients	557
D.2	Separating by central character	559
	Hints to selected exercises	563
	References	569
	Index	603

Chapter 1

Affine Algebraic Groups



Before Genesis, God gave the devil free reign to construct much of creation, but dealt with certain matters himself, including semisimple groups.

attributed to Chevalley by
Harish-Chandra

We briefly recall some of the basic notions related to affine algebraic groups. If the reader is comfortable with reductive groups then this chapter can probably be skipped and then referred to later as needed.

1.1 Introduction

Someone beginning to learn the theory of automorphic representations does not need to internalize the content of this chapter. For the most part, one can get by with much less. For example, one can think of GL_n , which is really an affine group scheme over the integers, as a formal notational device already familiar from an elementary course in algebra: for commutative rings R , the notation $\mathrm{GL}_n(R)$ refers to matrices with coefficients in R with determinant in $R^\times$. Likewise, one knows that, for example, $\mathrm{Sp}_{2n}(R)$ is the group of matrices with coefficients in R that fix the standard symplectic form on R^{2n} . One can think of a reductive group G as some gadget of this type that assigns a group to every ring of some type (say $\mathbb{Q}$ -algebras) and behaves like GL_n . In fact, for a first reading of this book, whenever one sees the assumption that G is a reductive group, one could assume that $G = \mathrm{GL}_n$ for some integer n and not lose a lot of the flavor.

That said, at some point, it is not a bad idea for any student of automorphic representation theory (and really, any mathematician) to come to grips

with the “modern” (post-1960s) perspective on algebraic groups and more generally algebraic geometry. It is useful, beautiful, and despite its reputation, not really that counterintuitive or difficult. Thus we will briefly introduce it in this chapter, mostly as a means of fixing notation and conventions, and make use of it in the remainder of the book. We have chosen to focus on affine schemes because (for the most part) this is all we require and they are simpler.

Much of the material in this chapter can be found in one form or another in the three standard references for algebraic groups [Bor91, Hum75, Spr09]. Unfortunately they were all written in the archaic language of Weil used before Grothendieck’s profound reimagining of the foundations of algebraic geometry. The most complete treatment is [DG74]. Since Demazure and Grothendieck treat group schemes over an arbitrary base scheme, this reference is hard to penetrate if one does not have solid preparation in algebraic geometry. Fortunately, Milne has reworked the three standard references in modern terminology. He restricts attention to algebraic groups over a base field as opposed to a base scheme, so the account is more accessible than [DG74]. His book [Mil17] was instrumental in the preparation of this chapter.

To aid the reader in understanding the general theory exposed in this chapter, we have appended §A. It contains a family of examples of algebraic groups and explicit descriptions of some of the objects attached to general algebraic groups in this chapter.

1.2 Affine schemes

Throughout this chapter, we let k be a commutative ring with identity. Let $\mathbf{Alg}_k$ denote the category of commutative k -algebras with identity. For a k -algebra A , one obtains a functor

$$\begin{aligned} \mathrm{Spec}(A) : \mathbf{Alg}_k &\longrightarrow \mathbf{Set} \\ R &\longmapsto \mathrm{Hom}_k(A, R), \end{aligned} \tag{1.1}$$

where the subscript k denotes k -algebra homomorphisms. If $\phi : A \rightarrow B$ is a morphism of k -algebras, then we obtain, for each k -algebra R , a map

$$\begin{aligned} \mathrm{Spec}(B)(R) &\longrightarrow \mathrm{Spec}(A)(R) \\ \varphi &\longmapsto \varphi \circ \phi. \end{aligned} \tag{1.2}$$

This collection of maps is an example of what is known as a natural transformation of functors. More precisely, to give a natural transformation $X \rightarrow Y$ of set-valued functors on $\mathbf{Alg}_k$ is the same as giving for each k -algebra R a map

$$X(R) \longrightarrow Y(R) \tag{1.3}$$

such that, for all morphisms of k -algebras $R \rightarrow R'$, the following diagram commutes:

$$\begin{array}{ccc} X(R) & \longrightarrow & Y(R) \\ \downarrow & & \downarrow \\ X(R') & \longrightarrow & Y(R') \end{array} \tag{1.4}$$

An isomorphism of set-valued functors on $\mathbf{Alg}_k$ is a natural transformation of functors $X \rightarrow Y$ such that the maps $X(R) \rightarrow Y(R)$ are bijections for all k -algebras R .

One can check that (1.2) defines a natural transformation of functors, and it is an isomorphism if and only if ϕ is an isomorphism of rings.

Definition 1.2.1. An **affine scheme** over k or an **affine k -scheme** is a functor on the category of k -algebras isomorphic to $\mathrm{Spec}(A)$ for some k -algebra A . A **morphism** of affine k -schemes is a natural transformation of functors.

This definition gives us a category $\mathbf{AffSch}_k$ of affine schemes over k , equipped with a contravariant equivalence of categories

$$\mathrm{Spec} : \mathbf{Alg}_k \longrightarrow \mathbf{AffSch}_k$$

(see Exercise 1.3). In other words, the category of affine schemes over k is the category of k -algebras “with the arrows reversed.” If k is understood, we often omit explicit mention of it. By way of terminology, if X is an affine scheme then

$$X(R)$$

is called its set of **R -valued points** or **R -points**.

Definition 1.2.2. A functor $S : \mathbf{Alg}_k \rightarrow \mathbf{Set}$ is **representable** by a ring A if $S = \mathrm{Spec}(A)$. In this case, we write

$$\mathcal{O}(S) := A$$

and refer to it as the **coordinate ring** of S .

To ease comparison with other references, we note that we are defining schemes using their functors of points, whereas the usual approach is to define a scheme to be a topological space with a sheaf of rings on it satisfying certain desiderata and then associate a functor of points to the scheme. The two approaches are equivalent. The usual approach is desirable for many purposes, but the approach via the functor of points is especially useful for the study of algebraic groups. For more details on the usual approach and on the scheme theoretic concepts mentioned below, we point the reader to [Har77, EH00, Mum99, GW10]. The functor of points approach is used in [DG74].

In order to work with affine schemes in practice, one must usually impose additional restrictions on the representing ring A .

Definition 1.2.3. An affine k -scheme $\mathrm{Spec}(A)$ is of **finite type** if A is a finitely generated k -algebra. It is **finite** if A is a finitely generated k -module. It is of **finite presentation** if there is an isomorphism of k -algebras $A \cong k[t_1, \dots, t_n]/(f_1, \dots, f_m)$ for some finite set of indeterminants $t_1, \dots, t_n$ and a finite set of polynomials $f_1, \dots, f_m \in k[t_1, \dots, t_n]$.

Any affine k -scheme $\mathrm{Spec}(A)$ of finite presentation is of finite type, and the converse holds if k is Noetherian. An important example of an affine scheme of finite type over k is affine n -space:

$$\mathbb{G}_a^n := \mathrm{Spec}(k[t_1, \dots, t_n]).$$

This is also denoted by $\mathbb{A}^n$, but we avoid this notation because we will use the symbol $\mathbb{A}$ for the adeles (which will be defined in §2.1). Incidentally $\mathbb{A}$ provides an example of a ring that is not Noetherian.

Here are three other nice properties that the schemes of interest to us will often enjoy:

Definition 1.2.4. An affine scheme X is **reduced** if $\mathcal{O}(X)$ has no nonzero nilpotent elements and **irreducible** if the nilradical of $\mathcal{O}(X)$ is prime. It is **integral** if $\mathcal{O}(X)$ is an integral domain.

It is important to note that an affine scheme can be both reduced and irreducible (see Exercise 1.4).

An affine k -scheme $\mathrm{Spec}(A)$ of finite presentation is **smooth** (resp. **étale**) if the coordinate ring A is formally smooth (resp. formally étale). Here a k -algebra A is said to be **formally smooth** (resp. **formally étale**) if for every k -algebra B with ideal $I \leq B$ of square zero and any k -algebra homomorphism $A \rightarrow B/I$, one has a morphism $A \rightarrow B$ (resp. there is a unique morphism $A \rightarrow B$) such that the diagram

$$\begin{array}{ccc} A & \longrightarrow & B/I \\ \uparrow & \searrow & \uparrow \\ k & \longrightarrow & B \end{array}$$

commutes [Sta16, tag 02GZ, tag 02HF].

For example, an affine k -scheme

$$\mathrm{Spec}(k[t_1, \dots, t_n]/(f_1, \dots, f_c)), \quad c \leq n, \quad (1.5)$$

such that $\det(\partial f_i / \partial t_j)_{1 \leq i, j \leq c}$ maps to a unit in $k[t_1, \dots, t_n]/(f_1, \dots, f_c)$ is smooth. Not all smooth affine k -schemes are of this form. However all smooth morphisms are locally of the form (1.5) in a sense made precise in [Sta16, tag 01V4].

Let $g, f \in k[x]$ with f monic and assume that f' is invertible in the localization $k[x]_g/(f)$. Then

$$\mathrm{Spec}(k[x]_g/(f)) \quad (1.6)$$

is an affine étale k -scheme. Not all affine étale k -schemes are of the form (1.6), but this is true locally as explained in [Sta16, tag 02GH].

The concepts of finite type, finite, finite presentation, smooth, and étale are meaningful in a relative context as well. In more detail, a morphism $\mathrm{Spec}(A) \rightarrow \mathrm{Spec}(B)$ of affine k -schemes gives A the structure of a B -algebra. We say that a morphism $\mathrm{Spec}(A) \rightarrow \mathrm{Spec}(B)$ of k -schemes is of finite type (resp. finite, finite presentation, smooth, étale) if $\mathrm{Spec}(A)$ is of finite type (resp. finite, finite presentation, smooth, étale) as a B -scheme.

We now discuss closed subschemes.

Definition 1.2.5. A morphism of affine k -schemes

$$X \longrightarrow Y$$

is a **closed immersion** if the associated map $\mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ is surjective.

If $X \rightarrow Y$ is a closed immersion then $X(R) \rightarrow Y(R)$ is injective for all k -algebras R (see Exercise 1.5).

If $X \rightarrow Y$ is a closed immersion then $\mathcal{O}(X) \cong \mathcal{O}(Y)/I$ for some ideal I of $\mathcal{O}(Y)$. Motivated by this observation, one defines a **closed subscheme** of the affine scheme Y to be an affine scheme of the form $\mathrm{Spec}(\mathcal{O}(Y)/I)$ for some ideal $I \subseteq \mathcal{O}(Y)$. Thus affine schemes of finite type over k are all closed subschemes of $\mathbb{G}_a^n$ for some n . When k is an algebraically closed field, one thinks of the closed subscheme $\mathrm{Spec}(k[t_1, \dots, t_n]/(f_1, \dots, f_m))$ as being the zero locus of $f_1, \dots, f_m$.

The definition of a closed subscheme can also be given from a topological perspective as we now explain. There is a topological space attached to $\mathrm{Spec}(A)$ which is again denoted $\mathrm{Spec}(A)$. It is the collection of all primes $\mathfrak{p} \subseteq A$ equipped with the **Zariski topology**. To define this topology, for every subset $S \subseteq A$, we let

$$V(S) = \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \supseteq S\}. \quad (1.7)$$

One checks that the sets $V(S)$ are closed under arbitrary intersections and finite unions. Thus we can define the Zariski topology on $\mathrm{Spec}(A)$ to be the topology with $V(S)$ as its closed sets. A morphism of schemes induces a morphism of the associated topological spaces. If I is an ideal of A then the image of the injective map of topological spaces

$$\mathrm{Spec}(A/I) \longrightarrow \mathrm{Spec}(A)$$

is $V(I)$ (see Exercise 1.1).

Since we have mentioned the associated topological space of an affine scheme, we state two definitions that use it.

Definition 1.2.6. A morphism of affine k -schemes $X \rightarrow Y$ is **dominant** if the induced map of topological spaces has dense image, and **surjective** if the induced map of topological spaces is surjective.

The following is [Sta16, Tag 0CC1]:

Lemma 1.2.7. *If X and Y are integral, then $X \rightarrow Y$ is dominant if and only if the map $\mathcal{O}(Y) \rightarrow \mathcal{O}(X)$ is injective.* $\square$

Lemma 1.2.8. *Assume k is a field and X and Y are affine schemes of finite type over k . Then $X \rightarrow Y$ is surjective if and only if $X(\bar{k}) \rightarrow Y(\bar{k})$ is surjective.*

Here and below, if k is a field, we denote by $\bar{k}$ an algebraic closure of k .

Proof. By Chevalley's theorem, the image of the topological space of X in Y is constructible [GW10, Theorem 10.20]. Since the map $X(\bar{k}) \rightarrow Y(\bar{k})$ is surjective, this constructible set contains the set of closed points of Y [GW10, Proposition 3.33]. The set of closed points in Y is very dense in Y (see [GW10, §3.12] for this assertion and the definition of very dense). It follows that the only constructible subset of Y that contains the set of closed points of Y is Y itself [GW10, Remark 10.15]. $\square$

We also give a useful criterion for checking that a morphism of affine schemes is an isomorphism. An integral affine scheme X is **normal** if $\mathcal{O}(X)$ is integrally closed in its field of fractions.

Proposition 1.2.9 *Assume that k is a characteristic zero field, that X and Y are integral affine schemes of finite type over k , and that Y is normal. A morphism $f : X \rightarrow Y$ is an isomorphism if and only if it induces a bijection $f : X(\bar{k}) \rightarrow Y(\bar{k})$.*

Proof. One direction is trivial. For the other, assume that $f : X(\bar{k}) \rightarrow Y(\bar{k})$ is bijective. Then f is quasi-finite (i.e. has finite fibers) [GW10, Remark 12.16]. Combining this with Lemma 1.2.8 and Zariski's main theorem in the form of [GW10, Corollary 12.87] we see that f is a finite morphism. By [GW10, Corollary 12.88] we see that to prove the proposition it suffices to check that f is birational, that is, induces an isomorphism of the fraction field of $\mathcal{O}(X)$ with the fraction field of $\mathcal{O}(Y)$. Since k is of characteristic zero, f is finite, and f induces a bijection on $\bar{k}$ -points, [Har95, Proposition 7.16] implies f is birational. $\square$

We close this section with the notion of fiber products. This is a simple and useful method to create new schemes from existing ones. Let

$$X = \operatorname{Spec}(A), \quad Y = \operatorname{Spec}(B), \quad Z = \operatorname{Spec}(C)$$

be affine k -schemes equipped with morphisms $f : X \rightarrow Y$ and $g : Z \rightarrow Y$. We can then form the **fiber product**

$$X \times_Y Z := \operatorname{Spec}(A \otimes_B C), \quad (1.8)$$

where A and C are regarded as B -algebras via the maps $B \rightarrow A$ and $B \rightarrow C$ induced by f and g , respectively. This scheme has the property that for k -algebras R one has that

$$(X \times_Y Z)(R) = X(R) \times_{Y(R)} Z(R),$$

where the fiber product on the right is in the category of sets:

$$X(R) \times_{Y(R)} Z(R) := \{(x, z) \in X(R) \times Z(R) : f(x) = g(z)\}. \quad (1.9)$$

The case that will appear most frequently in this book is the so-called **absolute product** in the category of affine k -schemes. To define it, note that every affine k -scheme $X = \operatorname{Spec}(A)$ comes equipped with a morphism $X \rightarrow \operatorname{Spec}(k)$, namely, the morphism induced by the ring homomorphism $k \rightarrow A$ that gives A the structure of a k -algebra. This morphism $X \rightarrow \operatorname{Spec}(k)$ is called the **structure morphism**. Thus for any pair of affine k -schemes X and Z , we can form the absolute product

$$X \times Z := X \times_k Z := X \times_{\operatorname{Spec}(k)} Z.$$

It follows from (1.9) that for k -algebras R ,

$$(X \times Z)(R) = X(R) \times Z(R).$$

1.3 Affine group schemes

Definition 1.3.1. An **affine group scheme** over k (or affine k -group scheme) is a functor

$$\mathbf{Alg}_k \longrightarrow \mathbf{Group}$$

representable by a k -algebra. A **morphism** of affine k -group schemes $H \rightarrow G$ is a natural transformation of functors from H to G .

Here a natural transformation of functors is defined as in the previous section, though this time we require that for each k -algebra R the associated map

$$H(R) \longrightarrow G(R) \quad (1.10)$$

is a group homomorphism, not just a map of sets. An **isomorphism** of affine group schemes is a natural transformation of functors $H \rightarrow G$ such that the

associated map (1.10) is an isomorphism for all k -algebras R . It is clear that an affine group scheme over k is in particular an affine scheme over k . In this book, we will only be interested in affine group schemes (as opposed to elliptic curves for instance).

An affine group scheme G over k is **abelian** if $G(R)$ is abelian for all k -algebras R . Some examples of abelian group schemes follow:

Example 1.1. The **additive group** $\mathbb{G}_a$ is the functor assigning to each k -algebra R its additive group,

$$\mathbb{G}_a(R) := (R, +).$$

It is represented by the polynomial algebra $k[x]$:

$$\mathrm{Hom}_k(k[x], R) = R.$$

Example 1.2. The scheme of $n \times n$ matrices M_n is the functor assigning to each k -algebra R the set

$$M_n(R)$$

of $n \times n$ matrices, viewed as a group under addition. It is represented by

$$k[\{x_{ij} : 1 \leq i, j \leq n\}].$$

It is isomorphic as an affine k -group scheme to $\mathbb{G}_a^{n^2}$.

Example 1.3. The **multiplicative group** $\mathbb{G}_m$ is the functor assigning to each k -algebra R its multiplicative group,

$$\mathbb{G}_m(R) = R^\times.$$

It is represented by $k[x, y]/(xy - 1)$.

A basic example of a nonabelian group scheme is the general linear group GL_n for $n \geq 2$. It is the functor taking a k -algebra R to the group of $n \times n$ matrices (x_{ij}) with coefficients x_{ij} in R satisfying $\det(x_{ij}) \in R^\times$. The representing ring is

$$k[\{x_{ij} : 1 \leq i, j \leq n\}][y]/(y \det(x_{ij}) - 1).$$

Note that $\mathrm{GL}_1 = \mathbb{G}_m$. If one wishes to be coordinate free, then for any finite rank free k -module V , one can define

$$\mathrm{GL}_V(R) := \{R\text{-module automorphisms } V \otimes_k R \longrightarrow V \otimes_k R\}.$$

A choice of isomorphism $V \cong k^n$ induces an isomorphism $\mathrm{GL}_V \cong \mathrm{GL}_n$.

We isolate a particularly important class of morphisms with the following definition:

Definition 1.3.2. A **representation** of an affine group scheme G is a morphism of affine group schemes $r : G \rightarrow \mathrm{GL}_V$. It is **faithful** if it is a closed immersion.

Definition 1.3.3. An affine group scheme G is said to be **linear** if it admits a faithful representation $G \rightarrow \mathrm{GL}_V$ for some V .

We will usually be concerned with linear group schemes. We shall see in Theorem 1.5.2 that this is not much loss of generality if k is a field.

Definition 1.3.4. Let G be an affine group scheme over k . An **affine subgroup scheme** (or simply a subgroup) H of G is an affine subscheme H of G such that $H(R)$ is a subgroup of $G(R)$ for all k -algebras R .

We write $H \leq G$ to indicate that H is a subgroup scheme of G .

We have never defined a subscheme. One first defines an open subscheme, and then defines a subscheme to be a closed subscheme of an open subscheme. We refer to any standard text on algebraic geometry for the definition of an open subscheme. One should be aware that an open subscheme of an affine scheme need not be affine, and this is why we have avoided defining the notion of an open subscheme. Happily, when k is a field and G and H are affine group schemes of finite type over k , a subgroup $H \leq G$ is automatically a closed subscheme of G (see Corollary 1.3.7).

Much of the basic theory of abstract groups remains valid for affine group schemes of finite type over a field once properly interpreted. We will not develop the theory; we refer the reader to [Mil17] for details. However, it is useful to define monomorphisms and quotients in this category:

Definition 1.3.5. Let k be a field. A morphism $H \rightarrow G$ of affine group schemes of finite type over k is a **monomorphism** (resp. **quotient map**) if the corresponding map $\mathcal{O}(G) \rightarrow \mathcal{O}(H)$ is surjective (resp. faithfully flat).

For this paragraph continue to assume that k is a field and that $H \rightarrow G$ is a morphism of affine group schemes of finite type over k . If $H \rightarrow G$ is both a monomorphism and a quotient map then it is an isomorphism [Mil17, Proposition 5.33]. By definition, $H \rightarrow G$ is a monomorphism if and only if it is a morphism of affine group schemes and is a closed immersion.

Proposition 1.3.6 *Assume $H \rightarrow G$ is a morphism of affine group schemes of finite type over a field k . Then $H \rightarrow G$ is a monomorphism if and only if $H(R) \rightarrow G(R)$ is injective for all k -algebras R .*

Proof. See Exercise 1.5 and [Mil17, Proposition 5.31]. □

Corollary 1.3.7 *Subgroups of affine group schemes of finite type over a field are closed.* □

The natural analogue of Proposition 1.3.6 for quotient maps is false. If $H \rightarrow G$ is a quotient map, then $H(R) \rightarrow G(R)$ need not be surjective for a given k -algebra R . A trivial example is the map $\mathbb{G}_m \rightarrow \mathbb{G}_m$ of affine group schemes over $\mathbb{Q}$ given on points by $x \mapsto x^n$ for an integer $n > 1$ (or $n < -1$). It is true, however, that if $H \rightarrow G$ is a quotient map then $H(\bar{k}) \rightarrow G(\bar{k})$ is surjective, and the converse holds whenever G is reduced (see Lemma 1.2.8 and [Mil17, Proposition 1.70]).

Example 1.4. Let k be a field. Then the homomorphism $\mathbb{G}_a \rightarrow \mathrm{GL}_2$ given on points in a k -algebra R by

$$\begin{aligned} R &\longrightarrow \mathrm{GL}_2(R) \\ x &\longmapsto \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \end{aligned}$$

is a monomorphism. The homomorphism $\mathrm{GL}_n \rightarrow \mathbb{G}_m$ given on points in a k -algebra by the determinant is a quotient map.

Let $\phi : H \rightarrow G$ be a morphism of affine group schemes over k (which is not necessarily a field). The **kernel** of ϕ is the subgroup scheme of H whose points in a k -algebra R are given by

$$(\ker \phi)(R) := \{h \in H(R) : \phi(h) = 1\}. \quad (1.11)$$

We can alternately construct $\ker \phi$ as the fiber product

$$\ker \phi = H \times_G \mathrm{Spec}(k),$$

where the left map is ϕ and the right map is the map $e : \mathrm{Spec}(k) \rightarrow G$ associated to the identity element of $G(k)$. Since G is affine, e is a closed immersion [Sta16, Tag 045W]. Thus $\ker \phi$ is a closed subscheme of H and hence a subgroup scheme. If k is a field and H and G are of finite type over k then Proposition 1.3.6 implies that $\ker \phi = 1$ if and only if ϕ is a monomorphism.

We say that a subgroup $N \leq H$ is **normal** if $N(R)$ is normal in $H(R)$ for all k -algebras R . It is clear that $\ker \phi$ is a normal subgroup.

Definition 1.3.8. Let N be a normal subgroup scheme of the affine k -group scheme H . A **quotient of H by N** is an affine k -group scheme Q equipped with a morphism $H \rightarrow Q$ of group schemes over k satisfying the following universal property: for any morphism of k -group schemes $a : H \rightarrow G$ with kernel containing N , there is a unique morphism $b : Q \rightarrow G$ of k -group schemes such that the following diagram commutes:

$$\begin{array}{ccc} H & \longrightarrow & Q \\ & \searrow a & \downarrow b \\ & & G \end{array}$$

When the quotient of H by N exists we denote it by H/N or $N \backslash H$. The definition implies that if a quotient of H by N exists then it is unique up to unique isomorphism.

The following is [Mil17, Theorems 5.13, 5.14]:

Theorem 1.3.9 *Let N be a normal subgroup of the affine group scheme H of finite type over a field k . Then a quotient of H by N exists, and the map $H \rightarrow H/N$ is a quotient map. Conversely, if $H \rightarrow Q$ is a quotient map with kernel N , then Q is a quotient of H by N .* $\square$

In the setting of the theorem, the surjection $H(\bar{k}) \rightarrow (H/N)(\bar{k})$ provided by Lemma 1.2.8 induces a bijection $H(\bar{k})/N(\bar{k}) = (H/N)(\bar{k})$. We will describe $(H/N)(k)$ when N is smooth as a special case of Proposition 1.10.7.

Finally we discuss normalizers, centralizers, and centers. Let G be an affine group scheme over k and let $Y \subseteq G$ be a closed subscheme. Consider the set-valued functors $N_G(Y)$ and $C_G(Y)$ on k -algebras R given by

$$\begin{aligned} N_G(Y)(R) &:= \{g \in G(R) : g \in N_{G(R')}(Y(R')) \text{ for all } R\text{-algebras } R'\}, \\ C_G(Y)(R) &:= \{g \in G(R) : g \in C_{G(R')}(Y(R')) \text{ for all } R\text{-algebras } R'\}. \end{aligned}$$

Here $N_{G(R')}(Y(R'))$ and $C_{G(R')}(Y(R'))$ are the usual set-theoretic normalizer and centralizer of $Y(R')$ in $G(R')$. We observe that

$$N_G(Y)(R) \leq N_{G(R)}(Y(R)) \quad \text{and} \quad C_G(Y)(R) \leq C_{G(R)}(Y(R)) \quad (1.12)$$

but the containment may be strict. When $N_G(Y)$ (resp. $C_G(Y)$) is represented by a closed subscheme of G then we call it the **normalizer** (resp. **centralizer**) of Y in G . We set

$$Z_G := C_G(G). \quad (1.13)$$

When Z_G is represented by a closed subscheme of G , we call it the **center** of G .

In §1.7 we will give an example of a class of subgroups that admit normalizers. Centralizers exist in greater generality:

Theorem 1.3.10 *If k is a field and G is an affine group scheme over k then the centralizer $C_G(Y)$ exists. If G is an affine group scheme of finite type over a field k with separable closure k^{sep} and $H \leq G$ is a smooth subgroup then $C_H(k) = H(k) \cap C_{G(k^{\text{sep}})}(H(k^{\text{sep}}))$.*

Proof. The first assertion is a special case of [Con14, Exercise 2.4.4(ii)]. The second is [Mil17, Proposition 1.93]. $\square$

The representability of normalizers and centralizers is delicate in general; we refer to [Con14, §2.1, 2.2] for a discussion.

1.4 Extension and restriction of scalars

Let $k \rightarrow k'$ be a homomorphism of commutative rings with identity. Given a k -algebra R , one obtains a k' -algebra $R \otimes_k k'$. Moreover, given a k' -algebra R' , one can view it as a k -algebra in the tautological manner. This gives rise to a pair of functors

$$\mathbf{Alg}_k \longrightarrow \mathbf{Alg}_{k'} \quad \text{and} \quad \mathbf{Alg}_{k'} \longrightarrow \mathbf{Alg}_k$$

known as **base change** and **restriction of scalars**, respectively. These functors are useful in that they allow us to change the base ring k .

For affine schemes, we similarly have a **base change** functor

$$k' : \mathbf{AffSch}_k \longrightarrow \mathbf{AffSch}_{k'}$$

given by $X_{k'}(R') = X(R')$; the ring representing $X_{k'}$ is simply $\mathcal{O}(X) \otimes_k k'$. In fact, this is a special case of the fiber product construction of (1.8). In the notation of loc. cit. one takes $Y = \mathrm{Spec}(k)$ and $Z = \mathrm{Spec}(k')$ and then takes $g : Z \rightarrow Y$ to be the morphism induced by $k \rightarrow k'$.

The functor in the opposite direction is a little more subtle. For any affine k' -scheme X' one can always define a set-valued functor

$$\mathrm{Res}_{k'/k} X'(R) := X'(R \otimes_k k')$$

called the **Weil restriction of scalars**. A priori this is just a set-valued functor on $\mathbf{Alg}_k$, but if k'/k satisfies certain conditions then it is representable, and hence we obtain a functor

$$\mathrm{Res}_{k'/k} : \mathbf{AffSch}_{k'} \longrightarrow \mathbf{AffSch}_k.$$

For example, it is enough to assume that k'/k is a field extension of finite degree, or more generally that k'/k is finite and locally free [BLR90, Theorem 4, §7.6].

Example 1.5. The **Deligne torus** is

$$\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_1.$$

One then has that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$ and $\mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times$. If V is a real vector space, then to give a representation $\mathbb{S} \rightarrow \mathrm{GL}_V$ is equivalent to giving a Hodge structure on V (see [Mil05] for instance).

Example 1.6. Let d be a square free integer and $L = \mathbb{Q}(\sqrt{d})$. If we view L as a 2-dimensional vector space over $\mathbb{Q}$, one obtains an injection of $L^\times$ into $\mathrm{GL}_2(\mathbb{Q})$. This can be refined into a monomorphism

$$\mathrm{Res}_{L/\mathbb{Q}} \mathbb{G}_m \longrightarrow \mathrm{GL}_2$$

of affine groups of finite type over $\mathbb{Q}$. One can choose the monomorphism so that the R -valued points of its image is

$$\left\{ \begin{pmatrix} a & db \\ b & a \end{pmatrix} : a, b \in R, a^2 - db^2 \in R^\times \right\}$$

for $\mathbb{Q}$ -algebras R .

1.5 Reductive groups

We now assume that k is a field and let $k^{\text{sep}} \leq \bar{k}$ be a separable (resp. algebraic) closure of k . Much of the theory simplifies in this case.

One has the following definition:

Definition 1.5.1. An **affine algebraic group** over a field k is an affine group scheme of finite type over k .

In other words, an affine k -group scheme G is an algebraic group if G is represented by a quotient of $k[x_1, \dots, x_n]$ for some n .

For all k -algebras R there is a natural action of $G(R)$ on

$$\mathcal{O}(G_R) = \mathcal{O}(G) \otimes_k R$$

via composition with right translation. For any k -subspace $V \leq \mathcal{O}(G)$ such that $V \otimes_k R$ is invariant under $G(R)$ for all k -algebras R we obtain a representation of G . This representation can be chosen to be faithful. If one fills in the details of this discussion, one obtains the following theorem [Mil17, §4.d]:

Theorem 1.5.2 *An affine algebraic group over a field admits a faithful representation.* $\square$

Thus affine algebraic groups are linear. Conversely, linear algebraic groups over a field are affine algebraic groups, since they admit closed immersions into the algebraic group GL_V for some vector space V .

Another nice feature of affine algebraic groups is that they are often smooth [Mil17, Proposition 1.26, Theorem 3.23]:

Theorem 1.5.3 *Let G be an affine algebraic group over a field k . If k has characteristic zero or if $G_{\bar{k}}$ is reduced, then G is smooth.* $\square$

We will also require the notion of connectedness. An affine scheme X of finite type over a field k is **connected** if the only idempotents in $\mathcal{O}(X)$ are 0 and 1. Though it is not true for general affine schemes, an affine algebraic group G over k is connected if and only if $G_{\bar{k}}$ is irreducible [Mil17, Summary 1.36]. Using this fact, one shows that if k is a subfield of $\mathbb{C}$, then G is connected if and only if $G(\mathbb{C})$ is connected in the analytic topology. Every

affine algebraic group has a normal subgroup $G^\circ \leq G$ that is the maximal connected subgroup of G . It is called the **neutral component** of G .

We now discuss the Jordan decomposition. Let M_n denote the affine scheme of n by n matrices over k . One reference for the following is [Mil17, §9.b].

Definition 1.5.4. Let k be a perfect field. An element $x \in M_n(k)$ is said to be **semisimple** if there exists $g \in \mathrm{GL}_n(\bar{k})$ such that $g^{-1}xg$ is diagonal, **nilpotent** if there exists a positive integer m such that $x^m = 0$, and **unipotent** if $x - I$ is nilpotent. For an arbitrary linear group, we say that an element $x \in G(k)$ is **semisimple** (resp. **unipotent**) if $r(x)$ is so for some faithful representation $r : G \rightarrow \mathrm{GL}_V$.

It turns out that if $x \in G(k)$ has the property that $r(x)$ is semisimple (resp. unipotent) for some faithful representation r then it is semisimple (resp. unipotent) for any faithful representation r .

Theorem 1.5.5 (Jordan decomposition) *Let G be an affine algebraic group over a perfect field k . Given $x \in G(k)$, there exist unique $x_s, x_u \in G(k)$ such that $x = x_s x_u = x_u x_s$, where x_s is semisimple and x_u is unipotent. $\square$*

This leads us to the notion of a unipotent group. An affine algebraic group G over k is **unipotent** if every nonzero representation of G has a nonzero fixed vector. Here we say that a nonzero representation V of G has a nonzero fixed vector if there is a $v \in V(k) - \{0\}$ such that $G(R)$ fixes the image of v in $V(R)$ for all k algebras R . If G is smooth then $G(\bar{k})$ consists of unipotent elements if and only if G is unipotent [Mil17, Corollary 14.12]. We remark that unipotent groups are always upper triangularizable. More precisely, an affine algebraic group G is unipotent if and only if there exists a faithful representation $r : G \rightarrow \mathrm{GL}_n$ such that the image of G is a subgroup of the group of upper triangular matrices in GL_n [Mil17, Theorem 14.5].

Let G be a smooth affine algebraic group over a field k . To define the notion of a solvable group, we recall that the **derived subgroup** $G^{\mathrm{der}} := \mathcal{D}G$ of G is the intersection of all normal subgroups $N \leq G$ such that G/N is commutative. If G is connected then G^{der} is connected. One has that

$$G^{\mathrm{der}}(\bar{k}) = \langle xyx^{-1}y^{-1} : x, y \in G(\bar{k}) \rangle.$$

In analogy with the case of abstract groups, we define

$$\mathcal{D}^n G := \mathcal{D}(\mathcal{D}^{n-1} G)$$

inductively for $n \geq 1$ and say that G is **solvable** if $\mathcal{D}^n G$ is the trivial group for n sufficiently large.

Definition 1.5.6. Let G be a smooth affine algebraic group. The **unipotent radical** $R_u(G)$ of G is the maximal connected normal unipotent subgroup

of G . The **(solvable) radical** $R(G)$ of G is the maximal connected normal solvable subgroup of G .

We remark that since a unipotent group is always solvable we have $R_u(G) \leq R(G)$.

Definition 1.5.7. A smooth connected affine algebraic group G over a field k is said to be **reductive** if $R_u(G_{\bar{k}}) = \{1\}$ and **semisimple** if $R(G_{\bar{k}}) = \{1\}$.

If k is a perfect field, then $R_u(G_{\bar{k}}) = R_u(G)_{\bar{k}}$. However, this is false in general for nonperfect fields. For more details, see [CGP10]. We emphasize that, throughout this entire book, our convention is that reductive groups are connected.

For $n > 1$ the group of upper triangular matrices in GL_n is not reductive. The most basic example of a reductive group is GL_n . It is not semisimple since $R(\mathrm{GL}_n) = Z_{\mathrm{GL}_n}$ and $Z_{\mathrm{GL}_n} \cong \mathbb{G}_m$. Here Z_G denotes the center of the affine algebraic group G . The subgroup $\mathrm{SL}_n \leq \mathrm{GL}_n$ is the derived group of GL_n and is semisimple (which implies reductive). In fact, for any reductive group G , the solvable radical $R(G)$ is the largest subgroup of the center such that $R(G)_{\bar{k}} \cong \mathbb{G}_m^n$ for some n (that is, $R(G)$ is a torus in the sense of Definition 1.7.1) [Mil17, Corollary 17.62]. Moreover G^{der} is semisimple [Mil17, Proposition 19.21]. In the special case where Z_G is smooth (which is automatic in characteristic zero by Theorem 1.5.3), the solvable radical $R(G)$ of a reductive group G is Z_G° . This last assertion follows from [Mil17, Remark 12.5].

Assume that H_1 and H_2 are closed subgroup schemes of an affine algebraic group G over k such that $H_1(R)$ normalizes $H_2(R)$ for all k -algebras R . We then have a product map $H_1 \times H_2 \rightarrow G$ given on points in a k -algebra R by

$$\begin{aligned} H_1(R) \times H_2(R) &\longrightarrow G(R) \\ (h_1, h_2) &\longmapsto h_1 h_2. \end{aligned}$$

We say $H_1 H_2 = G$ if this is a quotient map.

Suppose that G is a reductive group with center Z_G . Let Z_G^t be the largest subgroup of Z_G such that $(Z_G^t)_{\bar{k}} \cong \mathbb{G}_m^n$ for some nonnegative integer n . In other words Z_G^t is the maximal torus of Z_G in the sense of Definition 1.7.5. Then

$$G = Z_G^t G^{\mathrm{der}} \tag{1.14}$$

[Mil17, §21.f]. We note that

$$Z_G \cap G^{\mathrm{der}}$$

is the center of G^{der} . It is a finite group scheme [Mil17, Proposition 19.21].

We warn the reader that one has to be a bit careful with the statement that $G = Z_G^t G^{\mathrm{der}}$, and with similar statements involving products of affine algebraic groups in a larger group below. In particular,

$$Z_G^t(\bar{k}) \times G^{\text{der}}(\bar{k}) \longrightarrow G(\bar{k})$$

is surjective, but

$$Z_G^t(k) \times G^{\text{der}}(k) \longrightarrow G(k)$$

need not be. This is false even for $k = \mathbb{Q}$ and $G = \text{GL}_2$.

The following theorem of Mostow [Mos56] states that we can always break an affine algebraic group in characteristic zero into a reductive and unipotent part:

Theorem 1.5.8 *Let G be an affine algebraic group over a characteristic zero field k . Then there is a subgroup $M \leq G$ such that M° is reductive and*

$$G = MR_u(G).$$

All such subgroups M are conjugate under $R_u(G)(k)$. □

This theorem motivates the following definition:

Definition 1.5.9. A subgroup $M \leq G$ of a connected affine algebraic group G over a field k is a **Levi subgroup** if the restriction of the quotient map $G_{\bar{k}} \rightarrow G_{\bar{k}}/R_u(G_{\bar{k}})$ to $M_{\bar{k}}$ induces an isomorphism

$$M_{\bar{k}} \xrightarrow{\sim} G_{\bar{k}}/R_u(G_{\bar{k}}).$$

The reason that the definition is stated in terms of the algebraic closure $\bar{k}$ is that, in positive characteristic, one can have $R_u(G)_{\bar{k}} \subsetneq R_u(G_{\bar{k}})$. This leads to the subject of pseudo-reductive groups (see [CGP10]). If G has a Levi subgroup M and there is a subgroup $N \leq G$ such that $N_{\bar{k}} = R_u(G_{\bar{k}})$ then $G = MN$ and the product is semidirect [Mil17, §25.48]. In this case the decomposition $G = MN$ is called a **Levi decomposition**. Levi decompositions do not exist in general in positive characteristic but they do exist for parabolic subgroups (see §B.3).

1.6 Lie algebras

Now that we have defined reductive groups, we could ask for a classification of them, or more generally for a classification of morphisms $H \rightarrow G$ of reductive groups. The first step in this process is to linearize the problem using objects known as Lie algebras. After defining these objects in the current section, we will return to the question of classification in §1.7 and Theorem 1.8.8.

For simplicity we assume in this section that k is a field. For the general case we refer to [DG74, §II].

Definition 1.6.1. A **Lie algebra** (over k) is a k -vector space $\mathfrak{g}$ together with a bilinear pairing

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

satisfying the following assumptions:

- (a) $[X, X] = 0$ for all $X \in \mathfrak{g}$.
- (b) For $X, Y, Z \in \mathfrak{g}$,

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \quad (1.15)$$

Morphisms of Lie algebras are k -linear maps preserving $[\cdot, \cdot]$.

The pairing $[\cdot, \cdot]$ is known as the **Lie bracket** of the Lie algebra $\mathfrak{g}$ and the identity (1.15) is known as the **Jacobi identity**. We remark that if k'/k is a field extension then $\mathfrak{g} \otimes_k k'$ inherits the structure of a Lie algebra over k' in a natural manner.

Let $\mathbf{LAG}_k$ denote the category of linear affine group schemes over k and let $\mathbf{LieAlg}_k$ denote the category of Lie algebras over k . There exists a functor

$$\mathrm{Lie} : \mathbf{LAG}_k \longrightarrow \mathbf{LieAlg}_k$$

defined by

$$\mathrm{Lie} G = \ker(G(k[t]/(t^2)) \longrightarrow G(k)). \quad (1.16)$$

Here the implicit map is evaluation at $t = 0$. From the definition just given, it is not clear that $\mathrm{Lie} G$ is a Lie algebra, or even a k -vector space, for that matter. At the moment, it is only a set-valued functor. There is one important special case where it is easy to deduce a Lie algebra structure. Let $G = \mathrm{GL}_V$ for a finite dimensional vector space V . In this case it is not hard to see that

$$\begin{aligned} \mathrm{Lie} G &= \{I_V + Xt : X \in \mathrm{End}_{k\text{-linear}}(V)\} \xrightarrow{\sim} \mathrm{End}_{k\text{-linear}}(V) \\ I_V + Xt &\longmapsto X. \end{aligned}$$

This Lie algebra is traditionally denoted by $\mathfrak{gl}_V$, or simply $\mathfrak{gl}_n$ when $V = k^n$. One defines

$$[X, Y] = X \circ Y - Y \circ X$$

and verifies that it satisfies the axioms for a Lie bracket.

In order to deduce that $\mathrm{Lie} G$ is a Lie algebra in general, we essentially just embed $\mathrm{Lie} G$ in $\mathfrak{gl}_V$, but in a functorial way. The first step is to give $\mathrm{Lie} G$ a k -vector space structure. Let

$$\epsilon : \mathcal{O}(G) \longrightarrow k \quad (1.17)$$

be the coidentity, i.e. the homomorphism corresponding to the identity element of $G(k)$.

Lemma 1.6.2 *There is a canonical bijection*

$$\mathrm{Lie} G \xrightarrow{\sim} \mathrm{Hom}_{k\text{-linear}}(\ker(\epsilon)/\ker(\epsilon)^2, k).$$

In particular, Lie is a functor from $\mathbf{LAG}_k$ to the category of finite dimensional k -vector spaces.

Using the k -vector space structure of $\mathfrak{g}$ given above, we see that if k'/k is a field extension then we obtain a map

$$(\mathrm{Lie} G) \otimes_k k' \longrightarrow \mathrm{Lie} G_{k'}.$$

Since we have assumed k is a field, it is bijective [Mil17, §10.19]. In other words, the functor Lie is compatible with base change.

Proof. An element of $\mathrm{Lie} G$ is a k -algebra homomorphism

$$\varphi : \mathcal{O}(G) \longrightarrow k[t]/(t^2)$$

such that the composite with the natural map $k[t]/(t^2) \rightarrow k$ is ϵ . Thus φ maps the ideal $\ker(\epsilon)$ into (t) and hence factors through $\mathcal{O}(G)/\ker(\epsilon)^2$. Since $\mathcal{O}(G)/\ker(\epsilon)^2 = k \oplus \ker(\epsilon)/\ker(\epsilon)^2$ (see Exercise 1.9) we obtain a bijection

$$\begin{aligned} \mathrm{Lie} G &\longrightarrow \mathrm{Hom}_{k\text{-linear}}(\ker(\epsilon)/\ker(\epsilon)^2, k) \\ \varphi &\longmapsto \varphi|_{\ker(\epsilon)/\ker(\epsilon)^2} \end{aligned}$$

as claimed. We leave it to the reader to check the functoriality assertion. $\square$

The reader may have encountered the Lie algebra of a real Lie group. Briefly, the real Lie algebra of a real Lie group is the tangent space to the identity of the Lie group equipped with the Lie bracket of vector fields. The relationship between this definition and the definition in (1.16) when $k = \mathbb{R}$ is explained briefly in §4.2.

The group G always admits a representation

$$\mathrm{Ad} : G \longrightarrow \mathrm{GL}_{\mathrm{Lie} G}.$$

We define it briefly; more details are given in [Mil17, §10.d]. Let R be a k -algebra; we must construct the homomorphism $\mathrm{Ad} : G(R) \rightarrow \mathrm{GL}_{\mathrm{Lie} G}(R)$. Note that the map $R \rightarrow R[t]/(t^2)$ giving $R[t]/(t^2)$ its R -algebra structure gives rise to an injection $G(R) \hookrightarrow G(R[t]/(t^2))$. The conjugation action of $G(R[t]/(t^2))$ on itself restricts to yield an action of $G(R)$ on $G(R[t]/(t^2))$ that preserves

$$\ker(G(R[t]/(t^2)) \longrightarrow G(R)).$$

Arguing as in the proof of Lemma 1.6.2, the set $\ker(G(R[t]/(t^2)) \rightarrow G(R))$ can be given the structure of an R -module. Moreover the action of $G(R)$ on this module is R -linear. The natural map

$$(\mathrm{Lie} G) \otimes_k R \longrightarrow \ker(G(R[t]/(t^2)) \rightarrow G(R))$$

is an R -module isomorphism, and we obtain the action denoted Ad above. It is known as the **adjoint representation**.

We finally give the reader a definition of the bracket on $\text{Lie } G$. For this we note that, by functoriality of Lie as a k -vector space-valued functor, the morphism Ad gives rise to a morphism of finite dimensional k -vector spaces

$$\text{ad} : \text{Lie } G \longrightarrow \text{Lie } \text{GL}_{\text{Lie } G} = \mathfrak{gl}_{\text{Lie } G}.$$

We then define

$$[A, X] = \text{ad}(A)(X).$$

One checks that this is indeed a Lie bracket and that when $G = \text{GL}_V$ this recovers the earlier definition of the bracket.

In practice, to compute the Lie algebra of a linear affine group G , one just chooses an embedding $G \rightarrow \text{GL}_V$ and computes $\text{Lie } G$ in terms of the conditions that cut the group G from GL_V . One then obtains the bracket for free; it is just the restriction of the bracket on GL_V .

Example 1.7. If $G = \text{SL}_n$, the special linear group of matrices in GL_n of determinant 1, then

$$\det(I_n + Xt) = 1 + \text{tr}(X)t \pmod{t^2}$$

by Taylor expansion, so

$$\mathfrak{sl}_n := \text{Lie } \text{SL}_n = \{X \in \mathfrak{gl}_n : \text{tr}(X) = 0\}.$$

Notice how the use of the Taylor expansion amounted to differentiating some equation once and then taking the zero term; this method of determining Lie algebras works for at least all the classical groups, which are roughly subgroups of GL_V defined as the subgroup fixing various perfect pairings on V (see Exercise 1.10).

1.7 Tori

Throughout this section, we assume that k is a field with separable closure k^{sep} and algebraic closure $\bar{k}$. Set

$$\text{Gal}_k := \text{Gal}(k^{\text{sep}}/k).$$

Definition 1.7.1. An **algebraic torus** or simply **torus** is an affine algebraic group T over k such that $T_{\bar{k}} \cong \mathbb{G}_m^n$ for some n . The integer n is called the **rank of torus**.

We remark that if T is a torus of rank n over k then there is a finite separable extension L/k over which T **splits**, in other words, $T_L \cong \mathbb{G}_m^n$

[Con14, Lemma B.1.5] (see Definition 1.7.3 for the notion of a split torus).

Example 1.8. The subgroup $T \leq \mathrm{GL}_n$ given by

$$T(R) := \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} : t_1, \dots, t_n \in R^\times \right\}$$

is a torus. It is known as the subgroup of diagonal matrices in GL_n . The intersection $T \cap \mathrm{SL}_n$ is the subgroup of diagonal matrices in SL_n . It is again a torus.

Example 1.9. We define a special orthogonal group $\mathbb{S}^1 \leq \mathrm{SL}_2$ by stipulating that for $\mathbb{Q}$ -algebras R one has

$$\mathbb{S}^1(R) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in R \text{ and } a^2 + b^2 = 1 \right\}.$$

Since

$$\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} = \begin{pmatrix} a-bi & \\ & a+bi \end{pmatrix} \quad (1.18)$$

for any a and b in a $\mathbb{Q}(i)$ -algebra R we have $(\mathbb{S}^1)_{\mathbb{Q}(i)} \cong (\mathbb{G}_m)_{\mathbb{Q}(i)}$. We observe that $\mathbb{S}^1$ is not isomorphic to $\mathbb{G}_m$. Thus SL_2 contains two tori of the same rank that are not isomorphic.

Example 1.10. Let L/k be a finite separable field extension and let

$$\mathrm{N}_{L/k} : \mathrm{Res}_{L/k} \mathbb{G}_m \longrightarrow \mathbb{G}_m$$

be the norm map; it is given on points by

$$x \longmapsto \det(x : R \otimes_k L \rightarrow R \otimes_k L).$$

Then the kernel of $\mathrm{N}_{L/k}$ is an algebraic torus. When $k = \mathbb{Q}$ and $L = \mathbb{Q}(i)$, this torus is isomorphic to the group $\mathbb{S}^1$ of the previous example.

Definition 1.7.2. A **character** of an affine algebraic group G over k is an element of

$$X^*(G) = \mathrm{Hom}(G, \mathbb{G}_m).$$

A **cocharacter** is an element of

$$X_*(G) = \mathrm{Hom}(\mathbb{G}_m, G).$$

We observe that $X^*(G)$ is naturally an abelian group. If G is abelian the set of cocharacters $X_*(G)$ is again a group, but this is false for general G .

We warn the reader that often the notation $X^*(G)_k$ is used for what we call $X^*(G)$ (with an analogous notation for cocharacter groups). The reason is that authors want to emphasize that the characters are defined over k and

not over some field extension. Writing $X^*(G)_k$ for $X^*(G)$ is redundant so we will not use this notation.

Example 1.11. The group $X^*(\mathrm{GL}_n)$ is the free cyclic group generated by $\det$. The homomorphism $\mathbb{G}_m \rightarrow \mathrm{GL}_n$ given on points in a k -algebra R by

$$\begin{aligned} R^\times &\longrightarrow \mathrm{GL}_n(R) \\ t &\longmapsto \begin{pmatrix} t & & \\ & \ddots & \\ & & t \end{pmatrix} \end{aligned}$$

is a cocharacter, as is any $\mathrm{GL}_n(k)$ -conjugate of it.

Example 1.12. If L/k is any finite separable field extension, then $\mathrm{Res}_{L/k}\mathbb{G}_m$ is an algebraic torus. Moreover one can show that

$$X^*((\mathrm{Res}_{L/k}\mathbb{G}_m)_L) = \mathrm{Ind}_{\mathrm{Gal}_L}^{\mathrm{Gal}_k} \mathbb{Z}$$

as Gal_k -modules.

The examples above illustrate that though an algebraic torus T satisfies $T_{k^{\mathrm{sep}}} \cong \mathbb{G}_m^n$, it may not be the case that $T \cong \mathbb{G}_m^n$. This motivates the following definition:

Definition 1.7.3. An algebraic torus T over k is said to be **split** if $T \cong \mathbb{G}_m^n$ (over k) or equivalently if $X^*(T) \cong \mathbb{Z}^{\mathrm{rank}(T)}$. An algebraic torus T is said to be **anisotropic** if $X^*(T)$ is trivial.

Any torus T can be decomposed as $T = T_a T_s$ where T_s is the maximal split subtorus, T_a is the maximal anisotropic subtorus, and $T_a \cap T_s$ is finite [Mil17, Exercise 12-6].

Characters determine tori in a sense we now make precise. For a free $\mathbb{Z}$ -module M of finite rank, let $k[M]$ denote the k -group algebra of M . It is helpful to describe this group algebra explicitly. An isomorphism $M \xrightarrow{\sim} \mathbb{Z}^\ell$ for some positive integer ℓ induces an isomorphism

$$k[M] \xrightarrow{\sim} \underbrace{k[\mathbb{Z}] \otimes_k \cdots \otimes_k k[\mathbb{Z}]}_{\ell \text{ times}}.$$

The algebra $k[\mathbb{Z}]$ may be described concretely as follows:

$$\begin{aligned} k[\mathbb{Z}] &\xrightarrow{\sim} k[x, y]/(xy - 1) \xrightarrow{\sim} \mathcal{O}(\mathbb{G}_m) \\ \sum_n k_n n &\longmapsto \sum_n k_n x^n. \end{aligned} \tag{1.19}$$

Here the sum is over all integers n but we assume $k_n = 0$ for all but finitely many n . We see in particular that $\mathrm{Spec}(k[\mathbb{Z}]) \cong \mathbb{G}_m$ and $\mathrm{Spec}(k[\mathbb{Z}^\ell]) \cong \mathbb{G}_m^\ell$.

To obtain a similar description for nonsplit tori, we must incorporate a Galois action. Let M be a $\mathbb{Z}[\mathrm{Gal}_k]$ -module that is free of finite rank as a $\mathbb{Z}$ -module. We say M is continuous if the associated map

$$\mathrm{Gal}_k \longrightarrow \mathrm{GL}_M(\mathbb{Z})$$

is continuous when we give Gal_k the usual profinite topology and $\mathrm{GL}_M(\mathbb{Z})$ the discrete topology. Thus $\mathrm{Gal}_k \rightarrow \mathrm{GL}_M(\mathbb{Z})$ is continuous if and only if it factors through $\mathrm{Gal}_{k'}$ for some finite separable extension k'/k . Assume that M is a continuous $\mathbb{Z}[\mathrm{Gal}_k]$ -module that is free of finite rank as a $\mathbb{Z}$ -module, or in other words is finite rank $\mathbb{Z}$ -torsion free. Then the action of Gal_k on k^{sep} induces a k^{sep} -semilinear action of Gal_k on $k^{\mathrm{sep}}[M]$; we use this to define $(k^{\mathrm{sep}}[M])^{\mathrm{Gal}_k}$. For the proof of the following theorem we refer to [Mil17, §12.e, Theorem 12.23].

Theorem 1.7.4 *The association*

$$T \longmapsto X^*(T_{k^{\mathrm{sep}}})$$

defines a contravariant equivalence of categories between the category of algebraic tori defined over k and finite rank $\mathbb{Z}$ -torsion free continuous $\mathbb{Z}[\mathrm{Gal}_k]$ -modules. One has a canonical identification

$$\mathcal{O}(T) = (k^{\mathrm{sep}}[X^*(T_{k^{\mathrm{sep}}})])^{\mathrm{Gal}_k}.$$

□

We now consider tori that occur as subgroups of an affine algebraic group G over k .

Definition 1.7.5. A torus $T \leq G$ is **maximal** if $T_{\bar{k}}$ is a maximal element of the set of tori of $G_{\bar{k}}$, partially ordered by inclusion. It is a **maximal split torus** if it is a maximal element of the set of split tori of G , partially ordered by inclusion.

Thus a maximal split torus need not be a maximal torus. This terminology, though confusing, is standard.

Theorem 1.7.6 *Every connected smooth affine algebraic group G over k admits a maximal torus. All maximal split tori in G are conjugate under $G(k)$.*

Proof. The first result is due to Grothendieck (see [Con14, Appendix A] for instance). The second was announced by Borel and Tits; a reference is [CGP10, Theorem C.2.3]. □

In view of the second assertion of Theorem 1.7.6, the rank of a maximal split torus of G is an invariant of G ; it is known as the **rank** of G . For example, the rank of SL_n is $n - 1$, and the rank of GL_n is n . The rank cannot decrease under base change, but it can increase: For example, the group $\mathbb{S}^1$ defined in Example 1.9 has rank 0, but $(\mathbb{S}^1)_{\mathbb{Q}(i)}$ has rank 1.

Definition 1.7.7. A reductive group G is said to be **split** if there exists a maximal torus of G that is split.

Thus a reductive group over a field k is split if and only if its rank is equal to the rank of $G_{\bar{k}}$. For example, if $T \leq \mathrm{GL}_n$ is the torus of diagonal matrices then T is a split maximal torus, and hence GL_n is split.

For the remainder of this section, G is a reductive group and $T \leq G$ is a torus. Let $N_G(T)$ be the normalizer of T in G and let $C_G(T)$ be the centralizer of T in G . A priori these are just the set-valued functors defined in §1.3, but they are represented by subgroups of G by [Con14, Proposition 2.1.2] and Theorem 1.3.10. The torus T is maximal if and only if $C_G(T) = T$ [Mil17, Corollary 17.84]. The group schemes $N_G(T)$, $C_G(T)$ are algebraic subgroups of G and $C_G(T)$ is normal in $N_G(T)$. Moreover $N_G(T)$ and $C_G(T)$ are smooth [Con14, Proposition 2.1.2, Lemma 2.2.4].

Definition 1.7.8. The **Weyl Group** of T in G is $W(G, T) := N_G(T)/C_G(T)$.

It turns out that $W(G, T)$ is again a smooth (even étale) group scheme of finite type over k . In particular $W(G, T)(\bar{k})$ is finite. Moreover, one has that

$$W(G, T)(k^{\mathrm{sep}}) = N_G(T)(k^{\mathrm{sep}})/C_G(T)(k^{\mathrm{sep}}) \quad (1.20)$$

by Proposition 1.10.7.

In important special cases the Weyl group is not just étale, but constant in a sense we now describe. Let Γ be a finite abstract group. Consider the functor $\mathbf{Alg}_k \rightarrow \mathbf{Group}$ assigning to any k -algebra R the group of locally constant functions $\mathrm{Spec}(R) \rightarrow \Gamma$. Here we give $\mathrm{Spec}(R)$ the Zariski topology and Γ the discrete topology. This functor is an affine group scheme over k . It may be naturally identified with

$$\mathrm{Spec}(k^\Gamma), \quad (1.21)$$

where $k^\Gamma := \prod_{\gamma \in \Gamma} k$. An affine group scheme over k is **constant** if it is isomorphic to $\mathrm{Spec}(k^\Gamma)$ for some abstract group Γ . One motivation for this terminology is that if G is a constant group scheme over a field k , then for all field extensions k'/k , the map $G(k) \rightarrow G(k')$ is an isomorphism. In particular, $G(k)$ is canonically isomorphic to $G(k^{\mathrm{sep}})$.

Example 1.13. The **trivial** group scheme over k is the constant group scheme

$$\mathrm{Spec}(k^\Gamma) = \mathrm{Spec}(k),$$

where Γ is the trivial group. We denote this group scheme by 1.

Lemma 1.7.9 *If T is a maximal split torus of a reductive group G then the map $N_G(T)(k) \rightarrow W(G, T)(k)$ induces an isomorphism*

$$N_G(T)(k)/C_G(T)(k) \xrightarrow{\sim} W(G, T)(k).$$

Moreover $W(G, T)$ is a constant group scheme.

We emphasize that we are not assuming G is split in this lemma; thus T need not be a maximal torus.

Proof. See [Bor91, Chapter V, Theorem 21.2]. $\square$

To give some examples of Weyl groups of nonsplit tori, let L/k be an étale k -algebra of rank n (for example, a separable field extension of degree n). Choosing a basis for L we obtain a closed immersion

$$\mathrm{Res}_{L/k} \mathbb{G}_m \longrightarrow \mathrm{GL}_n.$$

In particular, if L/k is Galois then

$$W(\mathrm{GL}_n, \mathrm{Res}_{L/k} \mathbb{G}_m)(k) \cong \mathrm{Gal}(L/k).$$

Every maximal torus in GL_n arises in this manner for some L/k (see Exercise 1.11).

After spending all this time discussing tori, we give one important motivation for introducing them. Suppose that G is a split reductive group and that $T \leq G$ is a split maximal torus over k . Let

$$r : G \longrightarrow \mathrm{GL}_V$$

be a representation. Since T is abelian and reductive we can decompose V as

$$V = \bigoplus_{\alpha \in X^*(T)} V_\alpha, \tag{1.22}$$

where for k -algebras R

$$V_\alpha \otimes_k R = \{v \in V \otimes_k R : r(t)v = \alpha(t)v \text{ for all } t \in T(R)\}. \tag{1.23}$$

The α such that $V_\alpha \neq 0$ are known as **weights** and the V_α are known as **weight spaces**. The dimension $\dim V_\alpha$ is known as the **multiplicity** of the weight α .

One has the following basic theorem:

Theorem 1.7.10 *Suppose that G is a split reductive group and that V is a semisimple representation of G as above. The collection of weights α together with the weight multiplicities $\dim V_\alpha$ determines the representation V up to isomorphism.* $\square$

It is not hard to see that the set of weights is invariant under the natural action of $W(G, T)(k)$ on $X^*(T)$. Though the action does not preserve individual weights, it does preserve the multiplicities of weights.

One can place a partial ordering on $X^*(T)$ such that each irreducible representation of G admits a highest weight with respect to the ordering, unique up to the action of $W(T, G)(k)$. Moreover, one can characterize the set

of highest weights that can occur. This is known as **Cartan-Weyl highest weight theory** (see [Mil17, Chapter 22] for instance).

The key fact to take from this discussion is that the representation theory of G is determined by the restriction of the representation to a sufficiently large subgroup (in this case a maximal torus). Happily, the representation theory of a maximal torus is fairly simple and can be studied via linear algebra. The idea of studying representations by studying their restriction to subgroups is an incredibly useful one; it will come up in various guises throughout this book.

1.8 Root data

Let G be a split reductive group over a field k and let $T \leq G$ be a split maximal torus. Our next goal is to associate to such a pair (G, T) a root datum $\Psi(G, T) = (X^*(T), X_*(T), \Phi(G, T), \Phi^\vee(G, T))$. The root datum is a refinement of the related notion of a root system (also defined below) that Demazure introduced to systematically keep track of the center of G [DG74, Expose XXII]. The root datum characterizes G , and in fact Demazure proved that it characterizes G even in the case where k is replaced by $\mathbb{Z}$. This in some sense completed work of Chevalley who proved the analogous statement for semisimple groups over fields [Che58].

Let $\mathfrak{g}$ denote the Lie algebra of G . As we saw in §1.6, one has an adjoint representation

$$\mathrm{Ad} : G \longrightarrow \mathrm{GL}_{\mathfrak{g}}. \quad (1.24)$$

For example, when $G = \mathrm{GL}_n$ this is the usual action of GL_n on the space of $n \times n$ matrices $\mathfrak{gl}_n$ by conjugation.

By restriction we obtain an action of T on $\mathfrak{g}$ which we decompose into weight spaces as in (1.22). For a character $\alpha \in X^*(T)$, let $\mathfrak{g}_\alpha$ be the corresponding weight space. Thus for k -algebras R ,

$$\mathfrak{g}_\alpha \otimes_k R := \{X \in \mathfrak{g} \otimes_k R \mid \mathrm{Ad}(t)X = \alpha(t)X \text{ for all } t \in T(R)\}. \quad (1.25)$$

Definition 1.8.1. The nontrivial $\alpha \in X^*(T)$ such that $\mathfrak{g}_\alpha \neq 0$ are called the **roots** of T in G . We let $\Phi(G, T)$ be the (finite) set of all such roots α , and call the corresponding $\mathfrak{g}_\alpha$ **root spaces**.

Just as in (1.22) we have a decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi(G, T)} \mathfrak{g}_\alpha, \quad (1.26)$$

where $\mathfrak{t} := \mathrm{Lie} T$. It turns out that each of the root spaces $\mathfrak{g}_\alpha$ are one dimensional.

One turns the set of roots $\Phi(G, T)$ into a combinatorial gadget as follows. Let

$$V := \langle \Phi(G, T) \rangle \otimes_{\mathbb{Z}} \mathbb{R},$$

where $\langle \Phi(G, T) \rangle \subset X^*(T)$ denotes the ($\mathbb{Z}$ -linear) span of $\Phi(G, T)$. The pair $(\Phi(G, T), V)$ satisfies remarkable symmetry properties that are axiomatized in the following:

Definition 1.8.2. Let V be a finite dimensional $\mathbb{R}$ -vector space, and Φ a subset of V . We say that (Φ, V) is a **root system** if the following three conditions are satisfied:

- (a) Φ is finite, does not contain 0, and spans V .
- (b) For each $\alpha \in \Phi$ there exists a reflection s_α relative to α (i.e. an involution of V with $s_\alpha(\alpha) = -\alpha$ and restricting to the identity on a subspace of V of codimension 1) such that $s_\alpha(\Phi) = \Phi$.
- (c) For every $\alpha, \beta \in \Phi$, $s_\alpha(\beta) - \beta$ is an integer multiple of α .

A root system (Φ, V) is said to be of **rank** $\dim_{\mathbb{R}} V$ and is said to be **reduced** if for each $\alpha \in \Phi$, $\pm\alpha$ are the only multiples of α in Φ .

We will not need the notion until §1.9, but a subset

$$\Delta \subset \Phi \tag{1.27}$$

is a **base** or a **set of simple roots** if it is a basis of V and each $\alpha \in \Phi$ can be uniquely expressed as

$$\alpha = \sum_{\beta \in \Delta} c_\beta \beta,$$

where either $c_\beta \geq 0$ for all $\beta \in \Delta$ or $c_\beta \leq 0$ for all $\beta \in \Delta$. Any root system admits a base. We define

$$\Phi^+ \subset \Phi \quad (\text{resp. } \Phi^- \subset \Phi)$$

to be the set of roots expressible as a sum of positive (resp. negative) linear combinations of a nonempty subset of elements in the base. Then $\Phi = \Phi^+ \amalg \Phi^-$. Conversely, suppose that we are given a subset $\Phi^+ \subset \Phi$ such that

- (a) for every $\alpha \in \Phi$, exactly one of $\pm\alpha$ is in Φ^+ ;
- (b) if $\alpha, \beta \in \Phi^+$ satisfy $\alpha + \beta \in \Phi$ then $\alpha + \beta \in \Phi^+$.

We then call Φ^+ a **set of positive roots**. Setting $\Phi^- = \{-\alpha : \alpha \in \Phi^+\}$ we have $\Phi = \Phi^+ \amalg \Phi^-$. An element of Φ^+ is **simple** with respect to Φ^+ if it cannot be written as a sum of two elements of Φ^+ . The set of elements that are simple with respect to Φ^+ then forms a base (or in other words a set of simple roots). Thus there is a natural bijection between bases of Φ and sets of positive roots in Φ . We refer to [Hum78, §10] for proofs of the assertions in this paragraph.

The **Weyl group** of (Φ, V) is the subgroup of $\mathrm{GL}(V)$ generated by the reflections s_α :

$$W(\Phi, V) := \langle s_\alpha : \alpha \in \Phi \rangle \subseteq \mathrm{GL}(V).$$

The following is [Mil17, Corollary 21.12 and Theorem 21.37]:

Proposition 1.8.3 *If*

$$(\Phi, V) = (\Phi(G, T), \langle \Phi(G, T) \rangle \otimes_{\mathbb{Z}} \mathbb{R})$$

is the root system associated with the split maximal torus $T \leq G$ over k then (Φ, V) is reduced, the natural map $W(G, T)(k) \rightarrow \mathrm{GL}(V)$ is injective, and

$$W(\Phi, V) = \mathrm{im}(W(G, T)(k) \hookrightarrow \mathrm{GL}(V)).$$

□

If G is semisimple then $\Phi(G, T)$ does not change if we replace G by the adjoint group $G^{\mathrm{ad}} := G/Z_G$. In particular root systems do not provide enough information to classify reductive groups. Thus we incorporate more information from G in order to create an invariant, called the **root datum**, which determines G up to isomorphism. We refer to [Mil17, §21.c] for more details on the discussion that follows.

For $\alpha \in \Phi(G, T)$, let T_α be the maximal subtorus of $\ker(\alpha) < T$ and let

$$G_\alpha := C_G(T_\alpha) \tag{1.28}$$

be its centralizer. Then G_α is a split reductive group and $T < G_\alpha$ is a maximal torus. The derived group of G_α has rank 1 (and hence is either SL_2 or PGL_2). One has that

$$\mathrm{Lie} G_\alpha = \mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}.$$

There is a unique unipotent smooth connected subgroup

$$N_\alpha < G_\alpha \tag{1.29}$$

normalized by T with Lie algebra $\mathfrak{g}_\alpha$. The group N_α is called the **root group** of α . One has that $N_\alpha \cong \mathbb{G}_a$.

The Weyl group $W(G_\alpha, T)$ is the constant group scheme with

$$W(G_\alpha, T)(k) = \langle s_\alpha \rangle \cong \mathbb{Z}/2. \tag{1.30}$$

We have used s_α to denote the generator of this Weyl group. This is consistent with our earlier notation in the following sense. We have a homomorphism

$$W(G_\alpha, T)(k) \longrightarrow W(G, T)(k) \xrightarrow{\sim} W(\Phi, V), \tag{1.31}$$

where the first arrow is the natural map and the second is the isomorphism of Proposition 1.8.3. The image of s_α under (1.31) is the reflection in $W(\Phi, V)$ that was previously denoted s_α .

Given $(\alpha, \beta) \in X^*(T) \times X_*(T)$ we have a homomorphism $\alpha \circ \beta : \mathbb{G}_m \rightarrow \mathbb{G}_m$. Thus there is a perfect pairing

$$\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \longrightarrow \mathbb{Z} \quad (1.32)$$

given stipulating that

$$\alpha \circ \beta(a) = a^{\langle \alpha, \beta \rangle}$$

where $a \in R^\times$ for a k -algebra R . There is a unique $\alpha^\vee \in X_*(T)$ such that

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \quad (1.33)$$

for all $x \in X^*(T)$. The element $\alpha^\vee \in X_*(T)$ is called the **coroot** associated to α . One has that $\langle \alpha, \alpha^\vee \rangle = 2$. Let

$$\Phi^\vee(G, T) = \{\alpha^\vee : \alpha \in \Phi(G, T)\} \quad \text{and} \quad V^\vee := \langle \Phi^\vee(G, T) \rangle \otimes_{\mathbb{Z}} \mathbb{R}.$$

Then one has the following lemma:

Lemma 1.8.4 *The pair $(\Phi^\vee(G, T), V^\vee)$ is a root system.* □

The pair $(\Phi^\vee(G, T), V^\vee)$ is called the **dual root system** of $(\Phi(G, T), V)$.

Theorem 1.8.8 is the fundamental result that the quadruple

$$\Psi(G, T) = (X^*(T), X_*(T), \Phi(G, T), \Phi^\vee(G, T))$$

attached to $T \leq G$ contains enough information to characterize G among all split reductive k -groups. To make this precise, suppose we have quadruple $(X, Y, \Phi, \Phi^\vee)$ consisting of a pair of free abelian groups X, Y with a perfect pairing $\langle \cdot, \cdot \rangle : X \times Y \rightarrow \mathbb{Z}$, finite subsets $\Phi \subset X$ and $\Phi^\vee \subset Y$, and a bijective correspondence

$$\begin{aligned} \Phi &\longrightarrow \Phi^\vee \\ \alpha &\longmapsto \alpha^\vee. \end{aligned}$$

For each $\alpha \in \Phi$, we let $s_\alpha : X \rightarrow X$ and $s_{\alpha^\vee} : Y \rightarrow Y$ be the endomorphism given by

$$\begin{aligned} s_\alpha(x) &:= x - \langle x, \alpha^\vee \rangle \alpha, \\ s_{\alpha^\vee}(y) &:= y - \langle \alpha, y \rangle \alpha^\vee \end{aligned}$$

for any $x \in X$ and any $y \in Y$.

Definition 1.8.5. The quadruple $(X, Y, \Phi, \Phi^\vee)$ is a **root datum** if

- (a) for each $\alpha \in \Phi$, $\langle \alpha, \alpha^\vee \rangle = 2$ and
- (b) for each $\alpha \in \Phi$, $s_\alpha(\Phi) \subseteq \Phi$ and $s_{\alpha^\vee}(\Phi^\vee) \subseteq \Phi^\vee$.

We say that a root datum is **reduced** if for each $\alpha \in \Phi$ the only multiples of α that are in Φ are $\pm\alpha$.

For more information about root data, we recommend [Mil17, Appendix C]. One can find a proof in [Mil17, Corollary 21.12] that $\Psi(G, T)$ is a reduced root datum.

Definition 1.8.6. A **central isogeny** of root data

$$(X', Y', \Phi', \Phi'^\vee) \longrightarrow (X, Y, \Phi, \Phi^\vee)$$

is an injective morphism $f : X' \rightarrow X$ with finite cokernel such that there is a bijection $\iota : \Phi \rightarrow \Phi'$ satisfying

$$f(\iota(\alpha)) = \alpha \quad \text{and} \quad f^\vee(\alpha^\vee) = \iota(\alpha)^\vee$$

for all $\alpha \in \Phi$. Here $f^\vee : Y \rightarrow Y'$ is the dual of f . A central isogeny is an **isomorphism** of root data if f is an isomorphism of $\mathbb{Z}$ -modules.

This allows us to define the category

$$\mathbf{RRD} \tag{1.34}$$

of reduced root data; the objects are reduced root data and the morphisms are central isogenies of reduced root data.

The definition of a central isogeny of root data is motivated by the definition of a central isogeny of affine algebraic groups.

Definition 1.8.7. Let H and H' be connected affine algebraic groups over a field k . A **central isogeny** is a surjective homomorphism $H \rightarrow H'$ with finite kernel contained in the center of H .

Here when we say “finite kernel” we mean that the kernel is a finite group scheme. The condition that the kernel is contained in the center of H is automatic in characteristic 0, but the example of the Frobenius map shows that it is not automatic in positive characteristic [Mil17, §18.a]. We also point out that a composite of central isogenies between reductive groups is again a central isogeny [Mil17, Proposition 18.2].

For more information on the statements in this paragraph, we refer to [Mil17, §23.g]. A semisimple group G is **simply connected** if any central isogeny $H \rightarrow G$ is an isomorphism, and **adjoint** if any central isogeny $G \rightarrow H$ is an isomorphism. Any semisimple group G sits in a sequence of central isogenies

$$G^{\text{sc}} \longrightarrow G \longrightarrow G^{\text{ad}} := G/Z_G,$$

where G^{sc} is simply connected and G^{ad} is adjoint. If G is split the groups G^{sc} and G^{ad} are uniquely determined by the root system of G . If $G \rightarrow H$

is a central isogeny between two semisimple k -groups then G and H have the same root system, but not necessarily the same root datum. In general, simply connected groups have the largest center out of the set of semisimple groups with the same root system and adjoint groups have trivial center. For example, SL_n is simply connected and PGL_n is adjoint. The quotient map $\mathrm{SL}_n \rightarrow \mathrm{PGL}_n$ is a central isogeny.

We now return to the case where G is a split reductive group over k . We note that an isomorphism of reductive groups over k gives rise to an isomorphism of their underlying root data. Indeed, if $G \rightarrow G'$ is an isomorphism of reductive groups over k , then upon precomposing the isomorphism with an inner automorphism of G and postcomposing with an inner automorphism of G' , we can assume that the isomorphism maps a given maximal torus T of G to a given maximal torus T' of G' , and hence induces an isomorphism of the underlying root data. A similar statement is true for general central isogenies [Mil17, Proposition 23.5].

Let

$$\mathbf{Spl}_k \tag{1.35}$$

be the category whose objects are pairs (G, T) where G is a split reductive group over k and $T \leq G$ is a split maximal torus. The morphisms $(G, T) \rightarrow (G', T')$ are central isogenies taking T to T' up to equivalence, where we say two morphisms are **equivalent** if they differ by conjugation by an element of $(T'/Z_{G'})(k)$. A morphism $(G, T) \rightarrow (G', T')$ in $\mathbf{Spl}_k$ induces a central isogeny $\Psi(G', T') \rightarrow \Psi(G, T)$ [Mil17, Proposition 23.5]. The following result not only implies that every root datum is the root datum of a split reductive group, but also implies that every central isogeny of root data arises from a central isogeny of split reductive groups [Mil17, Theorem 23.25]:

Theorem 1.8.8 (Chevalley, Demazure) *The map*

$$\begin{aligned} \mathbf{Spl}_k &\longrightarrow \mathbf{RRD} \\ (G, T) &\longmapsto \Psi(G, T) \end{aligned}$$

is a contravariant equivalence of categories. □

It is interesting to note that though $\mathbf{Spl}_k$ a priori depends on k , $\mathbf{RRD}$ does not. In particular, if k'/k is an extension of fields, then we obtain a functor $\mathbf{Spl}_k \rightarrow \mathbf{Spl}_{k'}$ by base change that commutes with the functors to $\mathbf{RRD}$:

$$\begin{array}{ccc} & \mathbf{Spl}_{k'} & \\ \uparrow & \searrow & \\ & \mathbf{RRD} & \\ \uparrow & \nearrow & \\ \mathbf{Spl}_k & & \end{array} \tag{1.36}$$

It follows from Theorem 1.8.8 that all arrows in the diagram above are equivalences of categories.

Theorem 1.8.8 tells us that if we can classify root data, then we can classify split reductive groups. In fact, Killing obtained a classification of the underlying root systems (up to some mistakes) in [Kil88, Kil90]. Killing's work was revisited and corrected in E. Cartan's thesis. There are many references for the classification, including [Mil17, Appendix C.g]. One can essentially reduce the classification of root data to the classification of root systems as in [Mil17, Appendix C.f].

If $(X, Y, \Phi, \Phi^\vee)$ is a root datum, then so is $(Y, X, \Phi^\vee, \Phi)$. It is known as the **dual root datum** to $(X, Y, \Phi, \Phi^\vee)$. If $(G, T) \in \mathbf{Spl}_k$ then the root datum dual to $\Phi(G, T)$ gives rise to a pair

$$(\widehat{G}, \widehat{T}) \in \mathbf{Spl}_{\mathbb{C}}$$

called the **complex dual** of (G, T) (or sometimes simply G). We note that there is an isomorphism

$$\begin{aligned} W(G, T)(k) &\xrightarrow{\sim} W(\widehat{G}, \widehat{T})(\mathbb{C}) \\ s_\alpha &\longmapsto s_{\alpha^\vee}. \end{aligned} \tag{1.37}$$

The formation of the complex dual yields a contravariant equivalence of categories

$$\begin{aligned} \mathbf{Spl}_k &\longrightarrow \mathbf{Spl}_{\mathbb{C}} \\ (G, T) &\longmapsto (\widehat{G}, \widehat{T}). \end{aligned} \tag{1.38}$$

This implies, in particular, that if G is adjoint then $\widehat{G}$ is simply connected and if G is simply connected then $\widehat{G}$ is adjoint. The complex dual $\widehat{G}$ is a key component in the definition of Langlands dual group (see §7.3). We also note that if k'/k is an extension of fields then the dual groups of (G, T) and $(G_{k'}, T_{k'})$ may be canonically identified in view of the commutativity of the diagram (1.36).

Certainly, not every morphism between split reductive groups is a central isogeny. One might ask if one could extend the notion of a morphism of root data, and thereby use root data to classify morphisms between split reductive groups. We are unaware of such a definition. However, it is the case that a great deal of information about morphisms between split reductive groups can be deduced by considering root data. A systematic account of this for classical groups is given in Dynkin's work [Dyn52].

We now record arguably the most basic example of a root datum:

Example 1.14. Let $G = \mathrm{GL}_n$. For a k -algebra R , let

$$T(R) := \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mid t_i \in R^\times \right\}$$

be the group of diagonal matrices in $\mathrm{GL}_n(R)$. Then T is a split maximal torus in G . The groups of characters and of cocharacters of T are both isomorphic to $\mathbb{Z}^n$ via

$$(k_1, \dots, k_n) \mapsto \left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_1^{k_1} \cdots t_n^{k_n} \right) \quad (1.39)$$

and

$$(k_1, \dots, k_n) \mapsto \left(t \mapsto \begin{pmatrix} t^{k_1} & & \\ & \ddots & \\ & & t^{k_n} \end{pmatrix} \right), \quad (1.40)$$

respectively. Note that with these identifications, the natural pairing $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \rightarrow \mathbb{Z}$ corresponds to the standard “inner product” in $\mathbb{Z}^n$. We let $\varepsilon_i \in X^*(T)$ correspond to the elementary vector e_i under (1.39). Then the roots of G relative to T are the characters

$$\varepsilon_i - \varepsilon_j : \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \mapsto t_i t_j^{-1} \quad (1.41)$$

for every pair of integers (i, j) , $1 \leq i, j \leq n$ with $i \neq j$. Here we have followed tradition by writing multiplication in the character group $X_*(T)$ additively. The root space $(\mathfrak{gl}_n)_{\varepsilon_i - \varepsilon_j}$ is the linear span of the $n \times n$ matrix e_{ij} with all entries zero except the ij -th entry and a 1 in the ij -th entry. The coroot $(\varepsilon_i - \varepsilon_j)^\vee$ associated with $\varepsilon_i - \varepsilon_j$ is the map sending t to the diagonal matrix with t in the i th entry and t^{-1} in the j th entry and 1 in all other entries. The root group of $\varepsilon_i - \varepsilon_j$ is

$$N_{\varepsilon_i - \varepsilon_j}(R) := I_n + R e_{ij}. \quad (1.42)$$

1.9 Parabolic subgroups

We assume in this section that G is a reductive group over a field k with algebraic closure $\bar{k}$.

Definition 1.9.1. A subgroup $B \leq G$ is a **Borel subgroup** if $B_{\bar{k}}$ is a maximal smooth connected solvable subgroup of $G_{\bar{k}}$. A smooth subgroup $P \leq G$ is a **parabolic subgroup** if $P_{\bar{k}}$ contains a Borel subgroup of $G_{\bar{k}}$.

The group $G_{\bar{k}}$ trivially has Borel subgroups, and hence G always has at least one parabolic subgroup, namely, G itself. A **proper** parabolic subgroup is a parabolic subgroup of G not equal to G . In general, a reductive group need not have proper parabolic subgroups. For example, if D is a finite dimensional division algebra with center k and G is the reductive group defined by

$$G(R) = (D \otimes_k R)^\times \quad (1.43)$$

for a k -algebra R then G does not have a proper parabolic subgroup (see Exercise 1.12).

It is useful to isolate a weakening of the notion of a split reductive group:

Definition 1.9.2. A reductive group G is said to be **quasi-split** if it contains a Borel subgroup.

One can show that G is quasi-split if it is split [Spr09, Proposition 16.2.2] but the converse is not true. Indeed, take G to be the unitary group over the real numbers whose points in an $\mathbb{R}$ -algebra R are

$$G(R) = \{g \in \mathrm{GL}_2(\mathbb{C} \otimes_{\mathbb{R}} R) : \bar{g}^t \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} g = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\},$$

where the bar denotes complex conjugation. Then the subgroup of upper triangular matrices in G is a Borel subgroup of G . Thus G is quasi-split. It is not, however, split.

From the optic of finite dimensional representation theory, the behavior of a split reductive group is essentially as simple as a group over an algebraically closed field (see §1.7). Quasi-split groups are a little more technical to handle, but the existence of the Borel subgroup makes the theory not much more difficult. The fact that a general reductive group does not have a Borel subgroup (over the base field) creates more substantial problems. In this case, one has to make do with a **minimal parabolic subgroup**. Of course, in the quasi-split case, a minimal parabolic subgroup is simply a Borel subgroup.

Basic facts about parabolic subgroups come up constantly in the theory of automorphic representations. For example, they are used to understand the structure of adelic quotients at infinity (see §2.7) and are used to describe the representation theory of a reductive group inductively (see §4.9, §8.2, and Chapter 10). Thus we record some of the basic structural facts about the set of parabolic subgroups of G in this section. One reference is [Bor91, §20-21].

It is convenient to start with split tori in G . If there is no split torus contained in G then G is said to be **anisotropic**. Otherwise G is said to be **isotropic**. The derived (sub)group G^{der} is anisotropic if and only if the only parabolic subgroup of G is G itself. If G is isotropic then there exists a maximal split torus $T \leq G$ unique up to conjugation. Just as in the case when G is split (discussed in §1.8), we can decompose $\mathfrak{g}$ under the adjoint action (1.24) into eigenspaces under T :

$$\mathfrak{g} := \mathfrak{m} \oplus \bigoplus_{\alpha \in \Phi(G, T)} \mathfrak{g}_{\alpha}, \quad (1.44)$$

where $\mathfrak{g}_{\alpha}$ is defined as in (1.25). Here $\Phi(G, T)$ is the set of nontrivial weights occurring in the decomposition above, and $\mathfrak{m}$ is the trivial eigenspace. One has $\mathfrak{m} = \mathrm{Lie} C_G(T)$. As we recalled above Definition 1.7.8, $C_G(T)$ is equal to T if and only if T is a maximal torus, not just a maximal split torus. The set

$\Phi(G, T)$ is usually referred to as the set of relative roots (or more precisely, roots relative to k as opposed to $\bar{k}$), but we will avoid this terminology because we will later discuss relative trace formulae which have nothing to do with this notion of relative. The set $\Phi(G, T) \subset X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ is a root system, but it is not in general reduced. Just as in the split case, for each $\alpha \in \Phi(G, T)$ there is a unique unipotent smooth connected subgroup

$$N_{\alpha} < G \quad (1.45)$$

normalized by T with Lie algebra $\mathfrak{g}_{\alpha}$ [Mil17, 25.19]. The group N_{α} is known as the **root group** of α . The group N_{α} is not isomorphic to $\mathbb{G}_a$ in general, but it does admit a subnormal series with successive quotients isomorphic to $\mathbb{G}_a$ [Mil17, 25.19] (in positive characteristic, this is stronger than the assertion that N_{α} is unipotent).

Just as all the maximal split tori in G are conjugate under $G(k)$, the minimal parabolic subgroups are also conjugate under $G(k)$. Moreover, every minimal parabolic subgroup contains the centralizer $C_G(T)$ of a maximal split torus $T \leq G$. Thus to describe all minimal parabolic subgroups, it suffices to fix a maximal split torus $T \leq G$ and describe all minimal parabolic subgroups containing $C_G(T)$. Let $\Delta \subset \Phi(G, T)$ be a base. Then we can consider the set of positive roots Φ^+ with respect to Δ . Then there is a unique **minimal parabolic subgroup** P_0 containing $C_G(T)$ whose unipotent radical $N_0 < P_0$ satisfies

$$\mathrm{Lie} N_0 = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_{\alpha}.$$

Theorem 1.9.3 *The correspondence described above defines a bijection*

$$\{\text{bases } \Delta \subseteq \Phi(G, T)\} \xrightarrow{\sim} \{\text{minimal parabolic subgroups } P_0 \geq C_G(T)\}.$$

□

If we fix a minimal parabolic subgroup P_0 , then the parabolic subgroups containing P_0 are called **standard**. Every parabolic subgroup is conjugate under $G(k)$ to a unique standard parabolic subgroup, and two standard parabolic subgroups are $G(\bar{k})$ -conjugate if and only if they are equal. Thus to describe all parabolic subgroups of G , it suffices to fix a minimal parabolic subgroup P_0 of G and describe all standard parabolic subgroups (with respect to P_0).

To accomplish this, let $J \subseteq \Delta$ be a subset. Let

$$\Phi(J) := \mathbb{Z}J \cap \Phi(G, T). \quad (1.46)$$

There is a unique parabolic subgroup $P_J \geq P_0$ with unipotent radical N_J such that

$$\mathrm{Lie} N_J = \bigoplus_{\alpha \in \Phi^+ - (\Phi(J) \cap \Phi^+)} \mathfrak{g}_{\alpha}.$$

For the proof of the following theorem, see [Bor91, Proposition 21.12]:

Theorem 1.9.4 *There is a bijective correspondence*

$$\begin{aligned} \{J \subseteq \Delta\} &\xrightarrow{\sim} \{\text{standard parabolic subgroups of } G\} \\ J &\longmapsto P_J. \end{aligned}$$

□

We observe that the parabolic subgroup corresponding to $\emptyset$ is P_0 and the parabolic subgroup corresponding to Δ is G . For each $J \subset \Delta$, let

$$M_J \tag{1.47}$$

be the subgroup of G generated by $C_G(T)$ and the G_α with $\alpha \in \Phi(J)$ (see (1.28) for the definition of G_α). The following is [Spr09, Lemma 15.4.5]:

Lemma 1.9.5 *The subgroup M_J of G is a Levi subgroup of P_J .* □

The two bijections in theorems 1.9.3 and 1.9.4 allow us to define the notion of a parabolic subgroup opposite to a given parabolic subgroup. More precisely, suppose we are given a standard parabolic subgroup $P \leq G$ with unipotent radical N containing the centralizer of a maximal split torus T of G , i.e. $C_G(T) \leq P \leq G$. Then one has an **opposite parabolic** P^- constructed as follows: If $P = P_0$ is minimal, then we take P_0^- to be the minimal parabolic subgroup attached to the base

$$-\Delta := \{\alpha \in \Phi(G, T) : -\alpha \in \Delta\}.$$

If $P = P_J$, then we define P^- to be the parabolic subgroup containing the minimal parabolic subgroup P_0^- that is attached to the subset

$$-J := \{\alpha \in \Phi(G, T) : -\alpha \in J\} \subseteq -\Delta.$$

Still assuming $P = P_J$, we have

$$P_J \cap P^- = M_J.$$

The unipotent radical of P^- is often denoted N^- .

We collect the next two propositions for our use in §10.1. The first is [Spr09, Corollary 8.4.4]:

Proposition 1.9.6 *Let P be a parabolic subgroup of G and let $T \leq P$ be a maximal torus. There is a unique Levi subgroup of P containing T .* □

Proposition 1.9.7 *Let $P_0 \leq G$ be a minimal parabolic subgroup and let $M_0 \leq P_0$ be a Levi subgroup. If $P_0 \leq P \leq G$ then there is a unique Levi subgroup $M \leq P$ such that $M_0 \leq M$.*

Proof. Let $T \leq M_0$ be a maximal split torus and let $T' \leq M_0$ be a maximal torus containing T . Use T to define $\Phi(G, T)$ and P_0 to define a set of simple roots Δ . Thus the groups M_J and P_J are defined for all $J \subseteq \Delta$. We observe that M_0 and $C_G(T) = M_\emptyset$ both contain T' , hence $M_0 = M_\emptyset$ by Proposition 1.9.6.

We have $P = P_J$ for some subset $J \subseteq \Delta$ by Theorem 1.9.4, and hence $M := M_J$ is a Levi subgroup by Lemma 1.9.5. This proves existence. Clearly M_J contains T' , so we deduce the uniqueness assertion from Proposition 1.9.6. $\square$

Example 1.15. The subgroup $B \leq \mathrm{GL}_n$ of upper triangular matrices is a Borel subgroup. Throughout this book when we speak of standard parabolic subgroups in GL_n we will mean parabolic subgroups that are standard with respect to this choice of Borel subgroup. Let T be the torus of diagonal matrices. The base of $\Phi(\mathrm{GL}_n, T)$ and set of positive roots associated to B are

$$\Delta := \{\alpha_i := \varepsilon_i - \varepsilon_{i+1} : 1 \leq i \leq n-1\} \quad \text{and} \quad \Phi^+ := \{\varepsilon_i - \varepsilon_j : i < j\},$$

respectively. Here ε_i is defined as in Example 1.14. There are bijections

$$\left\{ n_1, \dots, n_d \in \mathbb{Z}_{>0} : \sum_{i=1}^d n_i = n \right\} \xrightarrow{\sim} 2^{\{1, \dots, n-1\}} \xrightarrow{\sim} 2^\Delta,$$

where 2^X is the set of subsets of X . Here the first bijection sends $(n_1, \dots, n_d)$ to $\Delta - \{n_1, n_1 + n_2, \dots, n_1 + \dots + n_{d-1}\}$ and the second is induced by the bijection $\{1, \dots, n-1\} \xrightarrow{\sim} \Delta$ sending i to $\varepsilon_i - \varepsilon_{i+1}$. Thus the standard parabolic subgroups of GL_n correspond bijectively to ordered tuples of positive integers $n_1, \dots, n_d$ such that $\sum_{i=1}^d n_i = n$. The corresponding parabolic subgroup is the product MB where

$$M(R) := \left\{ \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_d \end{pmatrix} : x_i \in \mathrm{GL}_{n_i}(R) \right\} \quad (1.48)$$

for k -algebras R . We refer to this parabolic subgroup as the standard parabolic subgroup of type $(n_1, \dots, n_d)$. The subgroup M is referred to as the standard Levi subgroup of type $(n_1, \dots, n_d)$.

Example 1.16. Consider the reductive group GL_3 . Instead of choosing the subset $\{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_3\}$ of $\Phi(\mathrm{GL}_3, T)$ to be our set of positive roots, suppose we choose the subset $\{\varepsilon_1 - \varepsilon_3, \varepsilon_3 - \varepsilon_2, \varepsilon_1 - \varepsilon_2\}$. The corresponding set of simple roots is $\{\varepsilon_1 - \varepsilon_3, \varepsilon_3 - \varepsilon_2\}$ and the corresponding Borel subgroup B' is the subgroup of GL_3 whose points in a k -algebra R are

$$B'(R) := \left\{ \begin{pmatrix} a_1 & x & y \\ 0 & a_2 & 0 \\ 0 & z & a_3 \end{pmatrix} \in \mathrm{GL}_3(R) \right\}. \quad (1.49)$$

There are four standard parabolic subgroups with respect to B' . Their points in a k -algebra R are given by

$$\begin{aligned}
P_{\{\varepsilon_1 - \varepsilon_3, \varepsilon_3 - \varepsilon_2\}}(R) &= \mathrm{GL}_3(R), \\
P_{\{\varepsilon_1 - \varepsilon_3\}}(R) &= \left\{ \begin{pmatrix} a_1 & x & y \\ 0 & a_2 & 0 \\ w & z & a_3 \end{pmatrix} \in \mathrm{GL}_3(R) \right\}, \\
P_{\{\varepsilon_3 - \varepsilon_2\}}(R) &= \left\{ \begin{pmatrix} a_1 & x & y \\ 0 & a_2 & w \\ 0 & z & a_3 \end{pmatrix} \in \mathrm{GL}_3(R) \right\}, \\
P_\emptyset(R) &= B'(R).
\end{aligned}$$

The relationship between parabolic subgroups of a reductive group and parabolic subgroups of a Levi subgroup is particularly straightforward [DG74, §XXVI, Proposition 1.20]:

Proposition 1.9.8 *Let P be a parabolic subgroup of the reductive group G . Let N be the unipotent radical of P and let M be a Levi subgroup of P . Given a parabolic subgroup $P' \leq P$ of G the group $P' \cap M$ is a parabolic subgroup of M .*

Conversely given a parabolic subgroup P' of M the group $P'N$ is a parabolic subgroup of G . This defines an inclusion-preserving bijection between the set of parabolic subgroups of G contained in P and the set of parabolic subgroups of M . Moreover, given a parabolic subgroup $P' \leq P$ the Levi subgroups of $P' \cap M$ are the Levi subgroups of P' contained in M . $\square$

We will make use of this result in §10.4.

It is sometimes useful to generalize the notion of roots still further. If T is any split torus in G , we can decompose $\mathfrak{g}$ into weight spaces as before. We let $\Phi(G, T)$ be the set of nonzero weights. Generalizing our previous terminology we refer to $\Phi(G, T)$ as the set of roots of T in G . Suppose that $P \leq G$ is a parabolic subgroup with Levi subgroup M and that T is a maximal split torus of Z_M . In this case, the set of roots $\Phi(G, T)$ need not be a root system [Kna86, §V.5]. However, we can still define a partition $\Phi^+ \amalg \Phi^- = \Phi(G, T)$ such that Φ^+ is the set of roots contained in the unipotent radical of P and Φ^- is the set of roots contained in the unipotent radical of the opposite parabolic P^- . The set Φ^+ is the **set of positive roots defined by P** . We will also require the notion of a positive Weyl chamber in this context. Let $\Lambda \subset X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ denote the union of the hyperplanes on which an element of $\Phi(G, T)$ vanishes. A **Weyl chamber** is a connected component of $X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} - \Lambda$. The **positive Weyl chamber attached to P** is the unique Weyl chamber whose elements are positive $\mathbb{R}$ -linear combinations of the positive roots defined by P .

1.10 Homogeneous spaces

Let G be an affine group scheme and $H \leq G$ a subgroup scheme. This section is devoted to making sense of what we should mean by a quotient of G by H . This requires more serious constructions in algebraic geometry. In particular, we must leave the category of affine schemes (and even schemes) and work

with more general objects. The reader is therefore invited to omit this section and refer back to it as necessary when it is used later in the book. Specifically the definition of the quotient will be used in §2.5 and §14.4. A more systematic study of quotients will play a role in Chapter 17.

We follow the presentation in [Poo17, §6.2 and 6.3]. Consider the set-valued functor on $\mathbf{Alg}_k$ given by

$$\mathcal{F}_{H \setminus G}(R) := H(R) \setminus G(R).$$

At the end of this section we give an example to illustrate that this is not a scheme in general. Instead we proceed as follows. First we embed the category of schemes into the larger category of presheaves which contains the functor $\mathcal{F}_{H \setminus G}$. It turns out that the presheaves obtained from schemes are fppf sheaves (see Proposition 1.10.2). On the other hand, every presheaf may be sheafified to produce an fppf sheaf that is “as close as possible” to the original presheaf. One then proves, for example, when k is a field and G is smooth, that $\mathcal{F}_{H \setminus G}$ is naturally isomorphic to the sheaf associated to a scheme. Of course everything in this section may be trivially modified to treat the functor $R \rightarrow G(R)/H(R)$.

A morphism of rings $R \rightarrow R'$ is **fppf** if it is faithfully flat of finite presentation. The fppf stands for *fidèlement plat de présentation finie*. A morphism $\mathrm{Spec}(R') \rightarrow \mathrm{Spec}(R)$ is fppf if the morphism $R \rightarrow R'$ is fppf. We recall that if R is Noetherian then $R \rightarrow R'$ is of finite presentation if and only if it is of finite type.

Example 1.17. If k is a field and E/k is a field extension of finite degree then $\mathrm{Spec}(E) \rightarrow \mathrm{Spec}(k)$ is fppf. One should think of general fppf morphisms as generalizations of this example.

Let R be a k -algebra. A **fppf cover**

$$\{\phi_i : \mathrm{Spec}(R_i) \longrightarrow \mathrm{Spec}(R)\}$$

of $\mathrm{Spec}(R)$ is a finite collection of fppf morphisms $\phi_i : \mathrm{Spec}(R_i) \rightarrow \mathrm{Spec}(R)$ of affine schemes over k such that $\coprod_i \mathrm{Spec}(R_i) \rightarrow \mathrm{Spec}(R)$ is fppf.

A **presheaf** on $\mathbf{AffSch}_k$ is a functor

$$\mathcal{F} : \mathbf{Alg}_k \longrightarrow \mathbf{Set}. \quad (1.50)$$

A **morphism** of presheaves is a natural transformation of functors, and an isomorphism is defined as in §1.2. We denote by

$$\mathbf{PreSh}_k \quad (1.51)$$

the category of presheaves on $\mathbf{AffSch}_k$.

By analogy with the case of presheaves on a topological space, $\mathcal{F}(R)$ is called the set of **sections** of $\mathcal{F}$ over $\mathrm{Spec}(R)$. For the same reason, if $R \rightarrow R'$

is a homomorphism of k algebras, then the induced map

$$\mathcal{F}(R) \longrightarrow \mathcal{F}(R') \quad (1.52)$$

is sometimes called the **restriction map** associated to $\mathrm{Spec}(R') \rightarrow \mathrm{Spec}(R)$.

Recall that if A, B, C are sets, then the diagram

$$A \xrightarrow{f_0} B \begin{array}{c} \xrightarrow{f_1} \\ \xrightarrow{f_2} \end{array} C$$

indicates that one has a map $f_0 : A \rightarrow B$ and two maps $f_i : B \rightarrow C$, $i = 1, 2$. The maps f_i are often omitted from notation for simplicity. The **equalizer** of f_1, f_2 is

$$\mathrm{Eq}(f_1, f_2) := \{b \in B : f_1(b) = f_2(b)\}. \quad (1.53)$$

The diagram is called **exact** if f_0 is injective and

$$f_0(A) = \mathrm{Eq}(f_1, f_2).$$

Definition 1.10.1. An **fppf sheaf** on $\mathbf{AffSch}_k$ is a presheaf $\mathcal{F}$ on $\mathbf{AffSch}_k$ such that for any k -algebra R and any fppf cover $\{\mathrm{Spec}(R_i) \rightarrow \mathrm{Spec}(R)\}$ the diagram

$$\mathcal{F}(R) \longrightarrow \prod_i \mathcal{F}(R_i) \rightrightarrows \prod_{(i,j)} \mathcal{F}(R_i \otimes_R R_j)$$

is exact. Here for each i the two arrows are induced by the canonical R -algebra homomorphisms $R_i \rightarrow R_i \otimes_R R_j$ and $R_i \rightarrow R_j \otimes_R R_i$ for each i, j .

A morphism (resp. isomorphism) of fppf sheaves is simply a morphism (resp. isomorphism) of the underlying presheaves. We denote by $\mathbf{Sh}_k$ the category of fppf sheaves on $\mathbf{AffSch}_k$.

The injectivity of the first arrow in Definition 1.10.1 is an analogue of the assertion that a section of a sheaf on a topological space is determined by its restrictions to the open sets in any open cover of the topological space. Moreover the assertion that the diagram is exact is the analogue of the assertion that sections of a sheaf over a topological space may be constructed from a collection of sections over an open cover of the topological space provided the sections agree on the intersections of the various open sets in the open cover.

Every affine k -scheme tautologically yields a presheaf on $\mathbf{AffSch}_k$. The same remains true for k -schemes as we now explain. The most convenient definition of the category of k -schemes from our point of view is that it is the full subcategory of $\mathbf{PreSh}_k$ containing $\mathbf{AffSch}_k$ and other presheaves that are glued from elements of $\mathbf{AffSch}_k$ in a suitable manner. We will not make this precise and instead refer the reader to [DG80, I, §1]. In the same reference one finds a proof that this definition is equivalent to the definition of a k -scheme that appears in standard modern textbooks on algebraic geometry.

Thus we have fully faithful functors

$$\mathbf{AffSch}_k \longrightarrow \mathbf{Sch}_k \longrightarrow \mathbf{PreSh}_k, \quad (1.54)$$

where $\mathbf{Sch}_k$ is the category of k -schemes, the first arrow is the obvious one, and the second is given by taking a k -scheme to its functor of points. The following is [Poo17, Proposition 6.3.19]:

Proposition 1.10.2 *The presheaf on $\mathbf{AffSch}_k$ defined by a k -scheme is an fppf sheaf.* $\square$

Thus the functor $\mathbf{Sch}_k \rightarrow \mathbf{PreSh}_k$ of (1.54) factors through the canonical fully faithful functor $\mathbf{Sh}_k \rightarrow \mathbf{PreSh}_k$.

Definition 1.10.3. A **sheafification** of a presheaf $\mathcal{F}$ is an fppf sheaf $\mathcal{F}^+$ equipped with a morphism $\mathcal{F} \rightarrow \mathcal{F}^+$ of presheaves with the following property: For every morphism $a : \mathcal{F} \rightarrow \mathcal{G}$ (of presheaves) from $\mathcal{F}$ to an fppf sheaf $\mathcal{G}$, there exists a unique map $b : \mathcal{F}^+ \rightarrow \mathcal{G}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ \\ & \searrow a & \downarrow b \\ & & \mathcal{G} \end{array}$$

The sheafification always exists [Sta16, tag 00W1]. Because it is defined using a universal property, it is unique up to unique isomorphism. In particular, if $\mathcal{F}$ is an fppf sheaf then the map $\mathcal{F} \rightarrow \mathcal{F}^+$ in the definition is an isomorphism.

Now consider the sheafification $\mathcal{F}_{H \backslash G}^+$. We say that this fppf sheaf is **representable** if the sheaf is naturally isomorphic to the functor of points of a k -scheme and write $H \backslash G$ for this k -scheme.

For a proof of the following theorem, see [Mil17, Theorem 7.18]:

Theorem 1.10.4 *Assume k is a field and that G is a smooth affine algebraic group over k . For any subgroup $H \leq G$ the sheaf $\mathcal{F}_{H \backslash G}^+$ is represented by a quasi-projective scheme $H \backslash G$ over k .* $\square$

Example 1.18. If H is a normal subgroup of the smooth affine algebraic group G , then the quotient $H \backslash G$ can be endowed with the structure of a group scheme using the universal property of sheafification. Then $H \backslash G$ is isomorphic to the quotient group G/H appearing in Theorem 1.3.9.

Example 1.19. If P is a parabolic subgroup of a reductive group G then $P \backslash G$ is projective. In particular if $B < \mathrm{GL}_2$ is a Borel subgroup then

$$B \backslash \mathrm{GL}_2 \cong \mathbb{P}^1.$$

This example demonstrates that to understand quotients, even of affine groups, we must leave the category of affine schemes and incorporate projective schemes.

Assume for the remainder of this section that k is a field and that G is smooth. It is important for us to be able to work with $H \backslash G$ at the level of the functor of points. For this we point out that the presheaf morphisms $G \rightarrow \mathcal{F}_{H \backslash G} \rightarrow H \backslash G$ induce a map

$$\begin{aligned} G(R) &\longrightarrow (H \backslash G)(R) \\ g &\longmapsto H(R)g. \end{aligned} \tag{1.55}$$

The nonempty fibers of this morphism are the left cosets of $H(R)$ in $G(R)$ [Mil17, §5.c]. The following is [Mil17, Proposition 7.15]:

Proposition 1.10.5 *If H is a smooth subgroup of G , then the map $G \rightarrow H \backslash G$ is smooth. $\square$*

It is important to point out that the map (1.55) need not be surjective for a given R , even for $R = k$. This leaves us with the question of providing a precise description of $(H \backslash G)(k)$. We accomplish this in Proposition 1.10.7. First we require a lemma [Sta16, Lemma 33.25.6]:

Lemma 1.10.6 *If X is a smooth scheme over a field k , then the set of closed points x in the underlying topological space of X whose residue field is a separable extension of k is dense in the underlying topological space of X . $\square$*

Proposition 1.10.7 *Let H and G be smooth algebraic groups over k with $H \leq G$. One has that*

$$(H \backslash G)(k) = \{H(k^{\text{sep}})g \in H(k^{\text{sep}}) \backslash G(k^{\text{sep}}) : g\xi(g^{-1}) \in H(k^{\text{sep}})\}$$

for all $\xi \in \text{Gal}_k$.

Proof. Using the action of Gal_k (see [GW10, §5.2]), one easily reduces the proposition to the case where $k = k^{\text{sep}}$.

We henceforth assume that $k = k^{\text{sep}}$. The morphism $G \rightarrow H \backslash G$ is faithfully flat and hence surjective [Mil17, Proposition 7.4(b)]. We claim that the map

$$G(k^{\text{sep}}) \longrightarrow (H \backslash G)(k^{\text{sep}}) \tag{1.56}$$

is surjective. Given $p \in (H \backslash G)(k^{\text{sep}})$ consider the fiber of $G \rightarrow H \backslash G$ over p . It is nonempty because the morphism $G \rightarrow H \backslash G$ is surjective. It is a smooth subscheme of G by Proposition 1.10.5. Hence the fiber has a k^{sep} -point by Lemma 1.10.6. This implies that (1.56) is surjective. Since the fibers are the cosets of $H(k^{\text{sep}})$ in $G(k^{\text{sep}})$ (see the discussion below (1.55)), we deduce the proposition. $\square$

Let us end the section by considering an example. We take $k = \mathbb{Q}$. For $\mathbb{Q}$ -algebras R and integers $n > 1$ let

$$\mu_n(R) := \{x \in R^\times : x^n = 1\}. \tag{1.57}$$

Thus μ_n is the scheme of n -th roots of unity. It is naturally a subgroup scheme of $\mathbb{G}_m$. We have a morphism $\mathcal{F}_{\mu_n \setminus \mathbb{G}_m} \rightarrow \mathbb{G}_m$ given on points in a $\mathbb{Q}$ -algebra R by $x \mapsto x^n$. It induces a morphism

$$\mu_n \setminus \mathbb{G}_m \longrightarrow \mathbb{G}_m \quad (1.58)$$

by the universal property of sheafification. It is an isomorphism (see Exercise 1.15). This implies that $\mathcal{F}_{\mu_n \setminus \mathbb{G}_m}$ is not a scheme. Indeed, if it were, then the universal property of sheafification would imply that the morphism of presheaves $\mathcal{F}_{\mu_n \setminus \mathbb{G}_m} \rightarrow \mu_n \setminus \mathbb{G}_m$ is an isomorphism, and hence that the morphism of presheaves $\mathcal{F}_{\mu_n \setminus \mathbb{G}_m} \rightarrow \mathbb{G}_m$ is an isomorphism. But the induced map $\mu_n(\mathbb{Q}) \setminus \mathbb{Q}^\times \rightarrow \mathbb{Q}^\times$ is not an isomorphism as its image is $(\mathbb{Q}^\times)^n \subsetneq \mathbb{Q}^\times$.

Exercises

1.1. Prove that a morphism of affine schemes induces a continuous map of the underlying topological spaces. If I is an ideal of A , prove that the map of topological spaces induced by the morphism

$$\mathrm{Spec}(A/I) \longrightarrow \mathrm{Spec}(A)$$

is a closed embedding with image $V(I)$. Here $V(I)$ is defined as (1.7).

1.2. Assume that $X = \mathrm{Spec}(\mathbb{C}[t_1, \dots, t_n]/(f_1, \dots, f_c))$ is a smooth scheme over the complex numbers. Show that $X(\mathbb{C})$ may be identified with a smooth complex manifold, namely, the zero locus of the f_i .

1.3. Let X and Y be affine k -schemes. Prove that giving a natural transformation of functors $X \rightarrow Y$ is equivalent to giving a morphism of k -algebras $\mathcal{O}(Y) \rightarrow \mathcal{O}(X)$. Deduce that Spec induces an contravariant equivalence of categories from the category of k -algebras to the category of affine k -schemes.

1.4. Give examples of affine schemes of finite type over $\mathbb{C}$ that are nonreduced, reducible, and reduced and irreducible.

1.5. Prove that if $X \rightarrow Y$ is a closed immersion of affine schemes over k then $X(R) \rightarrow Y(R)$ is injective for all k -algebras R .

1.6. Let X, Y, Z be affine k -schemes equipped with morphisms $f : X \rightarrow Y$ and $g : Z \rightarrow Y$. Prove that

$$(X \times_Y Z)(R) = X(R) \times_{Y(R)} Z(R)$$

for k -algebras R .

1.7. Let G be an affine scheme over k . Let $\text{Id} : G \rightarrow G$ denote the identity morphism, let

$$p_i : G \times G \longrightarrow G$$

denote the two projections, and let

$$\text{diag} : G \longrightarrow G \times G$$

denote the diagonal map. We say that G is a **group object** in the category of affine k -schemes if there exist morphisms of affine k -schemes

$$\begin{aligned} m : G \times G &\longrightarrow G, \\ e : \text{Spec}(k) &\longrightarrow G, \\ i : G &\longrightarrow G \end{aligned}$$

such that the following diagrams commute:

$$\begin{array}{ccccc} G \times G \times G & \xrightarrow{\text{Id} \times m} & G \times G & & \\ \downarrow m \times \text{Id} & & \downarrow m & & \\ G \times G & \xrightarrow{m} & G & & \end{array}$$

$$\begin{array}{ccccc} \text{Spec}(k) \times G & \xrightarrow{e \times \text{Id}} & G \times G & \xleftarrow{\text{Id} \times e} & G \times \text{Spec}(k) \\ & \searrow & \downarrow m & \swarrow & \\ & & G & & \end{array}$$

$$\begin{array}{ccccc} G & \xrightarrow{i \times \text{Id}} & G \times G & \xleftarrow{\text{Id} \times i} & G \\ \downarrow & & \downarrow m & & \downarrow \\ \text{Spec}(k) & \xrightarrow{e} & G & \xleftarrow{e} & \text{Spec}(k) \end{array}$$

Here the unlabeled diagonal arrows are the canonical isomorphisms and the unlabeled vertical arrows are the structural morphism. Prove that G is an affine group scheme over k if and only if it is a group object in the category of affine k -schemes.

1.8. For $\mathbb{R}$ -algebras R , define

$$U_n(R) := \{g \in \text{GL}_n(\mathbb{C} \otimes_{\mathbb{R}} R) : g\bar{g}^t = I_n\},$$

where the bar denotes the action of complex conjugation. Show that U_n is an affine algebraic group over $\mathbb{R}$, that $U_n(\mathbb{R})$ is compact, and that

$$U_{n\mathbb{C}} \cong \text{GL}_{n\mathbb{C}}.$$

The group U_n is called the **definite unitary group** over $\mathbb{R}$.

1.9. Let $\epsilon : \mathcal{O}(G) \rightarrow k$ be the coidentity defined in (1.17). Prove that

$$\mathcal{O}(G)/\ker(\epsilon)^2 = k \oplus \ker(\epsilon)/\ker(\epsilon)^2$$

as k -algebras.

1.10. Assume k is a field with $\text{char}(k) \neq 2$ and $J \in \text{GL}_n(k)$ is an invertible symmetric or skew-symmetric matrix. For k -algebras R , define

$$G(R) := \{g \in \text{GL}_n(R) : g^t J g = J\}.$$

Prove that G is an affine algebraic group over k . Moreover, show that

$$\text{Lie } G = \{X \in \mathfrak{gl}_n : X^t J + J X = 0\}.$$

1.11. View GL_n as a reductive group over k . Prove that the set of conjugacy classes of maximal tori $T \leq \text{GL}_n$ is in natural bijection with isomorphism classes of étale k -algebras of degree n .

1.12. Let D be a finite dimensional division algebra over a field k and G be the affine algebraic group defined by

$$G(R) = (D \otimes_k R)^\times$$

for k -algebras R . Prove G has no proper parabolic subgroups (over k).

1.13. Show that the complex dual of GL_n is $\text{GL}_n\mathbb{C}$.

1.14. Compute the complex duals of SL_n and PGL_n .

1.15. Let μ_n be as in (1.57). Let k be a field with $\text{char}(k) \nmid n$, and regard μ_n and $\mathbb{G}_m$ as schemes over k . Prove that the map

$$\mu_n \backslash \mathbb{G}_m \longrightarrow \mathbb{G}_m$$

induced by the map $R^\times \rightarrow R^\times$ given by $x \rightarrow x^n$ is an isomorphism.

Chapter 2

Adeles



Adeles make life possible.

J. Arthur

We review the adèle ring and recall the basic properties of the adelic points of affine schemes.

2.1 Adeles

The arithmetic objects of interest in this book are constructed using global fields and their adèle rings. We briefly summarize the construction of the adeles in this section; references include [CF86, Neu99, RV99].

Definition 2.1.1. A **global field** F is a field which is a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_p(t)$ for some prime p . Global fields over $\mathbb{Q}$ are called **number fields** while global fields over $\mathbb{F}_p(t)$ are called **function fields**.

To each global field F , one can associate an adèle ring $\mathbb{A}_F$. Before defining this ring, we introduce the related notion of a valuation of a global field.

Definition 2.1.2. Let F be a field. A (non-Archimedean) **valuation** on F is a map

$$v : F \longrightarrow \mathbb{R} \cup \infty$$

such that for all $a, b \in F$,

- (a) $v(a) = \infty$ if and only if $a = 0$.
- (b) $v(a) + v(b) = v(ab)$.
- (c) $v(a + b) \geq \min(v(a), v(b))$.

These axioms are designed so that if one picks $0 < \alpha < 1$ then

$$\begin{aligned} |\cdot|_v : F &\longrightarrow \mathbb{R}_{\geq 0} \\ x &\longmapsto \alpha^{v(x)} \end{aligned} \tag{2.1}$$

is a non-Archimedean absolute value on F in the sense of the following definition:

Definition 2.1.3. Let F be a field. An **absolute value**

$$|\cdot|_v : F \longrightarrow \mathbb{R}_{\geq 0}$$

is a function satisfying the following axioms:

- (a) $|a|_v = 0$ if and only if $a = 0$.
- (b) $|ab|_v = |a|_v |b|_v$.
- (c) There is a $C \in \mathbb{R}_{>0}$ such that $|1 + b|_v \leq C$ whenever $|b|_v \leq 1$.

It is **non-Archimedean** if we may take $C = 1$:

$$(c') \quad |a + b|_v \leq \max(|a|_v, |b|_v).$$

If $|\cdot|_v$ satisfies (c) but not (c') we say $|\cdot|_v$ is **Archimedean**.

We point out that (c') implies (c), but not conversely. We often implicitly exclude the **trivial absolute value** given by $|a|_v = 1$ for all $a \in F^\times$.

An absolute value $|\cdot|_v$ induces a metric on F known as the **v -adic metric**. The completion of F with respect to this metric is denoted F_v . Assume that F is a global field. Then F_v is a **local field**, that is, a field equipped with a nontrivial absolute value that is locally compact with respect to the induced metric (this convention is different from that of [Ser79]). All local fields arise as the completion of some global field [Lor08, §25, Theorem 2].

Two absolute values are said to be **equivalent** if they induce the same topology on F . We observe that the field F_v depends only on the equivalence class of the absolute value $|\cdot|_v$. Two absolute values $|\cdot|_v$ and $|\cdot|'_v$ on F are equivalent if and only if there is a $c \in \mathbb{R}_{>0}$ such that

$$|a|_v = |a|'^c_v$$

for all $a \in F$ [CF86, §II.4]. Thus one can always replace $|\cdot|_v$ with an equivalent valuation so that one may take $C = 2$ in (c), in which case it is equivalent to the triangle inequality [CF86, §II.1]. The reason for allowing general C is so that the square of the usual norm on $\mathbb{C}$ is an absolute value (see (2.2)).

Definition 2.1.4. A **place** of a global field F is an equivalence class of absolute values. A place is (non-)Archimedean if it consists of (non-)Archimedean absolute values.

We now describe the places of a global field and fix a representative absolute value in each place. These representative absolute values are said to be the **normalized absolute values**.

The places of a global field F are divided into the **finite** and **infinite** places. To define the finite places of a global field F we require rings of integers. If F is a number field, then the ring of integers $\mathcal{O}_F$ is unambiguously defined: it is the integral closure of $\mathbb{Z}$ in F . If F is a function field of characteristic p , we choose an embedding $\mathbb{F}_p(t) \rightarrow F$ such that $F/\mathbb{F}_p(t)$ is separable and declare $\mathcal{O}_F$ to be the integral closure of $\mathbb{F}_p[t]$ in F . The choice is non-canonical, but it has the virtue that it allows us to treat number fields and global fields on equal footing for the vast majority of this book.

The finite places of a global field F are in bijection with the nonzero prime ideals of $\mathcal{O}_F$. The place v associated to a prime $\mathfrak{p}$ of $\mathcal{O}_F$ is the equivalence class of an absolute value attached to the valuation

$$v(x) := \max \{k \in \mathbb{Z} : x \in \mathfrak{p}^k \mathcal{O}_F\}.$$

The normalized absolute value in this equivalence class is

$$|x|_v = q_v^{-v(x)},$$

where $q_v := |\mathcal{O}_F/\mathfrak{p}|$. These valuations are all non-Archimedean.

The infinite places of a number field are all Archimedean. They are indexed by embeddings $\tau : F \hookrightarrow \mathbb{C}$ up to complex conjugation; the associated normalized absolute value is

$$|x|_v := \begin{cases} |\tau(x)| & \text{if } \tau(F) \leq \mathbb{R}, \\ \tau(x)\overline{\tau(x)} & \text{if } \tau(F) \not\leq \mathbb{R}. \end{cases} \quad (2.2)$$

Here on the right the $|\cdot|$ denotes the usual absolute value on $\mathbb{R}$ and the bar denotes complex conjugation. Notice that in the complex case (i.e. where $\tau(F) \not\leq \mathbb{R}$), this is the square of the usual absolute value.

The infinite places of a function field are those attached to extensions of the absolute value

$$\left| \frac{f(t)}{g(t)} \right|_{\infty} := p^{\deg f - \deg g}$$

on $\mathbb{F}_p(t)$. If F_v is the completion of F with respect to such a valuation, the associated normalized absolute value is

$$|x|_v := |N_{F_v/\mathbb{F}_p((t^{-1}))}(x)|_{\infty}.$$

Here we are using the fact that the completion of $\mathbb{F}_p(t)$ at the infinite place is $\mathbb{F}_p((t^{-1}))$. These valuations are non-Archimedean, in contrast to the number field case. Henceforth we will always take $|\cdot|_v$ to be the unique normalized absolute value attached to v .

If v is finite, then the **ring of integers** of F_v is

$$\mathcal{O}_{F_v} = \{x \in F_v : |x|_v \leq 1\};$$

it is a local ring with a unique maximal ideal

$$\varpi_v \mathcal{O}_{F_v} := \{x \in F_v : |x|_v < 1\}.$$

Here ϖ_v is a **uniformizer** for F_v , that is, a generator for the maximal ideal of $\mathcal{O}_{F_v}$.

Let us make these constructions explicit when $F = \mathbb{Q}$. If $p \in \mathbb{Z}$ is a prime, then completing $\mathbb{Q}$ at the p -adic absolute value gives rise to the local field $\mathbb{Q}_p$. Its ring of integers is $\mathbb{Z}_p$ and the maximal ideal is $p\mathbb{Z}_p$. The residue field is $\mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z} \cong \mathbb{F}_p$, so the normalized absolute value is just the usual p -adic norm. There is only one infinite place of $\mathbb{Q}$, denoted ∞ , and $\mathbb{Q}_\infty \cong \mathbb{R}$. The normalized Archimedean norm $|\cdot|_\infty$ is the usual Euclidean norm on $\mathbb{R}$.

The normalized absolute values above are chosen so that the following **product formula** holds:

Proposition 2.1.5 *For $x \in F^\times$, one has that*

$$\prod_v |x|_v = 1,$$

where the product is over all places v of F . □

Definition 2.1.6. Let F be a global field. The ring of **adeles** of F , denoted by $\mathbb{A}_F$, is the restricted direct product of the completions F_v with respect to the rings of integers $\mathcal{O}_{F_v}$. In other words,

$$\mathbb{A}_F = \left\{ (x_v) \in \prod_v F_v : x_v \in \mathcal{O}_{F_v} \text{ for almost all } v \right\}. \quad (2.3)$$

Here, and throughout this book, “almost all” means all but finitely many. The restricted product is usually denoted by a prime:

$$\mathbb{A}_F = \prod'_v F_v.$$

Note that $\mathbb{A}_F$ is a subring of the full product $\prod_v F_v$. If S is a nonempty finite set of places of F then we write

$$\mathbb{A}_F^S = \prod'_{v \notin S} F_v := \left\{ (x_v) \in \prod_{v \notin S} F_v : x_v \in \mathcal{O}_{F_v} \text{ for almost all } v \notin S \right\}$$

and

$$F_S := \mathbb{A}_{F,S} = \prod_{v \in S} F_v.$$

Thus we may identify $F_S \times \mathbb{A}_F^S = \mathbb{A}_F$. For any finite sets of places $S' \subseteq S$, we set

$$F_S^{S'} := \prod_{v \in S - S'} F_v. \quad (2.4)$$

We endow $\mathbb{A}_F$ with the **restricted product topology**. This is defined by stipulating that open sets are the empty set together with all unions of sets of the form

$$U_S \times \prod_{v \notin S} \mathcal{O}_{F_v},$$

where S is a finite set of places of F including the infinite places and $U_S \subseteq F_S$ is an open set. We give $\mathbb{A}_F^S = 0 \times \mathbb{A}_F^S \subset \mathbb{A}_F$ the subspace topology and F_S the product topology. Then the identification $F_S \times \mathbb{A}_F^S = \mathbb{A}_F$ becomes an isomorphism of topological rings. We will often use the following useful notation: if S is a finite set of places of F including the infinite places,

$$\hat{\mathcal{O}}_F^S := \prod_{v \notin S} \mathcal{O}_{F_v}. \quad (2.5)$$

Let $\mathcal{O}_F^S$ denote the ring of S -integers of F , that is, elements of F that are integral outside of S . The notation $\hat{\mathcal{O}}_F^S$ is justified by the fact that $\hat{\mathcal{O}}_F^S$ is naturally isomorphic to the profinite completion of $\mathcal{O}_F^S$.

The topology on $\mathbb{A}_F$ is not the same as the topology induced on $\mathbb{A}_F$ by regarding it as a subset of the direct product $\prod_v F_v$. While $\prod_v F_v$ is not locally compact, one has the following result for $\mathbb{A}_F$:

Proposition 2.1.7 *The adele ring $\mathbb{A}_F$ of a global field F is a locally compact topological ring.*

Here, and throughout this book, we take the convention that a locally compact space is Hausdorff.

Proof. We prove that $\mathbb{A}_F$ is locally compact and leave the other details to the reader. For any finite set S of places of F including the infinite places, the subset

$$\prod_{v \in S} F_v \times \hat{\mathcal{O}}_F^S \quad (2.6)$$

is an open subring of $\mathbb{A}_F$ for which the induced topology coincides with the product topology. The open subring (2.6) is locally compact by Tychonoff's theorem. Every $x \in \mathbb{A}_F$ is contained in a set of the form (2.6), which shows that $\mathbb{A}_F$ is locally compact. $\square$

We note that the set of infinite places of F is often denoted by ∞ , and one often writes $v|\infty$ or $v \nmid \infty$ as shorthand for “ v is an infinite place of F ” and “ v is a finite place of F ,” respectively. As an example of this notation, the ring $\mathbb{A}_F^\infty$ is known as the **finite adeles**.

There is a natural diagonal embedding $F \hookrightarrow \mathbb{A}_F$ given by sending an element of F to all of its completions. We often identify F with its image

under the diagonal embedding. The quotient $\mathbb{Q} \backslash \mathbb{A}_{\mathbb{Q}}$ can be regarded as a generalization of $\mathbb{Z} \backslash \mathbb{R}$. Indeed, one has a homeomorphism

$$\begin{aligned} \mathbb{Z} \backslash \mathbb{R} &\longrightarrow \mathbb{Q} \backslash \mathbb{A}_{\mathbb{Q}} / \widehat{\mathbb{Z}} \\ \mathbb{Z} + r &\longmapsto \mathbb{Z} + (r, 0) + \widehat{\mathbb{Z}}, \end{aligned}$$

where we have identified

$$\widehat{\mathbb{Z}} := \prod_p \mathbb{Z}_p$$

with the subgroup $\{0\} \times \widehat{\mathbb{Z}}$ of $\mathbb{A}_{\mathbb{Q}}$ and $(r, 0)$ denotes the adele that is r at the infinite place and 0 at the finite places.

The subgroup $\mathbb{Z} < \mathbb{R}$ is closed and discrete. The same is true of $F < \mathbb{A}_F$:

Lemma 2.1.8 *The image of F in $\mathbb{A}_F$ under the diagonal embedding is closed and discrete.*

Proof. Choose $x \in F^\times$. We will construct a neighborhood of x in $\mathbb{A}_F$ containing no other element of F . Since $\mathbb{A}_F$ is a topological group under addition this will suffice to complete the proof.

For each finite place v of F , let $n_v = v(x)$, so that $x \in \varpi_v^{n_v} \mathcal{O}_{F_v}$ but $x \notin \varpi_v^{n_v+1} \mathcal{O}_{F_v}$ for all v . Note that $n_v = 0$ for almost all places. For each infinite place w , let

$$U_w := \left\{ y \in F_w : |y - x|_w < \prod_{v \nmid \infty} |x|_v^{-1/|\infty|} \right\}.$$

Here $|\infty|$ is the number of infinite places of F . Consider the open subset of $\mathbb{A}_F$ defined by

$$U = \prod_{v|\infty} U_v \times \prod_{v \nmid \infty} \varpi_v^{n_v} \mathcal{O}_{F_v}.$$

By construction, $x \in U$. Suppose that $y \in U \cap F$. Then $|x - y|_v \leq |x|_v$ for all finite places v . Thus

$$\prod_v |x - y|_v \leq \prod_{v \nmid \infty} |x|_v \times \prod_{v|\infty} |x - y|_v < 1.$$

By the product formula, we conclude that $x = y$. □

Given Lemma 2.1.8 the following theorem can be surprising the first time one sees it:

Theorem 2.1.9 (Strong Approximation) *If S is any finite nonempty set of places of F then F is dense in $\mathbb{A}_F^S$.* □

Thus omitting one place is enough to move F from being discrete to being dense. The proof can be found in any standard reference, see [Cas67, §15] for example.

Example 2.1. To understand Theorem 2.1.9, it is illustrative to consider the case of a real quadratic field $F = \mathbb{Q}(\sqrt{d})$. Here $d > 0$ is a square free integer and we write $\sqrt{d}$ for the positive square root of d . Suppose that $d \equiv 3 \pmod{4}$; thus $\mathcal{O}_F = \mathbb{Z}[\sqrt{d}]$. There are two infinite places $\sigma_{\pm}$, both real, that are characterized by

$$\sigma_{\pm}(\sqrt{d}) = \pm\sqrt{d}.$$

Let $\varepsilon > 0$ and consider the open set $(-\varepsilon, \varepsilon) \times \widehat{\mathcal{O}}_F$ in $F_{\sigma_-} \times \mathbb{A}_F^{\infty}$. Theorem 2.1.9 implies that there are infinitely many elements of F in this open set, which is equivalent to the statement that there are infinitely many $\alpha \in \mathbb{Z}[\sqrt{d}]$ satisfying $|\sigma_-(\alpha)| < \varepsilon$. This can be checked independently of Theorem 2.1.9 using the fact that $\sqrt{d}$ is irrational.

We close this section by remarking that one can construct analogues of the finite adèles in more general situations, for instance, schemes of finite type over the ring of integers of a global field [Hub91]. The construction is quite a bit more involved than that given above.

2.2 Adelic points of affine schemes

Weil and Grothendieck both gave approaches (under different hypotheses) to topologizing the points of schemes of finite type over a topological ring (for example, $\mathbb{A}_F$). Conrad gave a beautiful exposition and elaboration in [Con12b]. We lift Theorem 2.2.1 and its proof from loc. cit. with little modification.

The naïve idea is as follows. Given a topological ring R , R^n inherits a natural topology (this is a special case of a fiber product topology). Given an affine scheme X , we choose a closed immersion $X \rightarrow \mathbb{G}_a^n$. This induces an injection

$$X(R) \longrightarrow R^n.$$

Since we have endowed R^n with a topology already, we can then give $X(R)$ the subspace topology. The problem with this method is that it is not clear that the topology on $X(R)$ is canonical, and it is also not clear that it is functorial with respect to morphisms $X \rightarrow Y$ of affine schemes.

Conrad refines this naïve approach. He abstracts axiomatic properties we would like the topology on $X(R)$ to have, proves that these properties uniquely characterize a topology on $X(R)$ (if the topology exists) and then proves such a topology exists.

Theorem 2.2.1 *Let R be a topological ring. There exists a unique way to topologize $X(R)$ for all affine schemes X of finite type over R such that*

- (a) the topology is functorial in X , that is, if $X \rightarrow Y$ is a morphism then the induced map $X(R) \rightarrow Y(R)$ is continuous;
- (b) the topology is compatible with fiber products, that is, if $X \rightarrow Y$ and $Z \rightarrow Y$ are morphisms, then the topology on $(X \times_Y Z)(R)$ is the fiber product topology;
- (c) closed immersions $X \hookrightarrow Y$ induce topological embeddings $X(R) \hookrightarrow Y(R)$; and
- (d) if $X = \text{Spec}(R[t])$ then $X(R)$ is homeomorphic with R under the natural identification $X(R) \cong R$.

If R is Hausdorff or locally compact, then so is $X(R)$. Moreover, if R is Hausdorff then closed immersions induce closed embeddings, not just topological embeddings.

As above, we take the convention that locally compact spaces are Hausdorff. Assume that R is Hausdorff. In the sequel, if we need to distinguish between the Zariski topology on X and the topology on $X(R)$ provided by the theorem above, we will refer to the topology on $X(R)$ as the **Hausdorff topology** (the Zariski topology is rarely Hausdorff).

Proof. Let X be an affine R -scheme of finite type. Pick an R -algebra isomorphism

$$A := \mathcal{O}(X) \cong R[t_1, \dots, t_n]/I \quad (2.7)$$

for an ideal I , and identify $X(R)$ with the subset of R^n on which the elements of I (thought of as polynomials on R^n) vanish.

We start by proving uniqueness. By assumptions (b) and (d) the natural bijection

$$\text{Spec}(R[t_1, \dots, t_n])(R) \xrightarrow{\sim} R^n$$

is a homeomorphism provided that we give the right-hand side the product topology. By assumption (c), (2.7) induces a topological embedding $X(R) \hookrightarrow R^n$. This completes the proof of uniqueness and also shows that $X(R)$ is Hausdorff if R is Hausdorff. If R is Hausdorff, then $\{0\}$ is closed in R , so viewing $X(R)$ as the vanishing locus of $f \in I$ (viewed as polynomials on R^n), we see that $X(R)$ is closed. Thus if R is locally compact then $X(R)$ is as well.

We now prove existence. Consider the direct product R^A of “ A copies of R ,” or more formally the set of all set-theoretic maps from A to R . Since R is a topological ring, R^A comes equipped with the direct product topology. Note that there is a canonical injection

$$X(R) = \text{Hom}(A, R) \longrightarrow R^A. \quad (2.8)$$

We claim that the topology defined using (2.7) as above is the same as the subspace topology induced by the injection (2.8). In particular it is independent of the choice of (2.7). For $1 \leq i \leq n$ let $a_i \in A$ correspond to $t_i \pmod{I}$ via (2.7). Thus, the injection $X(R) \hookrightarrow R^n$ defined by (2.7) is the

composition of the natural injection $X(R) \hookrightarrow R^A$ and the map $R^A \rightarrow R^n$ given by projection to the factors indexed by $(a_1, \dots, a_n)$. Since $R^A \rightarrow R^n$ is continuous, every open set in $X(R)$ (with respect to the topology induced by (2.7)) is the inverse image of an open set in R^A . Since every element of A is an R -polynomial in $a_1, \dots, a_n$ and R is a topological ring, it follows that the map $X(R) \rightarrow R^A$ is also continuous. This completes the proof of the claim. It also implies that the formation of the topology on $X(R)$ is functorial (i.e. (a) is valid). Since the topologies defined using (2.7) and (2.8) are the same (d) is valid as well.

Consider a closed immersion

$$i : X := \operatorname{Spec}(A) \hookrightarrow \operatorname{Spec}(A') =: X'$$

corresponding to a surjective R -algebra map $h : A' \rightarrow A$. For $a \in A$ we have a diagonal embedding $R^{\{a\}} \rightarrow R^{h^{-1}(a)}$. Taking the product over all $a \in A$ we obtain a canonical map

$$j : R^A \longrightarrow R^{A'}.$$

The map j is visibly a topological embedding; it topologically identifies R^A with the subset of $R^{A'}$ cut out by a collection of equalities among components. Moreover j is a closed embedding when R is Hausdorff. We have

$$X'(R) \cap j(R^A) = j(X(R))$$

because a set-theoretic map $A \rightarrow R$ is an R -algebra homomorphism if and only if its composition with h is an R -algebra map. Hence $i : X(R) \rightarrow X'(R)$ is an embedding of topological spaces and it is a closed embedding when R is Hausdorff. In particular (c) is valid.

We are left with checking (b). First consider the special case when $Y = \operatorname{Spec}(R)$ and the maps $X \rightarrow Y$ and $Z \rightarrow Y$ are the structural morphisms. Choose two closed immersions $X \rightarrow \mathbb{G}_a^m$ and $Z \rightarrow \mathbb{G}_a^n$ for suitable positive integers m and n . The canonical map

$$(\mathbb{G}_a^m \times_{\operatorname{Spec}(R)} \mathbb{G}_a^n)(R) \longrightarrow R^m \times R^n$$

is a topological isomorphism. Hence the same is true of

$$(X \times_{\operatorname{Spec}(R)} Z)(R) \longrightarrow X(R) \times Z(R).$$

Now assume that $X \rightarrow Y$ and $Z \rightarrow Y$ are morphisms of affine R -schemes of finite type such that $Z \rightarrow Y$ a closed immersion. Then the projection $X \times_Y Z \rightarrow X$ is a closed immersion. Applying our previous work it induces a topological embedding on R -points, proving that the topology is compatible with fiber products in this special case.

To treat general fiber products, consider the (tautological) isomorphism

$$X \times_Y Z \cong (X \times_R Z) \times_{(Y \times_R Y)} Y$$

and its topological counterpart. The product map

$$\mathcal{O}(Y) \otimes_R \mathcal{O}(Y) \longrightarrow \mathcal{O}(Y)$$

is surjective and hence

$$Y \longrightarrow Y \times_R Y$$

is a closed immersion (i.e. affine schemes are separated). Since we have already checked compatibility with fiber products over $\mathrm{Spec}(R)$, we see that we are reduced to the case in which one of the maps defining the fiber product is a closed immersion. This special case was already treated above. $\square$

From now on in this book, whenever R is a topological ring and X is an affine R -scheme of finite type, we give $X(R)$ the topology described in Theorem 2.2.1.

We record the following easy consequence of Theorem 2.2.1:

Proposition 2.2.2 *If R is a topological ring and G is an affine group scheme of finite type over R , then $G(R)$ is a topological group.*

Proof. The morphisms $G \times G \rightarrow G$ and $G \rightarrow G$ given by multiplication and the inverse, respectively, induce continuous maps $G(R) \times G(R) \rightarrow G(R)$ and $G(R) \rightarrow G(R)$ by Theorem 2.2.1. $\square$

Since nonaffine schemes play only a minor role in this book, we have not defined them. However, in Chapter 17, we will require an analogue of Theorem 2.2.1 when X is no longer assumed to be affine, at least under suitable conditions on R . Here is one such analogue [Con12b, Proposition 3.1]:

Proposition 2.2.3 *Let R be a topological local ring such that $R^\times$ is open in R and the inversion map*

$$\begin{aligned} R^\times &\longrightarrow R^\times \\ x &\longmapsto x^{-1} \end{aligned}$$

is continuous. There is a unique way to topologize $X(R)$ for all schemes X locally of finite type over R such that (a)–(d) of Theorem 2.2.1 hold and the following additional property holds:

(e) Open immersions $X \hookrightarrow Y$ induce open embeddings $X(R) \hookrightarrow Y(R)$.

Moreover, if R is Hausdorff and X is separated over R then $X(R)$ is Hausdorff. If R is locally compact then so is $X(R)$. $\square$

We will use the topology provided by this proposition when R is a local field in §17.4 and §17.5.

The adèle ring $\mathbb{A}_F$ of a global field F does not satisfy the assumptions of Proposition 2.2.3. Therefore we cannot use the proposition to endow $X(\mathbb{A}_F)$

with a topology for nonaffine schemes X of finite type over F . We return to this point in Theorem 2.4.14.

2.3 Relationship with restricted direct products

In practice, we will be interested in the topology on $G(\mathbb{A}_F)$ given by Theorem 2.2.1 when G is an affine algebraic group over the global field F . Let us discuss this in more detail, starting with $G = \mathrm{GL}_n$. Consider the affine group scheme GL_n over $\mathbb{Z}$ (its base change to F is a reductive group over F). Let M_n be the affine $\mathbb{Z}$ -scheme of $n \times n$ matrices. There is a closed immersion $\mathrm{GL}_n \hookrightarrow M_n \times \mathbb{G}_a$ given on points in a $\mathbb{Z}$ -algebra R by

$$\begin{aligned} \mathrm{GL}_n(R) &\longrightarrow M_n(R) \times \mathbb{G}_a(R) \\ g &\longmapsto (g, \det g^{-1}). \end{aligned} \tag{2.9}$$

Thus for any topological ring R , the topology on $\mathrm{GL}_n(R)$ afforded by Theorem 2.2.1 is the subspace topology if we view $\mathrm{GL}_n(R)$ as a subspace of $M_n(R) \times \mathbb{G}_a(R)$ via (2.9). For example, if v is an Archimedean place of F , then by this recipe $\mathrm{GL}_n(F_v)$ acquires its usual topology.

To describe the topology for non-Archimedean v , we recall that to describe a topology on a space, it suffices to give a neighborhood base for every point x , that is, a set of open neighborhoods

$$x \in U_\alpha$$

such that every neighborhood of x contains some U_α . In a topological group, it suffices to just give a neighborhood base of the identity, since we can take translates of this neighborhood base to be neighborhood bases at every other point.

Now if ϖ_v is a uniformizer for $\mathcal{O}_{F_v}$ then

$$\{(I_n + \varpi_v^k M_n(\mathcal{O}_{F_v})) \times (1 + \varpi_v^k \mathcal{O}_{F_v}) : k \in \mathbb{Z}_{\geq 1}\}$$

forms a neighborhood base of the point $(I_n, 1) \in M_n(F_v) \times F_v$. It follows that

$$\{I_n + \varpi_v^k M_n(\mathcal{O}_{F_v}) : k \in \mathbb{Z}_{\geq 1}\}$$

forms a neighborhood base of the identity in $\mathrm{GL}_n(F_v)$. Note that this is the same topology we would obtain if we just gave $\mathrm{GL}_n(F_v) \subset M_n(F_v)$ the subspace topology.

On the other hand, if S is any finite set of places of F including the infinite places then

$$\{(I_n + m M_n(\widehat{\mathcal{O}}_F^S)) \times (1 + m \widehat{\mathcal{O}}_F^S) : m \in \mathcal{O}_F^S\}$$

forms a neighborhood basis for $(I_n, 1) \in M_n(\mathbb{A}_F^S) \times \mathbb{A}_F^S$. Here m runs over nonzero proper ideals of $\mathcal{O}_F^S$ and

$$mM_n(\widehat{\mathcal{O}}_F^S) := \prod_{v \notin S} \varpi_v^{v(m)} M_n(\mathcal{O}_{F_v}).$$

If we intersect one of these neighborhoods with the image of $\mathrm{GL}_n(\mathbb{A}_F^S)$ under (2.9) then we obtain

$$\prod_{\substack{v \notin S \\ v(m) \neq 0}} (I_n + mM_n(\mathcal{O}_{F_v})) \times \prod_{\substack{v \notin S \\ v(m)=0}} \mathrm{GL}_n(\mathcal{O}_{F_v}).$$

Here $mM_n(\mathcal{O}_{F_v}) = \varpi_v^{v(m)} M_n(\mathcal{O}_{F_v})$. This is not the same as the topology obtained by giving $\mathrm{GL}_n(\mathbb{A}_F^S) \subset M_n(\mathbb{A}_F^S)$ the subspace topology.

Finally, for any finite set S of places of F including the infinite places, it is not hard to see by modifying this argument that one has a topological isomorphism of locally compact groups

$$\mathrm{GL}_n(\mathbb{A}_F) = \mathrm{GL}_n(F_S) \times \mathrm{GL}_n(\mathbb{A}_F^S).$$

If F is a local field and G is an affine algebraic group over F , to define the topology on $G(F)$ we choose a monomorphism $G \hookrightarrow \mathrm{GL}_n$ and give $G(F)$ the subspace topology. Theorem 2.2.1 tells us that this topology is independent of the choice of monomorphism $G \hookrightarrow \mathrm{GL}_n$. If F is global instead we similarly obtain the topology on $G(\mathbb{A}_F)$.

It is helpful to organize these comments on the topology of $G(\mathbb{A}_F)$ by discussing the notion of a restricted direct product of topological spaces. This notion is implicit in the definition of the topology on $\mathbb{A}_F$ given in §2.1. Let $\{X_\alpha\}_{\alpha \in A}$ be a set of locally compact topological spaces indexed by a countable set A , and for all α outside a finite subset S_0 of A , let $K_\alpha \subseteq X_\alpha$ be a compact open subset of X_α . Then the **restricted direct product** of the X_α with respect to the K_α is the set

$$X := \prod'_{\alpha \in A} X_\alpha := \left\{ (x_\alpha) \in \prod_{\alpha \in A} X_\alpha : x_\alpha \in K_\alpha \text{ for almost all } \alpha \in A - S_0 \right\}.$$

We give X a topology by declaring a subset of X to be open if it is either empty or a union of sets of the form

$$U \times \prod_{\alpha \in A-S} K_\alpha,$$

where S is a finite subset of A including S_0 and

$$U \subseteq X_S := \prod_{\alpha \in S} X_\alpha$$

is an open subset. One verifies that this is indeed a topology and with respect to this topology, X is locally compact. Note that if the X_α are topological groups and the K_α are topological subgroups then X is again a topological group. We also note that the topology does not change if we replace S_0 by any finite subset of A containing S_0 .

Now assume that G is an affine algebraic group over the global field F . Choose a faithful representation

$$G \longrightarrow \mathrm{GL}_n.$$

Identify G with its image in GL_n and define, for all $v \nmid \infty$,

$$K_v := G(F_v) \cap \mathrm{GL}_n(\mathcal{O}_{F_v}). \quad (2.10)$$

Then K_v is a compact open subgroup of $G(F_v)$. Our discussion of the adelic topology on $G(\mathbb{A}_F)$ above can be summarized as follows:

Proposition 2.3.1 *One has an isomorphism of topological groups*

$$\begin{aligned} G(\mathbb{A}_F) &\xrightarrow{\sim} \prod'_v G(F_v) \\ (g_v) &\longmapsto (g_v), \end{aligned}$$

where the restricted direct product is defined with respect to the subgroups K_v . $\square$

In fact, in most references, $G(\mathbb{A}_F)$ is defined using Proposition 2.3.1. This has the advantage of being concrete, but it makes it awkward to rigorously prove that the topology satisfies good functorial properties.

2.4 Hyperspecial subgroups and models

Let G be an affine algebraic group over a non-Archimedean local field F and let $G \rightarrow \mathrm{GL}_n$ be a faithful representation. Identify G with its image in GL_n . In the previous section, we made use of the fact that $K := G(F) \cap \mathrm{GL}_n(\mathcal{O}_F)$ is a compact open subgroup of $G(F)$. This construction is actually algebraic in nature in the sense that K is the $\mathcal{O}_F$ -points of an affine group scheme over $\mathcal{O}_F$. We explain this in more detail in this section and use it as motivation to introduce the notion of a hyperspecial subgroup. We go into some detail, and this requires a little algebraic geometry over local rings. For most of this book the reader can manage with an understanding of Definition 2.4.4

and the statement of Corollary 2.4.12. Our presentation in this section was influenced by [Yu].

We start with a Dedekind domain $\mathfrak{o}$ with fraction field F . For example, $\mathfrak{o}$ could be the ring of S -integers $\mathcal{O}_F^S$ of a global field F for a finite set S of places of F including the infinite places or the ring of integers of a non-Archimedean local field. Let $\mathcal{Z}$ be an affine scheme over $\mathfrak{o}$. The **generic fiber** of $\mathcal{Z}$ is $\mathcal{Z}_F$. If F is local and $\mathfrak{o}$ is its ring of integers $\mathcal{O}_F$ then $\mathfrak{o}$ has a unique prime ideal $\mathfrak{p}$; in this case, we let $\kappa = \mathfrak{o}/\mathfrak{p}$ be the residue field. The scheme $\mathcal{Z}_\kappa$ is known as the **special fiber** of $\mathcal{Z}$.

Suppose we are given an affine scheme Y of finite type over F ; often it is useful to consider schemes $\mathcal{Y}$ that have Y as their generic fiber and satisfy certain desiderata:

Definition 2.4.1. A **model** $\mathcal{Y}$ of Y over $\mathfrak{o}$ is an affine scheme of finite type over $\mathfrak{o}$ isomorphic to $\mathrm{Spec}(A)$ where $A \leq \mathcal{O}(Y)$ is an $\mathfrak{o}$ -algebra and $A \otimes_{\mathfrak{o}} F = \mathcal{O}(Y)$.

The assertion that an affine $\mathfrak{o}$ -scheme $\mathrm{Spec}(A)$ is **flat** over $\mathfrak{o}$ in the usual sense is equivalent to the assertion that A is flat as an $\mathfrak{o}$ -module. This in turn is equivalent to the assertion that A is torsion free as an $\mathfrak{o}$ -module since $\mathfrak{o}$ is a Dedekind domain. In the situation of the definition, $\mathcal{O}(Y)$ may be regarded as an F -module by forgetting the algebra structure and A is an $\mathfrak{o}$ -submodule of this F -module. In particular, A is torsion free as an $\mathfrak{o}$ -module. Thus we see that models are flat over $\mathfrak{o}$. Suppose conversely we are given an affine scheme of finite type $\mathcal{Y} = \mathrm{Spec}(A)$ over $\mathfrak{o}$ together with an isomorphism $Y \xrightarrow{\sim} \mathcal{Y}_F$ of affine F -schemes. Assume moreover that $\mathcal{Y}$ is flat, which is to say that A is flat as an $\mathfrak{o}$ -module. Then we have an injection $A \rightarrow A \otimes_{\mathfrak{o}} F \rightarrow \mathcal{O}(Y)$ where the first map is canonical and the second is induced by $Y \xrightarrow{\sim} \mathcal{Y}_F$. Thus $\mathcal{Y}$ is a model of Y .

We defined smooth $\mathfrak{o}$ -schemes in §1.2 and hence it makes sense to speak of smooth models. We point out that an affine $\mathfrak{o}$ -scheme $\mathcal{Y}$ is smooth if it is of finite type, flat, and for all algebraically closed fields $\bar{\kappa}$ that admit a nonzero $\mathfrak{o}$ -algebra morphism $\mathfrak{o} \rightarrow \bar{\kappa}$, the fiber $\mathcal{Y}_{\bar{\kappa}}$ is smooth [Sta16, Tag 01V8].

When G is a group scheme we always assume that models $\mathcal{G}$ of G are again group schemes and that the generic fiber of the multiplication map $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ and the inversion map $\mathcal{G} \rightarrow \mathcal{G}$ are the multiplication map and inversion map on G , respectively. In brief, we assume that the group scheme structures on $\mathcal{G}$ and G are compatible.

Sometimes there are obvious models for affine schemes. For example, we can view GL_n as an affine group scheme over $\mathfrak{o}$, and it is clearly a model of its generic fiber GL_{nF} . However, there are many other models of GL_n (see Exercise 2.9 for an example).

We now describe a particular type of model of a reductive group. Assume until otherwise stated that F is a non-Archimedean local field.

Definition 2.4.2. A reductive group G over F is **unramified** if it is quasi-split and there is a finite degree unramified extension E/F such that G_E is split.

The following theorem is foundational for the theory of automorphic representations. It is an amalgamation of several results of Bruhat and Tits, see [Mac17, §1.2] for precise references.

Theorem 2.4.3 *Let F be a non-Archimedean local field. The reductive group G over F is unramified if and only if there exists a model $\mathcal{G}$ of G over $\mathcal{O}_F$ such that the special fiber of $\mathcal{G}$ is reductive. If $\mathcal{G}$ is a model of G over $\mathcal{O}_F$ such that the special fiber of $\mathcal{G}$ is reductive then $\mathcal{G}(\mathcal{O}_F)$ is a maximal compact subgroup of $G(F)$.* $\square$

Definition 2.4.4. A subgroup of $G(F)$ of the form $\mathcal{G}(\mathcal{O}_F)$ for a model $\mathcal{G}$ of G over $\mathcal{O}_F$ with reductive special fiber is called a **hyperspecial subgroup**.

We observe that the group scheme $\mathcal{G}$ in the definition of a hyperspecial subgroup is smooth.

Example 2.2. When $G = \mathrm{GL}_n$, it is clear that $\mathrm{GL}_n(\mathcal{O}_F)$ is a hyperspecial subgroup of $\mathrm{GL}_n(F)$. It turns out that all maximal compact subgroups of $\mathrm{GL}_n(F)$ are conjugate to $\mathrm{GL}_n(\mathcal{O}_F)$ [Ser06, Chapter IV, Appendix 1].

Though the theory is not as pleasant for ramified G , we still have the following theorem:

Theorem 2.4.5 *For any reductive group G over F , every compact subgroup of $G(F)$ is contained in a maximal compact subgroup. Every maximal compact subgroup $K \leq G(F)$ is of the form $\mathcal{G}(\mathcal{O}_F)$ where $\mathcal{G}$ is a smooth model of G . Maximal compact subgroups of $G(F)$ are open.*

Proof. See [Tit79, §3.2] for the assertion that every compact subgroup of $G(F)$ is contained in a maximal compact subgroup. The second assertion follows from [Tit79, §3.2, 3.4.1]. The final assertion follows from the second assertion and Exercise 2.2. $\square$

We now briefly discuss conjugacy classes of hyperspecial subgroups. The group G acts on G by conjugation, and the action factors through the map $G \rightarrow G/Z_G$. Hence the group $(G/Z_G)(F)$ acts on $G(F)$. In general, orbits under this action are larger than the orbits of $G(F)$ on itself by conjugation; for a discussion of group actions and orbits in a general context, we refer to §17.2. The set of hyperspecial subgroups is permuted transitively by the action of $(G/Z_G)(F)$ [Tit79, §2.5] (see also [Con14, Theorem 7.2.16] when G is split). However not all hyperspecial subgroups are conjugate under $G(F)$ in general. This plays a direct role in the definition of unramified L -packets (see §12.5).

Given a reductive group G over a global field F one can readily construct hyperspecial subgroups of $G(F_v)$ for all but finitely many places v . To explain this we first discuss schematic closures. We again revert to the assumptions at the beginning of this section, so $\mathfrak{o}$ is a Dedekind domain with fraction field F . Let Y be an affine scheme of finite type over F and let $\mathcal{Y}$ be a model of Y . Suppose in addition we are given a closed immersion of F -schemes

$$X \longrightarrow Y; \quad (2.11)$$

we identify X with a closed subscheme of Y . Let

$$A := \text{im}(\mathcal{O}(\mathcal{Y}) \longrightarrow \mathcal{O}(Y) \longrightarrow \mathcal{O}(X)).$$

Since $\mathcal{O}(\mathcal{Y}) \rightarrow \mathcal{O}(Y)$ is injective and (2.11) is a closed immersion, we see that $\mathcal{X} := \text{Spec}(A)$ comes equipped with a closed immersion $\mathcal{X} \rightarrow \mathcal{Y}$ and $\mathcal{X}_F = X$. We leave the proof of the following lemma as an exercise (see Exercise 2.5).

Lemma 2.4.6 *The affine $\mathfrak{o}$ -scheme $\mathcal{X}$ is a model of X and*

$$\mathcal{X}(\mathfrak{o}) = X(F) \cap \mathcal{Y}(\mathfrak{o}).$$

□

The scheme $\mathcal{X}$ earns its moniker as a schematic closure via a universal property (see Exercise 2.6).

Lemma 2.4.7 *Any affine scheme of finite type over F admits a model. In particular, any affine algebraic group G over F admits a model $\mathcal{G}$ that is again a group scheme such that the generic fibers of the multiplication map $\mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ and the inversion map $\mathcal{G} \rightarrow \mathcal{G}$ are the multiplication map and inversion map on G .*

Proof. The map $\mathfrak{o}^n \rightarrow F^n$ is clearly injective, hence $\mathbb{G}_a^n$, viewed as a scheme over $\mathfrak{o}$, is a model of $\mathbb{G}_{aF}^n$. If X is an affine scheme of finite type over F then choose a closed immersion $X \rightarrow \mathbb{G}_{aF}^n$. Its schematic closure $\mathcal{X}$ in $\mathbb{G}_a^n$ is then a model of X by Lemma 2.4.6.

Similarly, if we view GL_n as an affine group scheme over $\mathfrak{o}$ then the condition that $\mathcal{O}(\text{GL}_n) \rightarrow \mathcal{O}(\text{GL}_{nF})$ is injective is clearly satisfied. Thus GL_n is a model of GL_{nF} . Using Theorem 1.5.2 choose a faithful representation $G \rightarrow \text{GL}_{nF}$. Let $\mathcal{G}$ be the schematic closure of G in GL_n . It follows from Exercise 2.7 that it is a closed subgroup scheme of GL_n with generic fiber G .

□

In the special case where F is a local field and $\mathfrak{o} = \mathcal{O}_F$ is its ring of integers, the procedure in the proof yields

$$\mathcal{G}(\mathcal{O}_F) := G(F) \cap \text{GL}_n(\mathcal{O}_F)$$

which we denoted by K earlier. Thus we implicitly used schematic closures in our discussion of the restricted direct topology in the previous section.

Suppose that F is global and that we are given a faithful representation $G \rightarrow \mathrm{GL}_{nF}$ of a reductive group G . We can then take the schematic closure $\mathcal{G}$ of G in GL_n , where GL_n is viewed as a group scheme over $\mathcal{O}_F$. This is a model of G by the proof of Lemma 2.4.7. This model, or in fact any model, gives rise to hyperspecial subgroups for almost all v :

Proposition 2.4.8 *Let S be a finite set of places of F including the infinite places and let G be a reductive group over F . Let $\mathcal{G}$ be a model of G over $\mathcal{O}_F^S$. For almost all $v \notin S$, the subgroup $\mathcal{G}(\mathcal{O}_{F_v})$ is hyperspecial.*

Before giving the proof we state a preparatory lemma [Pool17, Theorem 3.2.1]:

Lemma 2.4.9 *Let X be a smooth affine F -scheme, S be a finite set of places of F including the infinite places, and let $\mathcal{X}$ be a model of X over $\mathcal{O}_F^S$. Then there is a finite set $S' \supseteq S$ such that $\mathcal{X}_{\mathcal{O}_{F_v}^{S'}}$ is smooth over $\mathcal{O}_{F_v}^{S'}$. $\square$*

Proof of Proposition 2.4.8: The scheme $\mathcal{G}_{\mathcal{O}_F^{S'}}$ is smooth over $\mathcal{O}_F^{S'}$ for a sufficiently large finite set $S' \supseteq S$ by Lemma 2.4.9. Thus the proposition follows from [Con14, Propositions 3.1.9 and 3.1.12]. $\square$

Models give us another perspective on the restricted direct product isomorphism in Proposition 2.3.1. Let X be an affine scheme of finite type over F . Suppose that we have a model $\mathcal{X}$ of X over $\mathcal{O}_F^S$, where S is a finite set of places including the infinite places. The following is a refinement of Proposition 2.3.1:

Proposition 2.4.10 *One has a homeomorphism*

$$\begin{aligned} X(\mathbb{A}_F) &\xrightarrow{\sim} \prod'_v X(F_v) \\ (x_v) &\longmapsto (x_v), \end{aligned}$$

where the restricted direct product is defined with respect to the subsets $\mathcal{X}(\mathcal{O}_{F_v})$. The set on the right, its topology, and the isomorphism are independent of the choice of model, and are unchanged if we replace S by S' (and $\mathcal{X}$ by $\mathcal{X}_{\mathcal{O}_F^{S'}}$) for any set of places $S' \supseteq S$. If X is an affine algebraic group then the homeomorphism is a group isomorphism.

Here when X is an affine algebraic group we assume (per our convention) that the model $\mathcal{X}$ is a group scheme and that the group scheme structures on $\mathcal{X}$ and X are compatible.

Proof. The homeomorphism assertion of the proposition is true for $\mathcal{X} = \mathbb{G}_a^n$ by part (b) of Theorem 2.2.1 and the definition of the topology on $\mathbb{A}_F$ given in §2.1.

We now reduce the homeomorphism assertion of the proposition to this special case. Since $\mathcal{X}$ is of finite type, we have a closed immersion

$$\iota : \mathcal{X} \longrightarrow \mathbb{G}_a^n$$

over $\mathcal{O}_F^S$. The scheme $\iota(\mathcal{X})$ is then the schematic closure of $\iota(X)$ in $\mathbb{G}_a^n$. By Theorem 2.2.1 the generic fiber of this map induces a closed embedding

$$X(\mathbb{A}_F) \xrightarrow{\iota} \mathbb{A}_F^n \xrightarrow{\sim} \prod'_v F_v^n, \quad (2.12)$$

where the homeomorphism is given by the special case of the proposition proved at the beginning of the proof. Since $\iota(\mathcal{X}(\mathcal{O}_{F_v})) = \mathcal{O}_{F_v}^n \cap \iota(X(F_v))$ for all non-Archimedean v by Lemma 2.4.6, the image of (2.12) is $\prod'_v X(F_v)$. This implies the isomorphism assertion of the proposition. The fact that the isomorphism is a group isomorphism when X is a group scheme is obvious. The independence of the restricted direct product and the isomorphism under replacing S by a larger set S' are clear.

Suppose that $\mathcal{X}'$ is another model of X . Then

$$\mathcal{O}(\mathcal{X}) \otimes_{\mathcal{O}_F^S} F = \mathcal{O}(\mathcal{X}') \otimes_{\mathcal{O}_F^S} F = \mathcal{O}(X)$$

and $\mathcal{O}(\mathcal{X})$ and $\mathcal{O}(\mathcal{X}')$ are of finite type over $\mathcal{O}_F^S$. Hence for a sufficiently large finite set of places $S' \supset S$, we have that

$$\mathcal{O}(\mathcal{X}') \otimes_{\mathcal{O}_F^S} \mathcal{O}_F^{S'} = \mathcal{O}(\mathcal{X}) \otimes_{\mathcal{O}_F^S} \mathcal{O}_F^{S'}. \quad (2.13)$$

This is equivalent to the assertion that the identity morphism $\mathcal{X}_F = X_F \rightarrow X_F = \mathcal{X}'_F$ extends to an isomorphism

$$\mathcal{X}_{\mathcal{O}_F^{S'}} \xrightarrow{\sim} \mathcal{X}'_{\mathcal{O}_F^{S'}}$$

for a sufficiently large set of places $S' \supset S$ (see [Poo17, Theorem 3.2.1] for many useful generalizations and variants). In particular the subsets $\mathcal{X}(\mathcal{O}_{F_v})$ and $\mathcal{X}'(\mathcal{O}_{F_v})$ of $X(F_v)$ are equal for $v \notin S'$. The assertion on the independence of the choice of model follows. $\square$

Corollary 2.4.11 *Let S be a finite set of places of F including the infinite places, let G be a smooth affine algebraic group over F , and let $K^S \leq G(\mathbb{A}_F^S)$ be a compact open subgroup. Let $\mathcal{G}$ be a model of G over $\mathcal{O}_F^S$. Then there is a finite set of places $S' \supseteq S$ such that $K^S = K_{S'}^S K^{S'}$ where*

$$K_{S'}^S \leq G(F_{S'}^S) \quad \text{and} \quad \mathcal{G}(\widehat{\mathcal{O}}_F^{S'}) = K^{S'}.$$

$\square$

Corollary 2.4.12 *Let S be a finite set of places of F including the infinite places, let G be a reductive group over F , and let $K^S \leq G(\mathbb{A}_F^S)$ be a compact open subgroup. Then the projection K_v of K^S to $G(F_v)$ is hyperspecial for almost all $v \notin S$.*

Proof. Choose a model $\mathcal{G}$ of G over $\mathcal{O}_F$ using Lemma 2.4.7. Then $\mathcal{G}(\mathcal{O}_{F_v}) = K_v$ for almost all $v \notin S$ by Corollary 2.4.11. Thus the corollary follows from Proposition 2.4.8. $\square$

It is helpful to discuss a possible point of confusion related to Proposition 2.4.10. Suppose we are given a reductive group G and a collection of hyperspecial subgroups $K_v \leq G(F_v)$ for almost all v . Using the fact that all hyperspecial subgroups of $G(F_v)$ are in the same $(G/Z_G)(F_v)$ -orbit, we can again construct an isomorphism of abstract groups

$$G(\mathbb{A}_F) \xrightarrow{\sim} \prod_v'' G(F_v), \quad (2.14)$$

where the double prime is to indicate that we are taking the restricted direct product with respect to the hyperspecial subgroups K_v . Now the isomorphism in Proposition 2.4.10 has good functorial properties, in particular, it commutes with the obvious embeddings of $G(F)$ into $G(\mathbb{A}_F)$ and $\prod_v' G(F_v)$. This is impossible to arrange in general for (2.14):

Example 2.3. Let

$$K_p = \begin{pmatrix} p & \\ & 1 \end{pmatrix} \mathrm{SL}_2(\mathbb{Z}_p) \begin{pmatrix} p^{-1} & \\ & 1 \end{pmatrix}.$$

Then $K_p \leq \mathrm{SL}_2(\mathbb{Q}_p)$ is hyperspecial and is not conjugate in $\mathrm{SL}_2(\mathbb{Q}_p)$ to $\mathrm{SL}_2(\mathbb{Z}_p)$ [Con14, Example 7.2.15]. We observe that for all p , the element $\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q})$ is not in K_p .

Thus we will always decompose $X(\mathbb{A}_F)$ as in Proposition 2.4.10, not with respect to some “exotic” collection of models chosen place by place.

As mentioned at the end of §2.2, Theorem 2.2.1 and Proposition 2.2.3 are insufficient to endow $X(\mathbb{A}_F)$ with a topology when X is not affine. We close this section by explaining how to adapt Proposition 2.4.10 to accomplish this. We will not use this discussion until §17.7.

Let X be a separated scheme of finite type over F . Then for all places v of F , the topology on $X(F_v)$ is defined by Proposition 2.2.3. Let S be a finite set of places of F including the infinite places. If S is large enough, we can choose a separated scheme $\mathcal{X}$ of finite type over $\mathcal{O}_F^S$ such that $\mathcal{X}_F = X$ [Pool17, Theorem 3.2.1]. For $v \notin S$, the topology on $\mathcal{X}(\mathcal{O}_{F_v})$ is also defined by Proposition 2.2.3. We isolate the following claim from the proof of [Con12b, Theorem 3.6]:

Lemma 2.4.13 *For $v \notin S$, the natural map $\mathcal{X}(\mathcal{O}_{F_v}) \rightarrow \mathcal{X}(F_v) = X(F_v)$ is an open embedding.*

Proof. The map is injective by the valuative criterion of separatedness. Let $\{U_i\}$ be an open cover of $\mathcal{X}$ by affine schemes. Then $U_i(\mathcal{O}_{F_v}) \rightarrow \mathcal{X}(\mathcal{O}_{F_v})$ and $U_i(F_v) \rightarrow \mathcal{X}(F_v)$ are open embeddings by Proposition 2.2.3. Moreover $\mathcal{X}(\mathcal{O}_{F_v}) = \cup_i U_i(\mathcal{O}_{F_v})$ and $\mathcal{X}(F_v) = \cup_i U_i(F_v)$. Thus it suffices to treat the case where $\mathcal{X}$ is affine. In this case, the lemma is a consequence of Exercise 2.2. $\square$

As explained in [Con12b, §3-4] there is a bijection

$$X(\mathbb{A}_F) \xrightarrow{\sim} \prod'_v X(F_v), \quad (2.15)$$

where the restricted direct product is defined with respect to the subsets $\mathcal{X}(\mathcal{O}_{F_v})$ for $v \notin S$. We define the topology on $X(\mathbb{A}_F)$ by giving the right-hand side the restricted direct product topology and declaring that the bijection (2.15) is a homeomorphism. Proposition 2.4.10 implies that this agrees with our earlier definition when X is affine. The following theorem can be extracted from [Con12b, §3-4]:

Theorem 2.4.14 *The definition of the topology on $X(\mathbb{A}_F)$ given above is independent of the choice of S and $\mathcal{X}$. With respect to this topology $X(\mathbb{A}_F)$ is locally compact. Moreover,*

- (a) *If $X \rightarrow Y$ is a morphism of separated schemes of finite type over F , then the map $X(\mathbb{A}_F) \rightarrow Y(\mathbb{A}_F)$ is continuous.*
- (b) *If $X \rightarrow Y$ and $Z \rightarrow Y$ are morphisms of separated schemes of finite type over F , then the topology on $(X \times_Y Z)(\mathbb{A}_F)$ is the fiber product topology.*
- (c) *Closed immersions $X \hookrightarrow Y$ of separated schemes of finite type over F induce closed embeddings $X(\mathbb{A}_F) \hookrightarrow Y(\mathbb{A}_F)$.*
- (d) *Proper morphisms $X \rightarrow Y$ of separated schemes of finite type over F induce topologically proper maps $X(\mathbb{A}_F) \rightarrow Y(\mathbb{A}_F)$.*
- (e) *Let $X \rightarrow Y$ be a smooth surjective morphism of separated schemes of finite type over F . If $X \rightarrow Y$ has geometrically connected fibers then the induced map $X(\mathbb{A}_F) \rightarrow Y(\mathbb{A}_F)$ is open.*

□

2.5 Approximation in affine algebraic groups

For a global field F and a nonempty finite set S of places of F , the image of F under the diagonal embeddings $F \rightarrow F_S$ is dense. This is fairly easy to prove and can be viewed as a generalization of the Chinese remainder theorem. Moreover, the image of F in $\mathbb{A}_F^S$ under the diagonal embedding is dense by Theorem 2.1.9. If X is a separated scheme of finite type over F and S is a nonempty finite set of places of F then one can ask if a similar phenomenon occurs:

Definition 2.5.1. Let S be a nonempty finite set of places of F . A separated scheme X of finite type over F satisfies **weak approximation** with respect to S if the image of the diagonal embedding $X(F) \rightarrow X(F_S)$ is dense. It satisfies **strong approximation** with respect to S if $X(F)$ is dense in $X(\mathbb{A}_F^S)$.

Here when we speak of density we are of course using the canonical topologies on $X(F_S)$ and $X(\mathbb{A}_F^S)$ afforded by Theorems 2.2.1 and 2.4.14.

Despite the relative ease of proving that $\mathbb{G}_a$ satisfies weak and strong approximation, establishing whether or not a general separated F -scheme enjoys these properties is very difficult. In general, they do not hold. We refer to [Har04] for a more detailed discussion.

Our goal in this section is to state when weak and strong approximation hold in settings related to affine algebraic groups. To simplify the discussion we often assume that F is a number field; additional complications come up in the general case. Our primary reference is [PR94, Chapter 7]. We start with the following proposition, the proof of which we leave as an exercise:

Proposition 2.5.2 *Let G be a connected affine algebraic group over a number field F with Levi decomposition $G = MN$. Then G admits weak (resp. strong) approximation with respect to a nonempty finite set S of places of F if and only if M does.* $\square$

Thus in the number field case, studying studying weak and strong approximation for connected affine algebraic groups is equivalent to studying it for the smaller class of reductive groups. It turns out that under suitable restrictions on the set of places S , weak approximation always holds:

Theorem 2.5.3 *Let G be a connected affine algebraic group over a number field F . There is a finite set S_0 of finite places of F such that G has weak approximation with respect to any nonempty finite set of places of F not containing S_0 .* $\square$

We refer to [PR94, §7.3] for the proof.

If we wish to guarantee that we can take S_0 to be empty, we need to assume additional arithmetic conditions on the group G . To state them, one needs the notions of simply connected and adjoint semisimple groups from §1.8. For the following theorem, we refer again to [PR94, §7.3]:

Theorem 2.5.4 *A simply connected or adjoint semisimple group G over a number field F admits weak approximation with respect to any nonempty finite set of places F .* $\square$

We finish our discussion of weak approximation by considering the case where X is not an affine algebraic group, but a particular type of homogeneous space. Let G be a connected affine algebraic group over a number field F and let $H \leq G$ be a subgroup. Then we may form the quotient scheme $H \backslash G$ as in §1.10. It need not be affine, but is always quasi-projective. If $\bar{F}$ is an algebraic closure of F and $F \leq L \leq \bar{F}$ is a subfield then

$$X(L) = (G(\bar{F})/H(\bar{F}))^{\text{Gal}(\bar{F}/L)}$$

with the natural Galois action (see Proposition 1.10.7).

We let

$$G^{\text{red}} := G/\mathcal{R}_u(G), \quad G^{\text{ss}} := (G^{\text{red}})^{\text{der}}, \quad G^{\text{tor}} := G^{\text{red}}/G^{\text{ss}}. \quad (2.16)$$

Thus G^{red} is the maximal reductive quotient of G and G^{tor} is the maximal quotient of G that is a torus. We recall that an algebraic torus T is **quasi-trivial** or **induced** if $X^*(T)$ is a permutation $\text{Gal}(\overline{F}/F)$ -module. We record the following theorem of Borovoi [Bor09, Theorem 3.12], which generalizes Theorem 2.5.3:

Theorem 2.5.5 *Let G be a connected affine algebraic group over a number field F such that G^{ss} is simply connected and G^{tor} is quasi-trivial. Assume that H is a connected subgroup of G . There is a finite set S_0 of finite places of F such that G/H has weak approximation with respect to any nonempty finite set of places of F not containing S_0 . $\square$*

In particular, G/H always has weak approximation with respect to ∞ . Borovoi gives a more explicit description of a set S_0 such that the theorem holds.

We also have the following generalization of Theorem 2.5.4:

Theorem 2.5.6 *Assume that G is a connected affine algebraic group over a number field F such that G^{ss} is simply connected and G^{tor} is quasi-trivial. Let H be a connected subgroup such that H^{tor} splits over a cyclic extension of F . Then G/H satisfies weak approximation with respect to any nonempty finite set of places of F . $\square$*

This is [Bor09, Corollary 3.14].

We now turn to a discussion of strong approximation. A nontrivial semisimple algebraic group G over a field F is **almost simple** if it has no nontrivial normal proper smooth connected subgroups. The basic result on strong approximation is the following. For the proof, see [Pra77]:

Theorem 2.5.7 *Let G be a connected affine algebraic group over a global field F such that G is almost simple and let S be a nonempty finite set of places of F . If G is simply connected and $G(F_S)$ is noncompact then G satisfies strong approximation with respect to S . $\square$*

At least in the number field case, there is a converse to this theorem, see [PR94, Theorem 7.12]. We point out in particular that the semisimplicity assumption is necessary. Even in the case $G = \text{GL}_1$ strong approximation is false in general. Indeed,

$$F^\times \backslash (\mathbb{A}_F^\infty)^\times / \widehat{\mathcal{O}}_F^\times$$

is the class group of $\mathcal{O}_F$ (see (2.17)). This is a finite group, but it is often nontrivial.

One has an analogue of Theorem 2.5.7 for certain homogeneous spaces [Rap14, Proposition 2.4]:

Theorem 2.5.8 *Assume that G is a simply connected semisimple group over a number field F such that $G_{\overline{F}}$ is almost simple. Let $H \leq G$ be a simply connected semisimple subgroup and let S be a nonempty finite set of places of F . Then G/H satisfies strong approximation with respect to S if and only if $(G/H)(F_S)$ is noncompact. $\square$*

2.6 The adelic quotient

Let G be an affine algebraic group over a global field F . By Exercise 2.2 the subgroup $G(F) < G(\mathbb{A}_F)$ is discrete and closed and we can consider the quotient $G(F) \backslash G(\mathbb{A}_F)$. In this section we state some basic facts on the topology of $G(F) \backslash G(\mathbb{A}_F)$. More precise results will be explained in the following section.

The first result to be discussed is the following (see [Con12a] for the proof):

Theorem 2.6.1 (Borel, Conrad, Oesterlé, Prasad) *For any finite set S of places of F containing the infinite places and for any compact open subgroup $K^S \leq G(\mathbb{A}_F^S)$, the quotient*

$$G(F) \backslash G(\mathbb{A}_F^S) / K^S$$

is finite. □

We note that one does not even have to assume that G is smooth (although this is automatic in the characteristic zero case, see Theorem 1.5.3). Theorem 2.6.1 is known as **finiteness of class numbers** for affine algebraic groups. It is a strict generalization of the assertion that the class group of F is finite. To see this, recall that the class group $\text{Cl}_F(\mathcal{O}_F)$ sits in an exact sequence

$$1 \longrightarrow \text{Prin}(\mathcal{O}_F) \longrightarrow \text{Div}(\mathcal{O}_F) \longrightarrow \text{Cl}_F(\mathcal{O}_F) \longrightarrow 1,$$

where $\text{Prin}(\mathcal{O}_F)$ is the group of principal divisors of $\mathcal{O}_F$ and $\text{Div}(\mathcal{O}_F)$ is the group of divisors of $\mathcal{O}_F$ (which can be identified with the group of fractional ideals). There is a surjection

$$\begin{aligned} \text{Div}(\mathcal{O}_F) &\longrightarrow (\mathbb{A}_F^\infty)^\times / \widehat{\mathcal{O}}_F^\times \\ \mathfrak{m} &\longmapsto (\mathfrak{m}\mathcal{O}_{F_v}). \end{aligned}$$

It induces a bijection

$$\text{Cl}_F(\mathcal{O}_F) \xrightarrow{\sim} F^\times \backslash (\mathbb{A}_F^\infty)^\times / \widehat{\mathcal{O}}_F^\times. \quad (2.17)$$

Hence taking $G = \text{GL}_1$, S to be the set of infinite places, and $K^\infty := \widehat{\mathcal{O}}_F^\times$ in Theorem 2.6.1 implies the finiteness of the class group of $\mathcal{O}_F$. Interestingly, if we view the class group as the Picard group of $\text{Spec}(\mathcal{O}_F)$, that is, the group of isomorphism classes of line bundles on $\text{Spec}(\mathcal{O}_F)$, this description of the class group naturally leads one to Weil's description of

$$\text{GL}_n(F) \backslash \text{GL}_n(\mathbb{A}_F^\infty) / \text{GL}_n(\widehat{\mathcal{O}}_F)$$

in terms of vector bundles over $\text{Spec}(\mathcal{O}_F)$ with rank n . We refer to [Gai03] for an introduction to what can be gained from this perspective in the function field case.

It is not hard to see that in general the quotient $G(F)\backslash G(\mathbb{A}_F)$ itself is infinite. However, we could ask when the quotient is compact or finite volume with respect to a suitable measure (see §3.6), and there is a complete answer to this question which we recall in Theorem 2.6.3. It is useful to first eliminate a trivial obstruction to the quotient $G(F)\backslash G(\mathbb{A}_F)$ being compact or finite volume.

The units $F^\times$ of F embed into $\mathbb{A}_F^\times$ diagonally as a discrete closed subgroup. The idelic norm

$$|\cdot| := \prod_v |\cdot|_v : F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{R}_{>0} \quad (2.18)$$

is continuous and nontrivial. In fact, its image is $\mathbb{R}_{>0}$ if F is a number field and of the form $p^{m\mathbb{Z}}$ for some nonzero integer m if F has characteristic p [RV99, Theorem 5-14]. Thus $F^\times \backslash \mathbb{A}_F^\times$ is noncompact and moreover is of infinite volume with respect to the Haar measure (for more on Haar measures, see §3.2). An analogous phenomenon occurs for other groups and this motivates the following definition:

$$G(\mathbb{A}_F)^1 := \bigcap_{\chi \in X^*(G)} \ker(|\cdot| \circ \chi : G(\mathbb{A}_F) \longrightarrow \mathbb{R}_{>0}). \quad (2.19)$$

Note that $G(F)$ is contained $G(\mathbb{A}_F)^1$ in virtue of the product formula stated in Proposition 2.1.5. Moreover, $G(F)$ is discrete and closed in $G(\mathbb{A}_F)^1$ by Exercise 2.2.

We now define a useful subgroup

$$A_G < G(F_\infty) < G(\mathbb{A}_F). \quad (2.20)$$

When F is a number field, we let A_G be the neutral component of the $\mathbb{R}$ -points of the greatest $\mathbb{Q}$ -split torus in $\text{Res}_{F/\mathbb{Q}} Z_G$. When F is a function field of characteristic p , temporarily write Z for the largest $\mathbb{F}_p(t)$ -split torus in

$$\text{Res}_{F/\mathbb{F}_p(t)} Z_G.$$

Then if Z has rank d , there is an isomorphism $Z \cong \mathbb{G}_m^d$ of tori over $\mathbb{F}_p(t)$ inducing an isomorphism $Z(\mathbb{F}_p(t)_\infty) \xrightarrow{\sim} (\mathbb{F}_p((t^{-1}))^\times)^d$. We let $A_G < Z(\mathbb{F}_p(t)_\infty)$ be the inverse image of the subgroup $(\{t^k : k \in \mathbb{Z}\})^d$. We warn the reader that the group A_G is noncanonical in the function field case. It depends both on the choice of embedding $\mathbb{F}_p(t) \rightarrow F$ used to define a notion of infinite places as in §2.1, and on the choice of isomorphism $Z(\mathbb{F}_p(t)_\infty) \xrightarrow{\sim} (\mathbb{F}_p((t^{-1}))^\times)^d$.

The groups A_G and $G(\mathbb{A}_F)^1$ may seem mysterious at first blush, but this is illusory. For example, when $G = \mathbb{G}_m$ over a number field F , the group $A_{\mathbb{G}_m}$ is just the image of the diagonal embedding $\mathbb{R}_{>0} \rightarrow F_\infty^\times$, regarded as a subgroup of $\mathbb{A}_F^\times$ via the natural map $F_\infty^\times \rightarrow \mathbb{A}_F^\times$. Moreover

$$A_{\text{GL}_n} = \{aI_n : a \in A_{\mathbb{G}_m}\},$$

where $I_n \in \mathrm{GL}_n(\mathbb{A}_F)$ is the identity matrix. Similarly

$$\mathrm{GL}_n(\mathbb{A}_F)^1 = \{g \in \mathrm{GL}_n(\mathbb{A}_F) : |\det g| = 1\}.$$

We leave the proof of the following lemma as Exercise 2.16:

Lemma 2.6.2 *Assume G is reductive. The group $A_G G(\mathbb{A}_F)^1$ has finite index in $G(\mathbb{A}_F)$ and $A_G \cap G(\mathbb{A}_F)^1 = 1$. If F is a number field, then $A_G G(\mathbb{A}_F)^1 = G(\mathbb{A}_F)$. $\square$*

The conclusions of Lemma 2.6.2 are false for a general affine algebraic group G over F (see Exercise 2.19).

The **adelic quotient** of G is the quotient

$$[G] := A_G G(F) \backslash G(\mathbb{A}_F). \quad (2.21)$$

We warn the reader that in the literature, $[G]$ often refers to $G(F) \backslash G(\mathbb{A}_F)$, but the convention (2.21) is more convenient for the purposes of this book. The canonical map $G(\mathbb{A}_F)^1 \rightarrow A_G \backslash G(\mathbb{A}_F)$ induces a continuous map

$$G(F) \backslash G(\mathbb{A}_F)^1 \longrightarrow [G] \quad (2.22)$$

that intertwines the right actions of $G(\mathbb{A}_F)^1$. Under the assumptions of Lemma 2.6.2, if F is a number field then (2.22) is a homeomorphism and if F is a function field there are only finitely many $G(\mathbb{A}_F)^1$ -orbits in the codomain of (2.22). Thus if G is reductive for many purposes one can work with either $A_G G(F) \backslash G(\mathbb{A}_F)$ or $G(F) \backslash G(\mathbb{A}_F)^1$. For any affine algebraic group G over F , the group $G(\mathbb{A}_F)^1$ is unimodular by Lemma 3.6.4, and hence admits a Haar measure (the notion of a Haar measure and a unimodular group will be discussed in §3.2). This in turn induces a measure on $G(F) \backslash G(\mathbb{A}_F)^1$ as we explain in §3.6.

The basic topological and measure theoretic properties of $G(F) \backslash G(\mathbb{A}_F)^1$ are summarized in the following theorem:

Theorem 2.6.3 (Borel, Conrad, Harder, Oesterlé) *Assume that G is smooth and connected. The quotient $G(F) \backslash G(\mathbb{A}_F)^1$ has finite volume. It is compact if and only if for every F -split torus $T \leq G$, one has that $T_{\overline{F}} \leq R(G_{\overline{F}})$.*

Proof. The first assertion is [Con12a, Theorem 1.3.6]. The “only if” portion of the last assertion is straightforward. The “if” portion of the last assertion is part of [Con12a, Theorem A.5.5]. $\square$

For comparison with the classical theory of automorphic forms it is convenient to relate the adelic quotient to locally symmetric spaces (see Chapter 6). For this purpose, if $K^\infty \leq G(\mathbb{A}_F^\infty)$ is a compact open subgroup let

$$h := h(K^\infty) = |G(F) \backslash G(\mathbb{A}_F^\infty) / K^\infty|.$$

By Theorem 2.6.1, $h < \infty$. Let $t_1, \dots, t_h$ denote a set of representatives for $G(F) \backslash G(\mathbb{A}_F^\infty) / K^\infty$. We then have a homeomorphism

$$\prod_{i=1}^h \Gamma_i(K^\infty) \backslash G(F_\infty) \longrightarrow G(F) \backslash G(\mathbb{A}_F) / K^\infty \quad (2.23)$$

given on the i th component by

$$\Gamma_i(K^\infty) g_\infty \longmapsto G(F) g_\infty t_i K^\infty,$$

where

$$\Gamma_i(K^\infty) := G(F) \cap t_i K^\infty t_i^{-1}$$

which is viewed as a subgroup of $G(F_\infty)$ via the map $G(F) \rightarrow G(F_\infty)$. For each i the group $\Gamma_i(K^\infty)$ is a discrete subgroup of $G(F_\infty)$. It is moreover arithmetic in the following sense:

Definition 2.6.4. Let $G \leq \mathrm{GL}_n$ be a linear algebraic group. A subgroup $\Gamma \leq G(F)$ is **arithmetic** if it is commensurable with $G(F) \cap \mathrm{GL}_n(\mathcal{O}_F)$.

Here one says that two subgroups Γ_1 and Γ_2 of a group are **commensurable** if $\Gamma_1 \cap \Gamma_2$ is of finite index in both Γ_1 and Γ_2 .

The notion of arithmeticity does not depend on the choice of representation $G \leq \mathrm{GL}_n$ (see Exercise 2.14). The subgroups of $G(F)$ that are intersections of a compact open subgroup of $G(\mathbb{A}_F^\infty)$ and $G(F)$ are known as **congruence subgroups**. Not every arithmetic group is a congruence subgroup in general, although for some groups G , this is the case. For more information, the reader can consult the literature on the so-called congruence subgroup problem.

We end this section by considering the special case where $G = \mathrm{GL}_2$ over $\mathbb{Q}$. Then $K^\infty = \mathrm{GL}_2(\widehat{\mathbb{Z}})$ is a maximal compact open subgroup, where $\widehat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ is the profinite completion of $\mathbb{Z}$. Let $N \in \mathbb{Z}_{>0}$. If we denote by

$$K_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : N \mid c \right\} \quad \text{and} \quad \Gamma_0(N) := K_0(N) \cap \mathrm{GL}_2(\mathbb{Z})$$

then (2.23) becomes

$$\Gamma_0(N) \backslash \mathrm{GL}_2(\mathbb{R}) = \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) / K_0(N).$$

If we let $K_\infty = \mathrm{SO}_2(\mathbb{R})$ then we have a bijection

$$A_{\mathrm{GL}_2} \backslash \mathrm{GL}_2(\mathbb{R}) / K_\infty \longrightarrow \mathbb{C} - \mathbb{R} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{ai+b}{ci+d}.$$

Hence

$$\Gamma_0(N) \backslash (\mathbb{C} - \mathbb{R}) \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{Q}) A_{\mathrm{GL}_2} \backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) / K_\infty K_0(N) \\ = \mathrm{GL}_2(\mathbb{Q}) \backslash ((\mathbb{C} - \mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_\mathbb{Q}^\infty) / K_0(N)),$$

where on the left $\Gamma_0(N)$ acts via Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}.$$

2.7 Reduction theory

Reduction theory allows us to control the adelic quotient $[G]$ when it is non-compact. To be more precise about this, assume that G is a reductive group over a global field F . In the number field case, this assumption can be weakened to allow for arbitrary connected affine algebraic groups over F at the cost of introducing more notation [PR94, §4.3].

The following theorem provides what is known as the **Iwasawa decomposition**:

Theorem 2.7.1 *Let G be a reductive group over a global field F and let $P \leq G$ be a parabolic subgroup. There exists a maximal compact subgroup $K \leq G(\mathbb{A}_F)$ such that*

$$G(\mathbb{A}_F) = P(\mathbb{A}_F)K. \quad (2.24)$$

We sketch the proof of this theorem in Appendix B.

Ideally one would like to use the Iwasawa decomposition to give a well-behaved fundamental domain for the action of $G(F)$ on $G(\mathbb{A}_F)^1$. Describing such fundamental domains is a difficult task in general. Fortunately for many purposes, it is unnecessary because one can make do with approximate fundamental domains known as Siegel sets.

We now prepare to give the definition of these sets. Assume that P is a minimal parabolic subgroup of G with unipotent radical N . Fix a Levi subgroup $M \leq P$. Let T_0 be a maximal split torus of G^{der} contained in M and let Z_0 be the maximal split torus of Z_G . Then $T := T_0 Z_0$ is a maximal split torus of G . As mentioned in §1.9, the set of nonzero weights of T acting on $\text{Lie } P$ is a set of positive roots with respect to a unique base $\Delta \subset \Phi(G, T)$ for the set of roots of T in G . We choose a maximal compact subgroup $K \leq G(\mathbb{A}_F)$ so that $G(\mathbb{A}_F) = P(\mathbb{A}_F)K$ using Theorem 2.7.1.

A **Siegel set** in $G(\mathbb{A}_F)^1$ is a set of the form

$$\mathfrak{S}(t) := \Omega_N \Omega_M A_{T_0}(t)K, \quad (2.25)$$

where $\Omega_N \subset N(\mathbb{A}_F)$ is a compact subset, $\Omega_M \subset M(\mathbb{A}_F)^1$ is a compact subset, $t \in \mathbb{R}_{>0}$, and

$$A_{T_0}(t) := \{x \in A_{T_0} : |\alpha(x)|_\infty \geq t \text{ for all } \alpha \in \Delta\}. \quad (2.26)$$

Here we are using the fact that $\alpha(x) \in \mathbb{G}_m(F_\infty) = F_\infty^\times$ and

$$|\cdot|_\infty := \prod_{v|\infty} |\cdot|_v.$$

The main result of reduction theory is the following:

Theorem 2.7.2 (Reduction theory) *There exists a Siegel set $\mathfrak{S}(t)$ such that*

$$G(F)\mathfrak{S}(t) = G(\mathbb{A}_F)^1.$$

Proof. See [Spr94]. To aid the reader, we note that in loc. cit. the group $G(F)$ acts on the right. Thus one has to take an inverse to move $G(F)$ to the left, and this changes the inequality in the analogue of $A_{T_0}(t)$ in loc. cit. to that we used in defining $A_{T_0}(t)$. $\square$

By the theorem, the set $\mathfrak{S}(t)$ can indeed be thought of as an approximate fundamental domain, as every element of $G(\mathbb{A}_F)^1$ is in the $G(F)$ -orbit of an element of $\mathfrak{S}(t)$. However, two elements of $\mathfrak{S}(t)$ can be in the same $G(F)$ -orbit. This is already the case for $G = \mathrm{GL}_2$ and $F = \mathbb{Q}$ (see Exercise 2.20).

To understand the content of Theorem 2.7.2, it is useful to consider the special case in which $G = \mathrm{GL}_2$ and $F = \mathbb{Q}$. Take $K = \mathrm{O}_2(\mathbb{R})\mathrm{GL}_2(\widehat{\mathbb{Z}})$ and let $B = P$ be the Borel subgroup of upper triangular matrices (since it is a Borel subgroup, it is a minimal parabolic subgroup). Let $T_0 \leq \mathrm{SL}_2$ and $T \leq \mathrm{GL}_2$ be the maximal tori of diagonal matrices. Then Δ consists of the single root

$$\begin{aligned} \alpha : T_0(R) &\longrightarrow R^\times \\ \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} &\longmapsto t_1 t_2^{-1} \end{aligned}$$

for $\mathbb{Q}$ -algebras R . In this setting, reduction theory tells us that there is a compact subset $\Omega \leq B(\mathbb{A}_{\mathbb{Q}})$ such that

$$\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{Q})A_{\mathrm{GL}_2}\Omega A_{T_0}(t)K$$

for small enough $t \in \mathbb{R}_{>0}$. This is the content of Exercise 2.20. We remark that it is this example that gives reduction theory its name. The key result in the reduction theory of positive definite binary quadratic forms over $\mathbb{Q}$ is the assertion that every form of a given discriminant is representable by a reduced form. A slight strengthening of Theorem 2.7.2 in the special case considered above implies this result.

We end this section by explaining how reduction theory is used in practice. Let $\varphi : [G] \rightarrow \mathbb{C}$ be a continuous function. Reduction theory implies that if we can control the restriction of φ to $A_{T_0}(t)$ then we can control the function on all of $[G]$. This observation motivates growth conditions we place on functions on $[G]$ (see Lemma 6.3.2) and plays a key role in convergence arguments. See §9.4, §9.5, and the proof of Proposition 14.3.3 for instance. We point out that in certain situations one requires a version of reduction theory in which the role of $G(F)$ is replaced by $P(F)$ for a parabolic subgroup P (not necessarily

minimal) and $A_{T_0}(t)$ is replaced by a larger subset of A_{T_0} . We refer to [MW95, §I.2.1] for a result of this type.

Exercises

2.1. Prove Proposition 2.1.5.

2.2. Let $R \rightarrow R'$ be a continuous map of topological rings and let X be an affine scheme of finite type over R . Show that $X(R) \rightarrow X(R')$ is continuous. Show moreover that if $R \rightarrow R'$ is a

- (a) topological embedding,
- (b) open topological embedding,
- (c) closed topological embedding,
- (d) topological embedding onto a discrete subset,

then so is $X(R) \rightarrow X(R')$.

2.3. Assume that F is a local field. Prove that a closed immersion $X \hookrightarrow Y$ of separated schemes of finite type over F induces a closed topological embedding $X(F) \hookrightarrow Y(F)$. Here we are using the topologies given by Proposition 2.2.3.

2.4. Let G be an affine algebraic group over a local field F , let $\rho : G \rightarrow \mathrm{GL}_n$ be a faithful representation, and let $K \leq \mathrm{GL}_n(F)$ be a compact open subgroup. Prove that $\rho(G(F)) \cap K$ is a compact open subgroup of $\rho(G(F))$.

2.5. Prove Lemma 2.4.6.

2.6. Let $\mathfrak{o}$ be a Dedekind domain with fraction field F and let $\mathcal{Y}$ be a flat affine scheme of finite type over $\mathfrak{o}$. Suppose that $X \rightarrow Y := \mathcal{Y}_F$ is a closed immersion. Let $\mathcal{X}$ be the schematic closure of X in $\mathcal{Y}$. Let $\mathcal{Z}$ be a flat affine scheme of finite type over $\mathfrak{o}$. For any closed immersion $\mathcal{Z} \rightarrow \mathcal{Y}$ whose generic fiber $\mathcal{Z}_F \rightarrow Y$ factors through $X \rightarrow Y$:

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow & \\ \mathcal{Z}_F & \longrightarrow & Y \end{array}$$

there is a unique closed immersion $\mathcal{X} \rightarrow \mathcal{Z}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{X} & & \\ \downarrow & \searrow & \\ \mathcal{Z} & \longrightarrow & \mathcal{Y} \end{array}$$

2.7. Let $\mathfrak{o}$ be a Dedekind domain with fraction field F . Let $\mathcal{G}$ be a flat affine group scheme over $\mathfrak{o}$ of finite type with generic fiber $G := \mathcal{G}_F$. Let H be an affine group scheme over F equipped with a morphism of group schemes $H \rightarrow G$ that is a closed immersion. Show that the schematic closure $\mathcal{H}$ of H in $\mathcal{G}$ is a closed subgroup scheme of $\mathcal{G}$.

2.8. Let S be a nonempty finite set of places of a global field F . Prove that F is dense in F_S .

2.9. Let $m > 1$ be an integer. Let

$$A := \mathbb{Z}[x_{ij}, t_{ij}, y]_{1 \leq i, j \leq n} / (\det(x_{ij})y - 1, \{x_{ij} - \delta_{ij} + mt_{ij} : 1 \leq i, j \leq n\}).$$

Here δ_{ij} is the Kronecker δ -function. Show that $\mathcal{G} := \text{Spec}(A)$ is a model of GL_n over $\mathbb{Z}$ and that

$$\mathcal{G}(\mathbb{Z}) = \{g \in \text{GL}_n(\mathbb{Z}) : g \equiv I_n \pmod{mM_n(\mathbb{Z})}\}.$$

2.10. Let X_1, X_2 be nonempty affine schemes of finite type over a global field F . Show that X_1 and X_2 admit weak (resp. strong) approximation with respect to a nonempty finite set S of places F if and only if $X_1 \times X_2$ does. Deduce that if G is an affine algebraic group over a number field F then G satisfies weak (resp. strong) approximation with respect to a finite set S of places F if and only if a Levi subgroup of G does.

2.11. Let F be a global field. Show that GL_n admits weak approximation with respect to any nonempty finite set of places of F .

2.12. Assume that the affine algebraic group G over the global field F satisfies strong approximation with respect to a finite set S of places of F including the infinite places. Show that

$$|G(F) \backslash G(\mathbb{A}_F^S) / K^S| = 1$$

for any compact open subgroup $K^S \leq G(\mathbb{A}_F^S)$.

2.13. Let F be a number field. Show that GL_n does not admit strong approximation with respect to the set of infinite places of F if the class number of F is not 1.

2.14. For $i = 1, 2$, let $\rho_i : G \rightarrow \text{GL}_{n_i}$ be a pair of faithful representations of the affine algebraic group G over a number field F . Let $\mathcal{G}_i$ be the schematic closure of $\rho_i(G)$ in GL_{n_i} over $\mathcal{O}_F$. Show that a subgroup $\Gamma \leq G(F)$ is commensurable with $\mathcal{G}_1(\mathcal{O}_F)$ if and only if it is commensurable with $\mathcal{G}_2(\mathcal{O}_F)$.

2.15. Let $B_n \leq \text{GL}_n$ be the Borel subgroup of upper triangular matrices. Show that

$$\text{GL}_n(\mathbb{R}) = B_n(\mathbb{R})\text{O}_n(\mathbb{R}) \quad \text{and} \quad \text{GL}_n(\mathbb{C}) = B_n(\mathbb{C})\text{U}_n(\mathbb{R})$$

and that for a non-Archimedean local field F ,

$$\mathrm{GL}_n(F) = B_n(F)\mathrm{GL}_n(\mathcal{O}_F).$$

Here for $\mathbb{R}$ -algebras R ,

$$\mathrm{O}_n(R) := \{g \in \mathrm{GL}_n(R) : gg^t = I_n\}$$

and

$$\mathrm{U}_n(R) := \{g \in \mathrm{GL}_n(\mathbb{C} \otimes_{\mathbb{R}} R) : g\bar{g}^t = I_n\}.$$

Deduce the adelic Iwasawa decomposition for $\mathrm{GL}_n(\mathbb{A}_F)$ for global fields F .

2.16. Prove Lemma 2.6.2.

2.17. Show that if F is a function field then there is a strict inequality $A_{\mathrm{GL}_2}\mathrm{GL}_2(\mathbb{A}_F)^1 < \mathrm{GL}_2(\mathbb{A}_F)$.

2.18. Let G be a reductive group over a global field F . Show that A_G is trivial if and only if the maximal torus of Z_G is anisotropic.

2.19. Give an example of a connected smooth affine algebraic group G over a global field F such that $A_G G(\mathbb{A}_F)^1$ is not of finite index in $G(\mathbb{A}_F)$.

2.20. Let $T_0 \leq \mathrm{SL}_2$ be the maximal torus of diagonal matrices and let

$$\Omega = \left\{ \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} : x \in [-\tfrac{1}{2}, \tfrac{1}{2}] \right\}.$$

Show that $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{Q})A_{\mathrm{GL}_2}\Omega A_{T_0}(t)\mathrm{O}_2(\mathbb{R})\mathrm{GL}_2(\widehat{\mathbb{Z}})$ for t sufficiently small. Show, on the other hand, that if t is chosen so that

$$\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) = \mathrm{GL}_2(\mathbb{Q})A_{\mathrm{GL}_2}\Omega A_{T_0}(t)\mathrm{O}_2(\mathbb{R})\mathrm{GL}_2(\widehat{\mathbb{Z}})$$

then there are at least two elements of

$$A_{\mathrm{GL}_2} \backslash A_{\mathrm{GL}_2} A_{T_0}(t) \Omega \mathrm{O}_2(\mathbb{R}) \mathrm{GL}_2(\widehat{\mathbb{Z}}) / \mathrm{O}_2(\mathbb{R}) \mathrm{GL}_2(\widehat{\mathbb{Z}})$$

that are in the same $\mathrm{GL}_2(\mathbb{Q})$ -orbit.

2.21. For each integer $N > 1$, let $\Gamma(N) \leq \mathrm{GL}_n(\mathbb{Z})$ be the kernel of the reduction map

$$\mathrm{GL}_n(\mathbb{Z}) \longrightarrow \mathrm{GL}_n(\mathbb{Z}/N).$$

Prove that a subgroup of $\mathrm{GL}_n(\mathbb{Q})$ is a congruence subgroup if and only if it contains $\Gamma(N)$ as a subgroup of finite index for some $N > 0$.

Chapter 3

Discrete Automorphic Representations



In mathematics you don't understand things. You just get used to them.

J. von Neumann

In this chapter, we give the definition of a discrete automorphic representation. These are objects in the category of unitary representations of the adelic points of an affine algebraic group. In Chapter 6, we will define general automorphic representations and explain how discrete automorphic representations give rise to automorphic representations.

3.1 Representations of locally compact groups

Our goal in this chapter is to define discrete automorphic representations. These are arguably the most important type of automorphic representations and are certainly the easiest to define. However, to study them, it is necessary to enlarge one's perspective and treat general automorphic representations. This requires more preparation. We will only complete the definition of a general automorphic representation in Chapter 6 (see definitions 6.3.5 and 6.4.2).

We begin by introducing some abstract representation theory, that is, representation theory that requires no more structure than a locally compact group. Our basic reference is [Fol95]. Throughout this chapter, we let G be a locally compact (Hausdorff) topological group (for example, $G = H(\mathbb{A}_F)$ for H an affine algebraic group over a global field F). Let V be a topological vector space over $\mathbb{C}$. Following [Rud91], we take the convention that topological vector spaces are Hausdorff. Let

$$\text{End}(V)$$

be the space of all continuous linear maps from V to itself. We let $\text{GL}(V) \leq \text{End}(V)$ be the group of invertible continuous linear maps with continuous inverse.

Often we will be concerned with the case where V is a **Hilbert space**, that is, a complex inner product space complete with respect to the metric induced by the inner product. We always assume that our Hilbert spaces are **separable**, that is, that they admit an orthonormal basis (in the Hilbert space sense) that is countable. The inner product will be denoted by $(\cdot, \cdot)$ and the norm by $\|\cdot\|_2$. When V is a Hilbert space, we let $\text{U}(V)$ be the subgroup of $\text{GL}(V)$ preserving $(\cdot, \cdot)$.

Definition 3.1.1. A **representation** (π, V) of G is a group action

$$\begin{aligned} G \times V &\longrightarrow V \\ (g, \varphi) &\longmapsto \pi(g)\varphi \end{aligned} \tag{3.1}$$

such that the map $G \times V \rightarrow V$ is (jointly) continuous. In particular, (π, V) defines a group homomorphism $\pi : G \rightarrow \text{GL}(V)$. If V is a Hilbert space and π maps G to $\text{U}(V)$, that is,

$$(\pi(g)\varphi_1, \pi(g)\varphi_2) = (\varphi_1, \varphi_2)$$

for all $\varphi_1, \varphi_2 \in V$ and $g \in G$, then π is said to be **unitary**.

Often one writes π (resp. V) for (π, V) if V (resp. π) is understood. If we wish to specify that π acts on a $\mathcal{V}$ -space (e.g. Fréchet, Hilbert, locally convex), we refer to π as a $\mathcal{V}$ -space representation.

A morphism of representations (or a **G -intertwining map**) is a morphism of topological vector spaces that is G -equivariant; it is an isomorphism if the underlying morphism of topological vector spaces is an isomorphism. If two representations are isomorphic, then they are also called **equivalent**. Two unitary representations (π_1, V_1) and (π_2, V_2) are **unitarily equivalent** if there is a G -equivariant isomorphism of Hilbert spaces $V_1 \rightarrow V_2$. Thus, unitary equivalence implies equivalence. The converse is also true under suitable assumptions by Theorem 4.4.10 and Proposition 5.3.8.

A **subrepresentation** of (π, V) is a closed subspace $W \leq V$ that is preserved by π , a **quotient** of (π, V) is a topological vector space W equipped with an action of G and a G -equivariant continuous linear surjection $V \rightarrow W$, and a **subquotient** of (π, V) is a subrepresentation of a quotient of (π, V) .

We will mostly be interested in unitary representations, and we need language to describe when a representation that is not originally presented as a unitary representation actually “is” unitary. First, we say that a representation on a Hilbert space is **unitarizable** if it is equivalent to a unitary representation. Sometimes, the natural representations to study are not on

Hilbert spaces, or even complete spaces (see Chapter 5, for example). However, one can still discuss unitarity in this situation as follows: A **pre-Hilbert space** is a complex inner product space that admits a countable orthonormal set with dense span. Thus, a Hilbert space is simply a pre-Hilbert space that is complete with respect to the metric induced by the inner product. We can still define $U(V)$ as before. A representation π of G on a pre-Hilbert space V extends by continuity to the completion $\bar{V}$ of V (which is a Hilbert space). With this in mind, we say that a representation π of G on a pre-Hilbert space V is **pre-unitary** if $\pi(G) \leq U(V)$. Thus, for every pre-unitary representation of G , we obtain, by continuity, a canonical unitary representation of G on $\bar{V}$. Finally, we say that a representation is **pre-unitarizable** if it is equivalent to a pre-unitary representation. In practice, these distinctions are often ignored, and one sometimes calls a unitarizable representation, or even a pre-unitarizable representation, simply a unitary representation.

The following lemma is helpful for checking the continuity of group actions in pre-Hilbert spaces:

Lemma 3.1.2 *Assume we are given a group action as in (3.1) where V is a pre-Hilbert space. If $\pi(g) \in U(V)$ for all $g \in G$, then the following are equivalent:*

(a) *For each $\varphi \in V$, the map*

$$\begin{aligned} G &\longrightarrow V \\ g &\longmapsto \pi(g)\varphi \end{aligned}$$

is continuous.

(b) *The action $G \times V \rightarrow V$ is continuous.*

Proof. It suffices to prove that (a) implies (b). We have

$$\begin{aligned} \|\pi(g_1)\varphi_1 - \pi(g_2)\varphi_2\|_2 &= \|\varphi_1 - \pi(g_1^{-1}g_2)\varphi_2\|_2 \\ &\leq \|\varphi_1 - \varphi_2\|_2 + \|\varphi_2 - \pi(g_1^{-1}g_2)\varphi_2\|_2 \end{aligned}$$

since $\pi(g_1)$ is an isometry of V . By (a), if we choose g_1 sufficiently close to g_2 and φ_1 sufficiently close to φ_2 , then the quantity above can be made to be arbitrarily small, proving (b). $\square$

We now discuss the basic means of constructing unitary representations. Assume one has a continuous right action

$$\begin{aligned} X \times G &\longrightarrow X \\ (x, g) &\longmapsto xg \end{aligned} \tag{3.2}$$

where X is a second countable, locally compact (Hausdorff) space equipped with a G -invariant Radon measure dx . The Radon measure gives rise to an

L^2 -space, namely the space $L^2(X, dx)$ of square integrable complex valued functions on X endowed with the inner product

$$(\varphi_1, \varphi_2) := \int_X \varphi_1(x) \overline{\varphi_2(x)} dx. \quad (3.3)$$

We obtain a left action of G on $L^2(X, dx)$ via

$$R(g)\varphi(x) := \varphi(xg). \quad (3.4)$$

We refer to the action of G on $L^2(X, dx)$ just constructed as the **natural action** or the **regular action**. It preserves the subspace

$$C_c(X) \subset L^2(X, dx)$$

of compactly supported continuous functions.

Theorem 3.1.3 *The action of G on $L^2(X, dx)$ defined as in (3.4) is continuous. It defines a unitary representation of G .*

For the proof, we follow [Fol95, Proposition 2.41]:

Proof. We first observe that, for any $\varphi \in L^2(X, dx)$, one has that

$$\int_X |\varphi(xg)|^2 dx = \int_X |\varphi(x)|^2 dx$$

since dx is right G -invariant. Thus, $R(g)$ is an isometry of $L^2(X, dx)$ for all $g \in G$. Let $U \subset G$ be an open neighborhood of the identity $1 \in G$ with compact closure such that $g \in U$ if and only if $g^{-1} \in U$. Let $\varphi \in C_c(X)$ and let $W = \text{supp}(\varphi)U \subset X$. Then W is relatively compact, and if $g \in U$, then $R(g)\varphi$ is supported in W . Thus,

$$\|R(g)\varphi - \varphi\|_2 \leq \text{meas}_{dx}(W)^{1/2} \|R(g)\varphi - \varphi\|_\infty.$$

The right-hand side can be made arbitrarily small by taking g to be in a sufficiently small neighborhood of 1 (see Exercise 3.1).

Since X is locally compact and dx is a Radon measure, $C_c(X)$ is dense in $L^2(X, dx)$ [Fol99, Proposition 7.9]. Thus, given $\varphi_1 \in L^2(X, dx)$ and $\varepsilon > 0$, we can choose $\varphi_2 \in C_c(X)$ such that $\|\varphi_1 - \varphi_2\|_2 < \varepsilon$. We have already checked that $R(g)$ is an isometry of $L^2(X, dx)$, so for all $g \in G$, we have that

$$\begin{aligned} \|R(g)\varphi_1 - \varphi_1\|_2 &\leq \|R(g)(\varphi_1 - \varphi_2)\|_2 + \|R(g)\varphi_2 - \varphi_2\|_2 + \|\varphi_2 - \varphi_1\|_2 \\ &\leq 2\varepsilon + \|R(g)\varphi_2 - \varphi_2\|_2. \end{aligned}$$

Since $\|R(g)\varphi_2 - \varphi_2\|_2$ can be made arbitrarily small by taking g to be in a sufficiently small neighborhood of 1, we deduce that for each $\varphi \in L^2(X, dx)$ the map

$$\begin{aligned} G &\longrightarrow L^2(X, dx) \\ g &\longmapsto R(g)\varphi \end{aligned}$$

is continuous. The theorem now follows from Lemma 3.1.2. $\square$

3.2 Haar measures on locally compact groups

To effectively study representations of locally compact groups, it is necessary to introduce the notion of Haar measure. If G is a locally compact group, then there exists a positive Radon measure $d_r g$ on G that is right invariant under the action of G :

$$\int_G f(gx) d_r g = \int_G f(g) d_r g \quad (3.5)$$

for all $x \in G$ and $f \in C_c(G)$, the space of compactly supported continuous functions $G \rightarrow \mathbb{C}$ [Fol95, Theorem 2.10].

Moreover, this measure is unique up to multiplication by an element of $\mathbb{R}_{>0}$. A **right Haar measure** is a choice of such a measure. There is also a left-invariant positive Radon measure, again unique up to multiplication by $\mathbb{R}_{>0}$. Such a measure is known as a **left Haar measure**. Indeed given a right Haar measure $d_r g$, we obtain a left Haar measure $d_\ell g$ by assigning to a measurable set $X \subseteq G$ the measure of $\{x^{-1} : x \in X\}$ with respect to $d_r g$. More briefly, $d_\ell g = d_r(g^{-1})$.

The existence of the right Haar measure gives via Theorem 3.1.3 a representation of G on $L^2(G) = L^2(G, d_r g)$ called the **regular representation**.

Definition 3.2.1. A locally compact group G is **unimodular** if any right Haar measure is also a left Haar measure.

A **Haar measure** is a left Haar measure that is also a right Haar measure. Thus, G admits a Haar measure if and only if it is unimodular.

We note that an abelian group is trivially unimodular. Moreover, a compact (Hausdorff) group is unimodular (see Exercise 3.2) and discrete groups are unimodular as well.

Example 3.1. Any finite dimensional real vector space is a group under addition. The usual Lebesgue measure on the vector space is a Haar measure with respect to this group structure.

Example 3.2. We can define a Haar measure on $\mathrm{GL}_n(\mathbb{R})$ by viewing it as an (open) subset of the n by n matrices $M_n(\mathbb{R})$ and then taking the measure

$$\frac{dX}{|\det X|^n}$$

where dX is a Haar measure on $M_n(\mathbb{R})$ (see Exercise 3.11). Since the measure described above is a Haar measure, $\mathrm{GL}_n(\mathbb{R})$ is unimodular. This is also a special case of Corollary 3.6.2 below.

Example 3.3. The group of points of a Borel subgroup is, in general, not unimodular. For example, if we regard GL_2 as an algebraic group over $\mathbb{R}$ and if $B \leq \mathrm{GL}_2$ is the Borel subgroup of upper triangular matrices, then

$$B(\mathbb{R}) = \left\{ \begin{pmatrix} a & \\ & b \end{pmatrix} \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} : a, b \in \mathbb{R}^\times \text{ and } t \in \mathbb{R} \right\}.$$

With respect to this decomposition, one can take

$$d_\ell g := \frac{dadbdtdt}{|a||b|} \quad \text{and} \quad d_r g := \frac{dadbdtdt}{|b|^2}$$

where da , db , and dt are the Lebesgue measure on $\mathbb{R}$.

We will define Haar measures on general local fields in §3.5. After this the two examples above can be readily generalized to the setting in which $\mathbb{R}$ is replaced by any local field F .

A **quasi-character** of G is a continuous homomorphism $G \rightarrow \mathbb{C}^\times$. Our convention is that a **character** of G is a continuous homomorphism $G \rightarrow \mathbb{C}^\times$ with image in $\{z : |z| = 1\}$. In other words, a character is a unitary quasi-character. We now define the modular quasi-character, which measures the failure of a group to be unimodular.

Assume that $d_r g$ is a right Haar measure on G and let $h \in G$. The map that sends a measurable set $X \subseteq G$ to the measure of hX with respect to $d_r g$ is another right Haar measure denoted $d_r(hg)$ (we use several obvious variants of this notation below). Since Haar measures are unique up to scaling, there exists a $\delta_G(h) \in \mathbb{R}_{>0}$ such that

$$d_r(hg) = \delta_G(h) d_r g. \tag{3.6}$$

The map

$$\delta_G : G \longrightarrow \mathbb{R}_{>0} \tag{3.7}$$

is obviously a homomorphism. It is known as the **modular quasi-character**. One can alternately define $\delta_G(g)$ via the relation

$$d_\ell(g^{-1}) = \delta_G(g) d_\ell g \tag{3.8}$$

by Exercise 3.3. Since $d_\ell(g^{-1})$ is a right Haar measure, we see that δ_G is trivial if and only if G is unimodular.

Some references define the modular quasi-character to be the multiplicative inverse of δ_G . In particular, $\Delta(g) := \delta_G^{-1}(g)$ in the notation of [Fol95]. Our convention in this book is consistent with most of the literature on automorphic representation theory. One motivation for our convention is provided by Proposition 3.6.1.

The following proposition provides a useful means of decomposing Haar measures:

Proposition 3.2.2 *Suppose that S and T are closed subgroups of G with compact intersection and that the product map $S \times T \rightarrow G$ is open with image of full measure with respect to $d_\ell g$. Then one can normalize the left and right Haar measures on S and T so that, for all $f \in C_c(G)$, one has that*

$$\int_G f(g) d_\ell g = \int_{S \times T} f(st) \frac{\delta_T(t)}{\delta_G(t)} d_\ell s d_\ell t = \int_{S \times T} \frac{f(st)}{\delta_G(t)} d_\ell s d_r t.$$

In particular, if G is unimodular then

$$\int_G f(g) dg = \int_{S \times T} f(st) d_\ell s d_r t.$$

Proof. The first equality is [Kna02, Theorem 8.32] in the case where G is a Lie group. The proof is the same in general. We also note that in the notation of loc. cit. one has $\Delta_G := \delta_G$, $\Delta_S := \delta_S$, and $\Delta_T := \delta_T$. The second equality follows from Exercise 3.3. $\square$

We will also require a slight generalization of the notion of a Haar measure. Let G be a locally compact group and let $H \leq G$ be a closed subgroup. The following theorem characterizes when $H \backslash G$ admits a (right) G -invariant Radon measure. See [Fol95, Theorem 2.49] for a proof:

Theorem 3.2.3 *There is a G -invariant Radon measure μ on $H \backslash G$ if and only if $\delta_G|_H = \delta_H$. In this case, μ is unique up to multiplication by an element of $\mathbb{R}_{>0}$, and for a suitable choice of right Haar measures $d_r g$ (resp. $d_r h$) on G (resp. on H), one has that*

$$\int_G f(g) d_r g = \int_{H \backslash G} \left(\int_H f(hg) d_r h \right) d\mu(Hg)$$

for $f \in C_c(G)$. $\square$

The measure $d\mu$ in the theorem will be denoted by $\frac{dg}{dh}$.

3.3 Gelfand-Pettis integrals

In the following section, we will require integration with values in topological vector spaces. Thus we pause to quickly state the relevant definitions and results. Let X be a locally compact (Hausdorff) topological space that is σ -compact (that is, a countable union of compact spaces). Let μ be a Radon measure on X . Since μ is a Radon measure, it is finite on compact sets, and since X is σ -compact μ is a regular measure [Fol99, Corollary 7.6].

Let V be a topological vector space. Assume that the continuous dual of V separates points. For example, V could be a locally convex space. Given a compactly supported continuous function

$$f : X \longrightarrow V,$$

the **Gelfand-Pettis integral**

$$\int_X f(x) d\mu(x)$$

is an element of V such that

$$\lambda \left(\int_X f(x) d\mu(x) \right) = \int_X \lambda(f(x)) d\mu(x)$$

for all continuous $\mathbb{C}$ -linear functionals $\lambda : V \rightarrow \mathbb{C}$. Since the continuous dual of V separates points, if $\int_X f(x) d\mu(x)$ exists, then it is unique.

A topological vector space V is **quasi-complete** if every bounded Cauchy net converges (thus complete topological vector spaces are quasi-complete). The following theorem is a rephrasing of [Gar18, Corollary 14.1.2]:

Theorem 3.3.1 *Assume V is locally convex and quasi-complete. If $f : X \rightarrow V$ is a compactly supported continuous function then the Gelfand-Pettis integral $\int_X f(x) d\mu(x)$ exists. The vector*

$$\int_X f(x) d\mu(x)$$

lies in $\int_{\text{supp}(f)} d\mu(x)$ times the closed convex hull of $f(X)$. □

Corollary 3.3.2 *Assume that $f : X \rightarrow V$ is a compactly supported continuous function and that the Gelfand-Pettis integral $\int_X f(x) d\mu(x)$ exists. If $\nu : V \rightarrow \mathbb{R}_{\geq 0}$ is a continuous seminorm then*

$$\nu \left(\int_X f(x) d\mu(x) \right) \leq \int_X \nu(f(x)) d\mu(x).$$

Proof. If $\int_X f(x) d\mu(x) = 0$, then the corollary is clear.

Assume $\int_X f(x) d\mu(x) \neq 0$. Let λ be the unique linear functional on the 1-dimensional $\mathbb{C}$ -vector subspace of V spanned by $\int_X f(x) d\mu(x)$ such that

$$\lambda \left(\int_X f(x) d\mu(x) \right) = \nu \left(\int_X f(x) d\mu(x) \right).$$

By the Hahn-Banach theorem, there is an extension of λ to a linear functional on V (still denoted by λ) such that $|\lambda(\varphi)| \leq \nu(\varphi)$ for all $\varphi \in V$. Since ν is continuous, $\lambda : V \rightarrow \mathbb{C}$ is continuous. Thus

$$\nu \left(\int_X f(x) d\mu(x) \right) = \lambda \left(\int_X f(x) d\mu(x) \right)$$

$$= \int_X \lambda(f(x)) d\mu(x).$$

This is bounded in complex norm by

$$\int_X |\lambda(f(x))| d\mu(x) \leq \int_X \nu(f(x)) d\mu(x).$$

□

Many topological vector spaces of interest are locally convex and quasi-complete. This is true, for example, of Hilbert spaces, Banach spaces, and Fréchet spaces. Every topological vector space we encounter in this book will be at least a subspace of a locally convex and quasi-complete vector space with one exception. When dealing with totally disconnected groups (see Chapter 5), we will have occasion to work with topological vector spaces V endowed with the discrete topology. In this case, the relevant integrals reduce to finite sums by Lemma 5.2.2. Thus, they can be interpreted directly without reference to Gelfand-Pettis integrals.

3.4 Convolution algebras of test functions

To any locally compact group G , we can associate the space

$$L^1(G) := \left\{ d_r g\text{-measurable } f : G \longrightarrow \mathbb{C} : \int_G |f(g)| d_r g < \infty \right\}$$

and the subspace $C_c(G)$ of compactly supported continuous functions. The space $L^1(G)$ is endowed with the structure of a $\mathbb{C}$ -algebra with product

$$(f_1 * f_2)(g) := \int_G f_1(gh^{-1}) f_2(h) d_r h. \quad (3.9)$$

The product (3.9) is associative (see Exercise 3.17) and $C_c(G)$ is a subalgebra of $L^1(G)$.

Example 3.4. When $G = \mathbb{R}$, the convolution $f_1 * f_2$ is exactly the usual convolution from Fourier analysis.

Given a representation $\pi : G \rightarrow \mathrm{GL}(V)$, for $f \in C_c(G)$, one obtains a linear map

$$\begin{aligned} \pi(f) : V &\longrightarrow V \\ \varphi &\longmapsto \int_G f(g) \pi(g) \varphi d_r g \end{aligned}$$

provided V is locally convex and quasi-complete (see Theorem 3.3.1). Here the integral is to be understood as a Gelfand-Pettis integral as in §3.3.

We observe that if π is unitary then $\pi(f)$ is bounded. In fact, in the unitary case, $\pi(f)$ is well-defined for all $f \in L^1(G)$ (see Exercise 3.18). The definition (3.9) is chosen so that

$$\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2)$$

(see Exercise 3.20). Thus, in particular, unitary representations (π, V) of G give rise to $C_c(G)$ -modules, that is, $\mathbb{C}$ -algebra homomorphisms

$$C_c(G) \longrightarrow \text{End}(V).$$

Definition 3.4.1. An **approximate identity** of $C_c(G)$ is a sequence of non-negative real valued functions $\{f_n\} \subset C_c(G)$ indexed by $n \in \mathbb{Z}_{>0}$ such that

- (a) for any open neighborhood $U \ni 1$ in G with compact closure, the support of f_n is contained in U for n large enough, and
- (b) one has that

$$\int_G f_n(g) d_r g = 1$$

for all n .

The following corollary of Theorem 3.3.1 explains this terminology:

Corollary 3.4.2 *Let (π, V) be a representation of G on a locally convex and quasi-complete vector space V . Then for any approximate identity $\{f_n\}$ of $C_c(G)$ and any $\varphi \in V$ one has that*

$$\lim_{n \rightarrow \infty} \pi(f_n)\varphi = \varphi.$$

□

3.5 Haar measures on local fields

Let F be a local field, that is, a field equipped with a nontrivial absolute value which is locally compact with respect to the topology induced by the absolute value (see §2.1). All infinite locally compact fields arise as the completion of some global field [Lor08, §25, Theorem 2] with respect to some absolute value. In this section, we compute Haar measures on the additive group of F .

Assume first that F is Archimedean, so F is isomorphic to $\mathbb{R}$ or $\mathbb{C}$. The standard Haar measure on $\mathbb{R}$ is the usual Lebesgue measure, and twice the standard Haar measure on $\mathbb{C}$. If $z = x + iy$ with $x, y \in \mathbb{R}$, then this Haar measure is $|dz \wedge d\bar{z}| = 2dx \wedge dy$, where dx and dy are the Lebesgue measure on $\mathbb{R}$.

Let F be a non-Archimedean local field with normalized absolute value $|\cdot|$ and ring of integers $\mathcal{O}_F$. Let ϖ be a uniformizer and set $q^{-1} := |\varpi|$. Since the additive group of F is a locally compact topological group, it admits a Haar measure dx (see §3.2). The set $\mathcal{O}_F \subset F$, being compact, has finite measure, so to specify the Haar measure, it suffices to specify the measure $dx(\mathcal{O}_F)$ of $\mathcal{O}_F$. We often assume $dx(\mathcal{O}_F) = 1$, but sometimes other normalizations are convenient.

Open sets in F of the form $a + \varpi^k \mathcal{O}_F$ where $a \in F$ and $k \in \mathbb{Z}_{>0}$ form a neighborhood base. Using additivity and invariance of the Haar measure, one can compute the measure of such sets:

Lemma 3.5.1 *If dx is a Haar measure on F then*

$$dx(a + \varpi^k \mathcal{O}_F) = q^{-k} dx(\mathcal{O}_F).$$

□

Thus, for every local field F , we have a Haar measure dx . Via the usual procedure, we then obtain a product measure on any F -vector space. It is again a Haar measure with respect to addition.

Given a Haar measure dx on F , we obtain a Haar measure

$$d^\times x := \frac{dx}{|x|}$$

on the unit group $F^\times$. With this normalization in the non-Archimedean case $d^\times x(\mathcal{O}_F^\times) = (1 - q^{-1})dx(\mathcal{O}_F)$ (see Exercise 3.7). Another normalization we will often use in the non-Archimedean case is $d^\times x = \frac{dx}{(1 - q^{-1})|x|}$, so that $d^\times x(\mathcal{O}_F^\times) = dx(\mathcal{O}_F)$.

3.6 Haar measures on the points of affine algebraic groups

Let G be a smooth affine algebraic group over a global field F . It is useful to more explicitly describe Haar measures on $G(R)$ for locally compact topological F -algebras R using differential forms.

Let $\mathfrak{g} := \text{Lie } G$ and let $\mathfrak{g}^*$ denote the linear dual of $\mathfrak{g}$, which we may identify with the cotangent space of G at the origin. Let $n = \dim G$. Then there is a nonzero top dimensional left-invariant differential form

$$\omega_\ell \in \wedge^n \mathfrak{g}^*$$

which is unique up to multiplication by an element of $F^\times$. For every place v of F , we have an injection

$$\wedge^n \mathfrak{g}^* \hookrightarrow \wedge^n \mathfrak{g}_{F_v}^*$$

and we continue to denote by ω_ℓ the image of ω_ℓ . This allows us to define for each v a Radon measure

$$\begin{aligned} C_c(G(F_v)) &\longrightarrow \mathbb{C} \\ f &\longmapsto \int_{G(F_v)} f(g) |\omega_\ell|_v(g). \end{aligned}$$

Since ω_ℓ is left $G(F_v)$ -invariant, the measure $|\omega_\ell|_v$ is left $G(F_v)$ -invariant, and hence is a left Haar measure. This is explained in more detail in the Archimedean case in [Kna02, Chapter VIII.2] and in the non-Archimedean case in [Oes84, §I.2].

Proposition 3.6.1 *One has that*

$$\delta_{G(F_v)}(g) = |\det(\mathrm{Ad}(g) : \mathfrak{g} \otimes_F F_v \longrightarrow \mathfrak{g} \otimes_F F_v)|_v. \quad (3.10)$$

Proof. The adjoint representation $\mathrm{Ad} : G \rightarrow \mathrm{GL}_{\mathfrak{g}}$ induces a representation

$$\mathrm{Ad}^* : G \longrightarrow \mathrm{GL}_{\mathfrak{g}^*}.$$

One has that

$$\mathrm{Ad}^*(g^{-1})\omega_\ell = c(g)\omega_\ell$$

where

$$c(g) := \det(\mathrm{Ad}(g) : \mathfrak{g} \otimes_F F_v \longrightarrow \mathfrak{g} \otimes_F F_v).$$

We write this in terms of Haar measures:

$$|\omega_\ell|_v(hg^{-1}) = |\omega_\ell|_v(ghg^{-1}) = |\mathrm{Ad}^*(g^{-1})\omega_\ell|_v(h) = |c(g)|_v |\omega_\ell|_v(h).$$

On the other hand

$$d_\ell(hg^{-1}) = \delta_{G(F_v)}(g)d_\ell h$$

for any left Haar measure $d_\ell h$ on $G(F_v)$ by Exercise 3.3. We deduce that $|c(g)|_v = \delta_{G(F_v)}(g)$. $\square$

Corollary 3.6.2 *If G° is reductive, then $G(F_v)$ is unimodular.*

Proof. Assume first that G is connected. Let Z_G be the center of G and G^{der} the derived group of G . Let Z_G^t be the maximal subtorus of Z_G . By (1.14), we then have a commutative diagram

$$\begin{array}{ccc} Z_G^t \times G^{\mathrm{der}} & \xrightarrow{a} & \mathbb{G}_m \\ \downarrow b & & \downarrow \mathrm{Id} \\ G & \xrightarrow{c} & \mathbb{G}_m \end{array}$$

where a and c are given on points in an F -algebra R by

$$\begin{aligned} a(z, g) &:= \det(\mathrm{Ad}(zg) : \mathfrak{g} \otimes_F R \longrightarrow \mathfrak{g} \otimes_F R), \\ c(g) &:= \det(\mathrm{Ad}(g) : \mathfrak{g} \otimes_F R \longrightarrow \mathfrak{g} \otimes_F R), \end{aligned}$$

the left vertical map b is the product map, and the right vertical map is the identity. The map a is trivial on Z_G^t and it is trivial on G^{der} since $X^*(G^{\mathrm{der}})$ is trivial. Indeed, the kernel of a character in $X^*(G^{\mathrm{der}})$ is a normal subgroup $N \leq G^{\mathrm{der}}$ such that G^{der}/N is commutative, hence $N = G^{\mathrm{der}}$. Since b is a quotient map, c is trivial as well. Thus, Proposition 3.6.1 implies that the modular character of $G(F_v)$ is trivial.

For the general case, we observe that Proposition 3.6.1 implies that $\delta_{G(F_v)}|_{G^\circ(F_v)} = \delta_{G^\circ(F_v)}$, which we have just shown is trivial. Thus, $\delta_{G(F_v)}$ is trivial on the finite index subgroup $G^\circ(F_v)$ of $G(F_v)$. It follows that $\delta_{G(F_v)}$ itself is trivial. $\square$

The construction at the beginning of this section yields left Haar measures $d_\ell g_v$ on $G(F_v)$ for every v . We similarly obtain right Haar measures $d_r g_v := d_\ell(g_v^{-1})$. We now construct left and right Haar measures on $G(\mathbb{A}_F)$. For this purpose, let S be a finite set of places of F including the infinite places and let

$$K^S = \prod_{v \notin S} K_v \leq G(\mathbb{A}_F^S)$$

be a compact open subgroup. We then define a left (resp. right) Haar measure on $G(\mathbb{A}_F)$ via

$$d_\ell g = \prod_v d_\ell g_v \quad \text{and} \quad d_r g = \prod_v d_r g_v \quad (3.11)$$

where we assume that $d_\ell g_v(K_v) = d_r g_v(K_v) = 1$ for all $v \notin S$. Then it is easy to see that the measures $d_\ell g$, $d_r g$ so defined are left and right Haar measures on $G(\mathbb{A}_F)$, respectively (see Exercise 3.6). If G is reductive, we can apply an analogous procedure and obtain left and right Haar measures on $A_G \backslash G(\mathbb{A}_F)$.

Lemma 3.6.3 *Any right Haar measure on $G(\mathbb{A}_F)$ is left-invariant under $G(F)$.*

Proof. Since the right Haar measure is constructed as the product of local Haar measures, one has that

$$\delta_{G(\mathbb{A}_F)}(g) = \prod_v \delta_{G(F_v)}(g_v) = |\det(\mathrm{Ad}(g) : \mathfrak{g} \otimes_F \mathbb{A}_F \longrightarrow \mathfrak{g} \otimes_F \mathbb{A}_F)|. \quad (3.12)$$

By the product formula, we deduce that $\delta_{G(\mathbb{A}_F)}(\gamma) = 1$ for $\gamma \in G(F)$. $\square$

The proof here amounted to observing that $\delta_{G(\mathbb{A}_F)}$ is trivial on $G(F)$. In fact, there is a much larger subgroup of $G(\mathbb{A}_F)$ on which the modular quasi-character is trivial, namely $G(\mathbb{A}_F)^1$, defined in (2.19):

Lemma 3.6.4 *The group $G(\mathbb{A}_F)^1$ is unimodular. If G° is reductive, then $G(\mathbb{A}_F)$ is unimodular.*

Proof. For any F -algebra R , we have a homomorphism

$$\begin{aligned} G(R) &\longrightarrow R^\times \\ g &\longmapsto \det(\mathrm{Ad}(g) : \mathfrak{g} \otimes_F R \rightarrow \mathfrak{g} \otimes_F R). \end{aligned} \quad (3.13)$$

This defines an element of $X^*(G)$, the character group of G . The equality (3.12) states that the modular quasi-character of $G(\mathbb{A}_F)$ is just (3.13) in the special case $R = \mathbb{A}_F$ followed by the idelic norm. This implies that $G(\mathbb{A}_F)^1$ is unimodular.

The modular quasi-character of $G(\mathbb{A}_F)$ is the product of the local modular quasi-characters by (3.12). Therefore, if G° is reductive, we deduce that $G(\mathbb{A}_F)$ is unimodular by Corollary 3.6.2. $\square$

Now $G(F)$ is a discrete subgroup of $G(\mathbb{A}_F)$. By Lemma 3.6.3, $\delta_{G(\mathbb{A}_F)}|_{G(F)}$ is trivial, and the modular quasi-character of any discrete group is trivial. Thus, by Theorem 3.2.3, a right Haar measure on $G(\mathbb{A}_F)$ and the counting measure on $G(F)$ together induce a $G(\mathbb{A}_F)$ -invariant Radon measure on

$$G(F) \backslash G(\mathbb{A}_F).$$

Similarly, a Haar measure on $G(\mathbb{A}_F)^1$ and the counting measure on $G(F)$ induce a $G(\mathbb{A}_F)^1$ -invariant Radon measure on $G(F) \backslash G(\mathbb{A}_F)^1$. The volume of $G(F) \backslash G(\mathbb{A}_F)^1$ is finite by Theorem 2.6.3.

We also have $\delta_{G(\mathbb{A}_F)}|_{A_G} = 1$. Moreover, $A_G G(F) \leq G(\mathbb{A}_F)$ is the direct product of A_G and $G(F)$. Hence, a right Haar measure on $G(\mathbb{A}_F)$, a Haar measure on A_G , and the counting measure on $G(F)$ together induce a right $G(\mathbb{A}_F)$ -invariant Radon measure on $[G] := A_G G(F) \backslash G(\mathbb{A}_F)$. In fact, choosing a right Haar measure on $G(\mathbb{A}_F)$ and a Haar measure on A_G individually is unnecessary to define a $G(\mathbb{A}_F)$ -invariant Radon measure on $[G]$. It is sufficient to choose a right Haar measure on $A_G \backslash G(\mathbb{A}_F)$.

If we choose a measure $d_r g$ on $G(\mathbb{A}_F)$ (resp. $A_G \backslash G(\mathbb{A}_F)$), we continue to denote by $d_r g$ the induced measure on $G(F) \backslash G(\mathbb{A}_F)$ (resp. $[G]$) described in the previous two paragraphs.

Our method of describing the Haar measure on $G(\mathbb{A}_F)$ is noncanonical. It would be more natural to define a Haar measure by directly using the product $\prod_v |\omega_\ell|_v$. However, for arbitrary compact subsets $C = \prod_v C_v \subset G(\mathbb{A}_F)$

$$\prod_v |\omega_\ell|_v(C_v)$$

need not converge. This is already the case for $G = \mathrm{GL}_1$.

If G is semisimple, then $\prod_v |\omega_\ell|_v$ does in fact yield a well-defined Haar measure on $G(\mathbb{A}_F)$. For reductive G , this measure must be corrected as follows.

One introduces a natural family of positive real numbers λ_v such that, for all compact subsets $C = \prod_v C_v \subset G(\mathbb{A}_F)$, the product

$$\prod_v \lambda_v |\omega_\ell|_v(C_v)$$

is convergent. In this way, one obtains the so-called **Tamagawa measure** on $G(\mathbb{A}_F)$. Upon choosing a suitable measure on A_G , we obtain the Tamagawa measure on $[G]$. The measure of $[G]$ with respect to the Tamagawa measure is called the **Tamagawa number** of G . This number is intimately connected with the arithmetic of G . We refer to the reader to [Kot88, Oes84] and the references therein for more details.

3.7 Discrete automorphic representations

We continue to assume that G is a smooth affine algebraic group over a global field F . Consider the quotient

$$[G] := A_G G(F) \backslash G(\mathbb{A}_F).$$

As explained in the previous section, a right Haar measure on $A_G \backslash G(\mathbb{A}_F)$ induces a $A_G \backslash G(\mathbb{A}_F)$ -invariant Radon measure on $[G]$. Thus, we have a well-defined Hilbert space

$$L^2([G])$$

with inner product defined as (3.3) (but with X replaced by $[G]$ and dx replaced by a $A_G \backslash G(\mathbb{A}_F)$ -invariant Radon measure).

Definition 3.7.1. A **discrete automorphic representation** of $A_G \backslash G(\mathbb{A}_F)$ is an irreducible unitary representation π of $A_G \backslash G(\mathbb{A}_F)$ that is equivalent to a subrepresentation of $L^2([G])$.

In general, $L^2([G])$ cannot be written as a direct sum of its irreducible subrepresentations (see §10.5). This is one of several reasons why a refined definition of an automorphic representation is necessary. We give this refinement in Chapter 6.

Note that, contrary to the usual convention, we have not assumed that G is reductive. Even without this assumption, the preceding definition makes sense. It is only when one starts discussing admissible representations that the assumption of reductivity is really necessary. A concrete example of this is given right after [BS98, Proposition 3.2.11]. The nonreductive case has not received as much attention as the reductive case, but see [Sli84].

We end this chapter with §3.8 through §3.11. Sections 3.8 through 3.10 discuss how one can decompose unitary representations of locally compact groups in general, and §3.11 discusses why we have restricted our attention

to affine algebraic groups instead of certain natural larger classes of groups. All of these sections can be omitted on a first reading.

Before delving into these sections, we note that the definition of an automorphic representation leaves much to be desired. It does not illuminate the place of automorphic representations in arithmetic and geometry and it is unclear how one would go about studying these objects. In the subsequent two chapters, we will develop the algebras of operators on automorphic representations that are used to study them. They are refinements of the algebra $L^1(A_G \backslash G(\mathbb{A}_F))$ discussed in §3.4 above. For a number field F , the decomposition

$$A_G \backslash G(F_\infty) \times G(\mathbb{A}_F^\infty)$$

naturally gives rise to a decomposition of $L^1(A_G \backslash G(\mathbb{A}_F))$; the first factor is the Archimedean factor which will be discussed in Chapter 4 and the second factor is the non-Archimedean factor which will be discussed in chapters 5 and 8.

This gives us tools to study automorphic representations, but it still does not shed much light on how they appear in geometry and arithmetic. The link with geometry will be made precise in Chapter 15. The primary link with arithmetic is much more profound, subtle, and conjectural; it is the content of the Langlands functoriality conjecture, explained in Chapter 12.

3.8 Decomposition of representations

If V is a finite dimensional complex representation of a finite group G , then it is completely reducible. In other words, it admits a direct sum decomposition

$$V = \oplus_i V_i$$

where the V_i are irreducible representations of G (see Exercise 3.15). Thus, a subquotient of V in this case is just a subrepresentation. When G is no longer a finite group, these sorts of decompositions may no longer hold (see Exercise 3.16).

However, there is a very general theory which guarantees some sort of decomposition exists if we restrict our attention to unitary representations. To elaborate we require the notion of direct integrals of representations. We follow the exposition of [Fol95, §7.4]. Let X be a measurable space. A **field of Hilbert spaces** over X is a collection of Hilbert spaces $\{V_x\}_{x \in X}$ indexed by $x \in X$. We have an obvious projection map

$$p : \prod_{x \in X} V_x \longrightarrow X$$

sending everything in $V_x \hookrightarrow \coprod_{x \in X} V_x$ to x . The inner product on V_x will be denoted by $\langle \cdot, \cdot \rangle_x$ and the induced norm by $\|\cdot\|_x$. A **section** or **vector field** is a map

$$s : X \longrightarrow \coprod_{x \in X} V_x$$

such that $(p \circ s)(x) = x$. Thus, a section may be identified with an element of $\coprod_{x \in X} V_x$.

These notions are uninteresting unless one imposes some measurability condition on the fibers of p and on sections s . This is introduced as follows. A **measurable field of Hilbert spaces** over X is a pair $(\{V_x\}_{x \in X}, \{e_j\})$ consisting of a field of Hilbert spaces $\{V_x\}_{x \in X}$ together with a sequence of sections $e_1, e_2, \dots$ such that

- (a) $x \mapsto \langle e_i(x), e_j(x) \rangle_x$ is measurable for all i, j ,
- (b) for each $x \in X$, the linear span of $e_1(x), e_2(x), \dots$ is dense in V_x .

Given a measurable field of Hilbert spaces over X , a section s is called **measurable** if $x \mapsto \langle s(x), e_j(x) \rangle_x$ is measurable on X for each j . For simplicity, in the sequel we will drop the $\{e_j\}$ from notation, despite the fact that it is an essential part of the data defining the measurable field of vector spaces.

Suppose that $\{V_x\}_{x \in X}$ is a measurable field of Hilbert spaces over X and suppose that μ is a measure on X . For measurable sections s, s' define

$$\langle s, s' \rangle_\mu := \int_X \langle s(x), s'(x) \rangle_x d\mu(x) \quad (3.14)$$

and $\|s\|_\mu := \langle s, s \rangle_\mu^{1/2}$. Then the **direct integral of $\{V_x\}_{x \in X}$ with respect to μ** is

$$\int_X^\oplus V_x d\mu(x) := \{\text{measurable sections } s : \|s\|_\mu < \infty\}. \quad (3.15)$$

As usual, we identify sections that agree almost everywhere with respect to μ . The space (3.15) is again a Hilbert space with inner product $\langle \cdot, \cdot \rangle_\mu$.

For each $x \in X$, let $\text{End}(V_x)$ denote the $\mathbb{C}$ -vector space of continuous linear operators from V_x to itself. A **field of operators** on a measurable field of Hilbert spaces $\{V_x\}_{x \in X}$ is an element

$$(A_x) \in \prod_{x \in X} \text{End}(V_x).$$

The field of operators is said to be **measurable** if $x \mapsto A_x(s(x))$ is a measurable section whenever s is a measurable section. If the essential supremum

$$\text{e.sup}_{x \in X} \|A_x\|_x$$

is finite, then we have a bounded operator

$$\int_X^\oplus A_x d\mu(x) : \int_X^\oplus V_x d\mu(x) \longrightarrow \int_X^\oplus V_x d\mu(x). \quad (3.16)$$

It sends a section s to the section $x \mapsto A_x(s(x))$. The operator (3.16) is called the **direct integral of** (A_x) . For example, a **diagonal operator** is an operator of the form

$$\begin{aligned} \int_X^\oplus V_x d\mu(x) &\longrightarrow \int_X^\oplus V_x d\mu(x) \\ s &\longmapsto (x \mapsto \phi(x)s(x)) \end{aligned} \quad (3.17)$$

for $\phi \in L^\infty(\mu)$.

Now let G be a locally compact (Hausdorff) second countable group (for example, $G(\mathbb{A}_F)$ or $A_G \backslash G(\mathbb{A}_F)$ for some affine algebraic group G over a global field F). A **measurable field of representations** on $\{V_x\}_{x \in X}$ is a collection of continuous representations

$$\pi_x : G \times V_x \longrightarrow V_x$$

indexed by $x \in X$ such that

- (a) $(\pi_x(g))$ is a measurable field of operators for each $g \in G$,
- (b) $\text{e. sup}_{x \in X} \|\pi_x(g)\|_x < \infty$ for each $g \in G$.

We say the measurable field of representations is **unitary** if π_x is unitary for all $x \in X$. In this case, condition (b) above is automatically satisfied. For any measurable field of representations on $\{V_x\}_{x \in X}$ and $g \in G$, the integral

$$\int_X^\oplus \pi_x(g) d\mu(x)$$

is defined. Hence, we obtain a representation of G on $\int_X^\oplus V_x d\mu(x)$ in the usual sense. This is the direct integral of the measurable field of representations, or more briefly, the **direct integral of the representations**.

Two motivating examples to keep in mind are the right regular representation of $\mathbb{R}$ on $L^2(\mathbb{Z} \backslash \mathbb{R})$ and $L^2(\mathbb{R})$. Fourier analysis tells us that

$$L^2(\mathbb{Z} \backslash \mathbb{R}) = \oplus_{n \in \mathbb{Z}} V_n \quad \text{and} \quad L^2(\mathbb{R}) = \int_{t \in \mathbb{R}}^\oplus V_t dt. \quad (3.18)$$

Here V_t is the 1-dimensional $\mathbb{C}$ -vector space on which $\mathbb{R}$ acts via $x \mapsto e^{2\pi i t x}$.

In some sense, all of the V_t occurring in (3.18) are “pieces” of the representation of $\mathbb{R}$ on $L^2(\mathbb{R})$. However, they are not subquotients:

Lemma 3.8.1 *For every $t \in \mathbb{R}$, there is a continuous linear $\mathbb{R}$ -intertwining map*

$$L^1(\mathbb{R}) \longrightarrow V_t.$$

There is no nontrivial continuous linear $\mathbb{R}$ -intertwining map

$$V_t \longrightarrow L^p(\mathbb{R}) \quad \text{for } p \in \{1, 2\}.$$

Moreover, there is no nontrivial $\mathbb{R}$ -intertwining map from V_t to any quotient of $L^2(\mathbb{R})$.

Proof. For each t , one has a continuous $\mathbb{R}$ -intertwining linear map

$$\begin{aligned} L^1(\mathbb{R}) &\longrightarrow V_t \\ f &\longmapsto \int_{\mathbb{R}} f(x) e^{-2\pi i t x} dx. \end{aligned}$$

Let $p \in \{1, 2\}$. An $\mathbb{R}$ -intertwining map

$$V_t \longrightarrow L^p(\mathbb{R})$$

is either zero or has image contained in the $\mathbb{C}$ -span of $x \mapsto e^{2\pi i t x}$. On the other hand, $x \mapsto e^{2\pi i t x}$ is not in $L^p(\mathbb{R})$, so we deduce that any linear $\mathbb{R}$ -intertwining map $V_t \rightarrow L^p(\mathbb{R})$ is zero.

Suppose that V_t is a subquotient of $L^2(\mathbb{R})$. Thus, there is a continuous surjective $\mathbb{R}$ -intertwining map $L^2(\mathbb{R}) \rightarrow W$ and a continuous injective $\mathbb{R}$ -intertwining map $V_t \rightarrow W$. By the Riesz representation theorem, this implies that there is a continuous injective intertwining map

$$V_t \longrightarrow W \longrightarrow L^2(\mathbb{R})$$

where the second arrow is the adjoint of $L^2(\mathbb{R}) \rightarrow W$. This contradicts the fact, proven above, that there is no nontrivial $\mathbb{R}$ -intertwining map $V_t \rightarrow L^2(\mathbb{R})$. $\square$

For any second countable locally compact group G and unitary representation (π, V) of G , one can obtain a direct integral decomposition [Fol95, Theorem 7.37] satisfying certain desiderata. Unfortunately, the direct integral decomposition is fairly badly behaved in this degree of generality. If one restricts to the class of groups that are of **type I**, then the situation is much more pleasant. We discuss this in §3.10.

3.9 The Fell topology

To describe the refined decomposition that exists for type I groups G , it is necessary to define the **unitary dual** $\widehat{G}$ of G . Our basic reference is [Fol95, §7.2]. As a set, $\widehat{G}$ is the set of unitary equivalence classes of unitary representations. The unitary dual $\widehat{G}$ is equipped with the **Fell topology**. The

procedure for defining this topology is very similar to the procedure we used in §1.2 to define the Zariski topology on an affine scheme.

We recall from §3.4 every representation (π, V) of G gives rise to an algebra homomorphism $C_c(G) \rightarrow \text{End}(V)$ which extends to $L^1(G)$ if π is unitary. We can extend this homomorphism still further by taking a suitable completion as follows. For each irreducible unitary representation π , denoted by $[\pi]$ its class in $\widehat{G}$. We set

$$\|f\|_* = \sup_{[\pi] \in \widehat{G}} \|\pi(f)\| \quad (3.19)$$

where $f \in L^1(G)$ and $\|\pi(f)\|$ denotes the operator norm of $\pi(f)$ acting on the space of π . One has $\|f\|_* \leq \|f\|_1$ where $\|\cdot\|_1$ is the norm on $L^1(G)$. The function $\|\cdot\|_*$ is obviously a seminorm and it is in fact a norm [Fol95, Corollary 7.2]. We define $C^*(G)$ to be the completion of $L^1(G)$ with respect to $\|\cdot\|_*$. The convolution operation extends by continuity to $C^*(G)$ and endows it with the structure of an algebra. The algebra $C^*(G)$ is called the **C^* -algebra** of the group G . This is the motivating example of a C^* -algebra. We will not give the definition of a C^* -algebra, referring instead to [Fol95, §1.1].

For every unitary representation (π, V) of G , the continuous homomorphism $L^1(G) \rightarrow \text{End}(V)$ extends to a continuous homomorphism

$$C^*(G) \longrightarrow \text{End}(V).$$

The kernel

$$\ker(\pi) = \{f \in C^*(G) : \pi(f) = 0\}$$

is a closed 2-sided ideal of $C^*(G)$. If π is irreducible, this ideal is called a **primitive ideal** of $C^*(G)$. In fact, it depends only on the equivalence class of π in the category of unitary representations. The set of all primitive ideals in $C^*(G)$ is called $\text{Prim}(G)$:

$$\text{Prim}(G) := \{\ker(\pi) : [\pi] \in \widehat{G}\}.$$

If we think of this as the analogue of the prime spectrum of a ring, then proceeding as follows is reasonable: For any subset $S \subset \text{Prim}(G)$, we let

$$V(S) := \left\{ \mathfrak{p} \in \text{Prim}(G) : \mathfrak{p} \supset \bigcap_{\mathfrak{q} \in S} \mathfrak{q} \right\}.$$

By convention, $V(\emptyset) = \emptyset$. Then one checks that the $V(S)$ just defined are the closed sets of a topology on $\text{Prim}(G)$, called the **hull-kernel** or **Jacobson topology**. Now we have a natural map

$$\begin{aligned} \widehat{G} &\longrightarrow \text{Prim}(G) \\ [\pi] &\longmapsto \ker(\pi) \end{aligned}$$

and we can pull back the topology on $\text{Prim}(G)$ to obtain a topology on $\widehat{G}$. In other words, the open sets in $\widehat{G}$ are just the inverse images of open sets in $\text{Prim}(G)$. This is the **Fell topology**. Since the Fell topology is a topology, the set of Borel sets on $\widehat{G}$ is defined; this gives $\widehat{G}$ the structure of a measurable space. When speaking of measures and measurable functions on $\widehat{G}$, we will always mean measures and measurable functions with respect to the σ -algebra of Borel sets with respect to the Fell topology.

3.10 Type I groups

We have not given (and will not give) the definition of a type I group. However, we can give a characterization using the following theorem [Fol95, Theorem 7.6]:

Theorem 3.10.1 *If G is a second countable locally compact group, then the following are equivalent:*

- (a) *The group G is of type I.*
- (b) *The Fell topology on $\widehat{G}$ is T_0 : for all distinct $x, y \in \widehat{G}$, there is an open set $U \subset \widehat{G}$ such that $x \in U$ and $y \notin U$ or $y \in U$ and $x \notin U$.*
- (c) *The map $\widehat{G} \rightarrow \text{Prim}(G)$ is injective.* □

Fortunately, the groups of primary concern to us in this book are of type I:

Theorem 3.10.2 *If G is an affine algebraic group over a characteristic zero local field F , then $G(F)$ is of type I.*

Proof. See [Sli84, Théorème 1.2.3]. □

One can find a proof of the following in [Clo07, Appendix]:

Theorem 3.10.3 *If G is a reductive group over a number field F , then $G(\mathbb{A}_F)$ is of type I.* □

It should be pointed out that if G is not reductive, then $G(\mathbb{A}_F)$ need not be of type I [Moo65, §7].

For $\pi \in \widehat{G}$, let V_π be the space of (a particular realization of) π . For type I groups, the following theorem describes the decomposition of a unitary representation into a direct integral of irreducible representations [Clo07, Theorem 3.3]:

Theorem 3.10.4 *Let G be a group of type I and let (ρ, V) be a unitary representation of G . Then there exists a Borel measurable multiplicity function*

$$\widehat{G} \longrightarrow \mathbb{Z}_{\geq 1}$$

$$\pi \longmapsto m(\pi)$$

and a positive measure μ on $\widehat{G}$ such that

$$(\rho, V) = \int_{\widehat{G}}^{\oplus} \left(\widehat{\bigoplus}^{m(\pi)} V_{\pi} \right) d\mu(\pi).$$

The decomposition is unique up to changing the measures and multiplicity functions on sets of measure zero. $\square$

Of course, in practice, it would be better to have an explicit description of the decomposition of Theorem 3.10.4 for naturally occurring representations V . Langlands' profound and foundational work on Eisenstein series provides such a decomposition in our primary case of interest, namely when $V = L^2([G])$. We state his results in §10.5 below.

3.11 Why affine groups?

In this section for simplicity, we assume that F is a number field as opposed to a general global field. In the definition of a discrete automorphic representation of $A_G \backslash G(\mathbb{A}_F)$, we assumed that G was an affine algebraic group. There are two possible avenues for generalization. First, we could look at covering groups of $G(\mathbb{A}_F)$ that are not algebraic. There is a great deal of beautiful theory that has developed around this topic and it is starting to become systematic. However, due to the authors' lack of experience with the theory and additional complications that come up, we have decided not to treat this topic.

If we forgo covering groups, we could still generalize from affine groups to algebraic groups that are not necessarily affine. To get some idea of how this enlarges the set of groups in question, we recall that a general connected algebraic group G over a number field F (not necessarily affine) has a unique connected linear algebraic subgroup $G^{\text{aff}} \trianglelefteq G$ such that $G^A := G/G^{\text{aff}}$ is an abelian variety by a famous theorem of Chevalley [Con02].

Now consider an abelian variety A over a number field F . Using Theorem 2.4.14, we can still define the topological space $A(\mathbb{A}_F)$. By the same theorem, the fact that A is proper over F implies that $A(\mathbb{A}_F)$ is compact. In particular, if $A(F)$ is infinite, it cannot be simultaneously discrete and closed in $A(\mathbb{A}_F)$. Thus $A(\mathbb{A}_F)$ and the adelic quotient $A(F) \backslash A(\mathbb{A}_F)$ are very different from their analogues in the affine case. We do not know whether or not it is sensible to define automorphic representations in this setting.

Exercises

3.1. In the situation of the proof of Theorem 3.1.3, prove that, if $\varphi \in C_c(X)$, then, for every $\varepsilon > 0$, there is a neighborhood $U_\varepsilon \subset G$ of 1 such that if $g \in U_\varepsilon$, then $\|R(g)\varphi - \varphi\|_\infty < \varepsilon$.

3.2. Prove that compact (Hausdorff) topological groups are unimodular.

3.3. Show that for a locally compact topological group G the following are valid:

$$\begin{aligned} d_r(hg) &= \delta_G(h)d_rg, \\ d_\ell(gh) &= \delta_G^{-1}(h)d_\ell g, \\ d_r(g^{-1}) &= \delta_G^{-1}(g)d_rg, \\ d_\ell(g^{-1}) &= \delta_G(g)d_\ell g. \end{aligned}$$

Show in addition that if the measures d_rg and $d_\ell g$ are normalized appropriately then

$$d_rg = \delta_G(g)d_\ell g.$$

3.4. Prove Lemma 3.5.1.

3.5. Let G be a reductive group over a global field F , let $P \leq G$ be a parabolic subgroup, and let $K \leq G(\mathbb{A}_F)$ be a maximal compact subgroup such that the Iwasawa decomposition

$$G(\mathbb{A}_F) = P(\mathbb{A}_F)K$$

holds (see Theorem 2.7.1). Show that the Haar measures on $G(\mathbb{A}_F)$, $P(\mathbb{A}_F)$, and K can be normalized so that

$$dg = d_\ell p dk.$$

3.6. Let $K^\infty \leq G(\mathbb{A}_F^\infty)$ be a compact open subgroup and let $\Omega \subset G(\mathbb{A}_F)$ be a compact set. Show there is a finite set S of places of F including the infinite places such that the projection of Ω to $G(\mathbb{A}_F^S)$ is contained in K^S , the projection of K^∞ to $G(\mathbb{A}_F^S)$. Deduce that the products (3.11) are left and right Haar measures on $G(\mathbb{A}_F)$, resp.

3.7. Let F be a non-Archimedean local field, let dx be a Haar measure on F , and let $d^\times x = \frac{dx}{|x|}$. Show that

$$(1 - q^{-1})dx(\mathcal{O}_F) = d^\times x(\mathcal{O}_F^\times)$$

where q is the order of the residue field of $\mathcal{O}_F$.

3.8. Let F be a non-Archimedean local field and let dg be the Haar measure on $\mathrm{GL}_2(F)$ such that $dg(\mathrm{GL}_2(\mathcal{O}_F)) = 1$. For $n \geq 0$, compute

$$dg(\{g \in \mathfrak{gl}_2(\mathcal{O}_F) \cap \mathrm{GL}_2(F) : |\det g| = q^{-n}\}).$$

3.9. Suppose that G is a reductive group over a global field F whose center contains a nontrivial split torus. Let dg be a Haar measure on $G(\mathbb{A}_F)$; it induces an invariant measure on $G(F) \backslash G(\mathbb{A}_F)$. Prove that the measure of $G(F) \backslash G(\mathbb{A}_F)$ is infinite.

3.10. Suppose that G is a reductive group over a global field F . Prove the first assertion of Theorem 2.6.3, namely that $G(F) \backslash G(\mathbb{A}_F)^1$ has finite measure, using Theorem 2.7.2.

3.11. Let F be a local field with absolute value $|\cdot|$. Prove that

$$d(x_{ij}) = \frac{\wedge_{i,j} dx_{ij}}{|\det(x_{ij})|^n}$$

is a (right and left) Haar measure on $\mathrm{GL}_n(F)$.

3.12. Let N be a unipotent group over a local field F . Prove that $N(F)$ is unimodular.

3.13. Let N be a unipotent group over a global field F . Prove that $N(\mathbb{A}_F)$ is unimodular.

3.14. Let F be a global field and let R be an F -algebra. For $(x_{ij}) \in \mathrm{GL}_2(R)$ let

$$\omega = \frac{dx_{11} \wedge dx_{21} \wedge dx_{12} \wedge dx_{22}}{(x_{11}x_{22} - x_{21}x_{12})^2}$$

viewed as a left-invariant top-dimensional differential form on G . For a non-Archimedean place v of F , compute

$$|\omega|_v(\mathrm{GL}_2(\mathcal{O}_{F_v})).$$

Prove that

$$\prod_{v \nmid \infty} |\omega|_v(\mathrm{GL}_2(\mathcal{O}_{F_v}))$$

does not converge but that

$$\prod_{v \nmid \infty} \frac{|\omega|_v(\mathrm{GL}_2(\mathcal{O}_{F_v}))}{1 - q_v^{-1}}$$

does converge.

3.15. Prove that a representation of a finite group on a finite dimensional complex vector space is completely reducible; that is, it decomposes into a direct sum of irreducible subrepresentations.

3.16. Let $B \leq \mathrm{GL}_2$ be the Borel subgroup of upper triangular matrices. Prove that the 2-dimensional representation given by the inclusion

$$B(\mathbb{C}) \longrightarrow \mathrm{GL}_2(\mathbb{C})$$

is not completely reducible; that is, it does not decompose into a direct sum of irreducible subrepresentations.

3.17. Let G be a locally compact group. Prove that the convolution product on $L^1(G)$ defined in (3.9) is associative.

3.18. If $\pi : G \rightarrow \mathrm{GL}(V)$ is a unitary representation of G , then show that $\pi(f) \in \mathrm{End}(V)$ for all $f \in L^1(G)$.

3.19. Let $\pi : G \rightarrow \mathrm{GL}(V)$ be a unitary representation of G . Let $W \leq V$ be a subrepresentation. Show that for each $f \in L^1(G)$ the operator $\pi(f)$ preserves W and defines an element of $\mathrm{End}(W)$.

3.20. Let G be a locally compact group and let $f_1, f_2 \in C_c(G)$. Prove that for any representation π of G , one has that

$$\pi(f_1 * f_2) = \pi(f_1) \circ \pi(f_2).$$



I have often pondered over the roles of knowledge or experience, on the one hand, and imagination or intuition, on the other, in the process of discovery. I believe that there is a certain fundamental conflict between the two, and knowledge, by advocating caution, tends to inhibit the flight of imagination. Therefore, a certain naivete, unburdened by conventional wisdom, can sometimes be a positive asset.

attributed to Harish-Chandra by
Langlands

In this chapter, we introduce the main players in the representation theory of real Lie groups. In particular, we define admissible representations, $(\mathfrak{g}, K)$ -modules and infinitesimal characters. The chapter ends with a brief discussion of the Langlands classification.

4.1 The passage between analysis and algebra

For the moment, let F be a number field and let G be an affine algebraic group over F . Consider a Hilbert space representation V of

$$G(F_\infty) = \prod_{v|\infty} G(F_v).$$

Since we are interested in automorphic representations, the example one should keep in mind is $L^2([G])$. The fact that V is generally not a space of smooth functions is often inconvenient. It is also not consonant with the classical theory of automorphic forms (to be discussed in Chapter 6). Indeed, the classical theory defines automorphic forms to be certain smooth, even real analytic functions satisfying specific differential equations.

The goal of this chapter is to explain how one passes from the abstract representation V to a space of smooth functions that determines the original space under suitable hypotheses (see Theorem 4.4.10). An added benefit of explaining this process is that it allows us to pass back and forth from the analytic theory of representations of $G(F_\infty)$ to the algebraic theory of $(\mathfrak{g}, K)$ -modules (to be defined in §4.4). Sections 4.2, 4.3, and 4.4 establish crucial vocabulary for the remainder of the book. At the end of §4.4, we will use this vocabulary to describe the remaining sections of the chapter. One could omit these sections on a first reading and refer back to them as needed.

Before continuing, we note that much of [Bum97, §2.4] is reproduced here without essential change. The presentation in loc. cit. seemed nearly perfect so we did not feel the need to modify it. We have of course added material that frames these basic results in a form suitable for our purposes later in the book.

There is a substantial literature dealing solely with Archimedean representation theory, and we have passed over much of it in silence. We cannot, however, omit explicit mention of Harish-Chandra. Anyone studying reductive Lie groups must at some point walk the roads laid by him. His original papers are still the only reference for certain results. We mention that Knapp [Kna86] and Wallach [Wal88, Wal92] have written lucid introductions to the theory in complementary styles.

4.2 Smooth vectors

In this section, G denotes an affine algebraic group over the Archimedean local field F . Let $\mathfrak{g}$ denote its Lie algebra. If F is real then $G(F)$ is a real Lie group and $\mathfrak{g}$ is a real Lie algebra. When F is complex, we use the canonical identification

$$\mathrm{Res}_{F/\mathbb{R}} G(\mathbb{R}) = G(F)$$

to view $G(F)$ as the group of $\mathbb{R}$ -points of an affine algebraic group over $\mathbb{R}$. Similarly, when F is complex, then we identify the complex Lie algebra $\mathfrak{g}$ with the real Lie algebra $\mathrm{Lie} \mathrm{Res}_{F/\mathbb{R}} G$ by viewing $\mathfrak{g}$ as a real vector space via the inclusion $\mathbb{R} \hookrightarrow \mathbb{C}$. In this setting, the Lie algebra $\mathfrak{g}$, defined as in §1.6, may be identified with the tangent space at the identity of the real Lie group $G(F)$. Indeed, this follows from the discussion in [EH00, §VI.1.3] identifying

two notions of the tangent space in algebraic geometry together with the following lemma:

Lemma 4.2.1. *Let X be a smooth affine scheme of finite type over $\mathbb{R}$. If $X(\mathbb{R})$ is nonempty, then it is a finite union of smooth connected manifolds. The tangent space at $x \in X(\mathbb{R})$ in the sense of algebraic geometry may be canonically identified with the tangent space in the sense of differential geometry.*

Proof. By [BCR98, Theorem 2.3.6], the $\mathbb{R}$ -points of any affine $\mathbb{R}$ -scheme of finite type have finitely many components in the Hausdorff topology. The remainder of the lemma follows from the discussion of tangent spaces in [Mum99, §III.4]. $\square$

Recall that there exists an exponential map

$$\exp : \mathfrak{g} \longrightarrow G(F).$$

In the case where $G = \mathrm{GL}_{nF}$, the Lie algebra $\mathfrak{gl}_{nF}$ is the F -vector space of $n \times n$ matrices. The exponential is simply the matrix exponential in this case:

$$\exp(X) := \sum_{m=0}^{\infty} \frac{X^m}{m!}.$$

In general, one chooses a faithful representation $G \hookrightarrow \mathrm{GL}_n$ (which exists by Theorem 1.5.2). This induces an injection

$$\mathfrak{g} \longrightarrow \mathfrak{gl}_{nF}$$

and the exponential on $\mathfrak{g}$ is just obtained by restriction. Of course, there is an intrinsic definition of the exponential map [Kna02, Proposition 1.84], and using it, one proves that the exponential map is independent of the choice of representation.

Though we are most interested in the case of Hilbert space representations, to study them, it is often necessary to study related representations on Fréchet spaces. We recall that **Fréchet spaces** are complete metrizable locally convex topological vector spaces. Every Hilbert space is a Fréchet space, but not conversely. To give a Fréchet space structure on a complex vector space V is equivalent to giving a countable separating family of seminorms

$$\{\nu_i : V \longrightarrow \mathbb{R}_{\geq 0}\}$$

such that V is complete with respect to the set of seminorms. Here the set of seminorms is **separating** if and only if $\nu_i(\varphi) = 0$ for all i implies $\varphi = 0$. The topology on the Fréchet space is defined so that $\varphi_n \rightarrow \varphi$ in V if and only if $\nu_i(\varphi_n) \rightarrow \nu_i(\varphi)$ for all i .

Example 4.1. For any $k, n \in \mathbb{Z}_{>0}$ and $f \in C^\infty(\mathbb{R})$ we may consider the seminorm

$$\nu_{k,n}(f) := \sup \left\{ |f^{(k)}(x)| : n \geq x \geq -n \right\}.$$

These seminorms give $C^\infty(\mathbb{R})$ the structure of a Fréchet space. It is not a Hilbert space.

For any affine algebraic group over an Archimedean local field F the space of smooth functions $C^\infty(G(F))$ is a Fréchet space. To define the relevant seminorms, we recall that for any $X \in \mathfrak{g}$ and $f \in C^\infty(G(F))$ we can define the left-invariant vector field

$$\tilde{X}f(g) := \frac{d}{dt}f(g \exp(tX)) \Big|_{t=0} = \lim_{t \rightarrow 0} \frac{f(g \exp(tX)) - f(g)}{t}. \quad (4.1)$$

Here the limit is taken over real numbers t . For all $X_1, \dots, X_n \in \mathfrak{g}$ and all compact sets $\Omega \subseteq G(F)$, we define

$$\nu_{\Omega, X_1, \dots, X_n}(\varphi) := \sup \left\{ |\tilde{X}_1 \dots \tilde{X}_n f(g)| : g \in \Omega \right\}.$$

This is not a countable set of seminorms, but it can be replaced by a countable set of seminorms inducing the same topology since $\mathfrak{g}$ is finite dimensional and $G(F)$ is the union of a countable set of compact subsets.

More generally, for any smooth real manifold Y , the Fréchet space $C^\infty(Y)$ of smooth functions $f : Y \rightarrow \mathbb{C}$ is defined. In the special case that $Y = G(F)$, it turns out that this Fréchet space is equal to the space described above, though the definitions of smoothness and the seminorms involved are slightly different. We refer to [God15, §25], especially [God15, Theorem 39] for a discussion.

As in Definition 3.1.1, a Fréchet representation is a Fréchet space V equipped with a group action $G(F) \times V \rightarrow V$ that is (jointly) continuous. The archetypical example is the regular representations induced by right multiplication by $G(F)$:

$$\begin{aligned} G(F) \times C^\infty(G(F)) &\longrightarrow C^\infty(G(F)) \\ (g, f) &\longmapsto (x \mapsto f(xg)). \end{aligned} \quad (4.2)$$

This realizes $C^\infty(G(F))$ as a Fréchet space representation of $G(F)$.

Let (π, V) be a Fréchet space representation of $G(F)$. Given $\varphi \in V$ and $X \in \mathfrak{g}$ we write

$$\begin{aligned} \pi(X)\varphi &= \frac{d}{dt}\pi(\exp(tX))\varphi \Big|_{t=0} \\ &= \lim_{t \rightarrow 0} \frac{\pi(\exp(tX))\varphi - \varphi}{t} \end{aligned}$$

if the limit exists. Here as in (4.1) the limit is taken over $t \in \mathbb{R}$. We will sometimes simply write $X\varphi$ for $\pi(X)\varphi$. We say that a vector $\varphi \in V$ is C^1 if for all $X \in \mathfrak{g}$, the derivative $X\varphi$ is defined. We define C^j inductively by stipulating that $\varphi \in V$ is C^j if φ is C^{j-1} and $X\varphi$ is C^{j-1} for all $X \in \mathfrak{g}$. A vector $\varphi \in V$ is C^∞ if it is C^j for all $j \geq 1$.

Definition 4.2.2. A vector $\varphi \in V$ is said to be **smooth** if φ is C^∞ . The subspace of smooth vectors is denoted by $V_{\text{sm}} \leq V$.

It is helpful to explicitly relate this notion of smoothness to more general notions of smoothness. Let X be a smooth real manifold that is a countable union of compact subsets. The Fréchet space $C^\infty(X)$ of smooth functions $f : X \rightarrow \mathbb{C}$ is defined. The definition of a smooth function $f : X \rightarrow \mathbb{C}$ is given in terms of local coordinates on X , so the definition is still meaningful if we replace $\mathbb{C}$ by a Fréchet space V . We thereby obtain the space

$$C^\infty(X, V) \tag{4.3}$$

of smooth functions on X with values in V . Moreover, one can mimic the definition of the Fréchet structure on $C^\infty(X)$ to give $C^\infty(X, V)$ the structure of a Fréchet space (see [War72, Appendix 2]). Grothendieck showed that the natural map

$$C^\infty(X) \otimes V \longrightarrow C^\infty(X, V)$$

induces a topological isomorphism of the completed tensor product $C^\infty(X) \widehat{\otimes} V$ with $C^\infty(X, V)$. Grothendieck moreover proved the following theorem:

Theorem 4.2.3 *A function $f : X \rightarrow V$ lies in $C^\infty(X, V)$ if and only if for all continuous $\mathbb{C}$ -linear maps $\lambda : V \rightarrow \mathbb{C}$ the map $\lambda \circ f$ lies in $C^\infty(X)$. $\square$*

We refer to [War72, Appendix 2] for references and [Gar18, §15.1] for the important special case in which X is an open interval.

Specializing to the case $X = G(F)$ we see that it would also be reasonable to define V_{sm} to be the subspace of $\varphi \in V$ such that $g \mapsto \pi(g)\varphi$ lies in $C^\infty(G(F), V)$. Fortunately, this definition coincides with the earlier one:

Theorem 4.2.4 *Let $\varphi \in V$. The function $g \mapsto \pi(g)\varphi$ is in $C^\infty(G(F), V)$ if and only if $\varphi \in V_{\text{sm}}$.*

Proof. See [God15, §25-26], especially [God15, Theorem 39]. In loc. cit. it is assumed that V is a Banach space, but this assumption is never used. $\square$

Lemma 4.2.5 *The space V_{sm} is invariant under $G(F)$.*

Proof. Let $g \in G(F)$ and let $X \in \mathfrak{g}$. An elementary calculation implies that

$$g^{-1} \exp(X)g = \exp(g^{-1}Xg) = \exp(\text{Ad}(g^{-1})X).$$

Thus, we have that

$$\begin{aligned}
X(\pi(g)\varphi) &= \lim_{t \rightarrow 0} \frac{\pi(\exp(tX))\pi(g)\varphi - \pi(g)\varphi}{t} \\
&= \lim_{t \rightarrow 0} \frac{\pi(g \exp(tg^{-1}Xg))\varphi - \pi(g)\varphi}{t} \\
&= \pi(g) \left(\lim_{t \rightarrow 0} \frac{\pi(\exp(t \operatorname{Ad}(g^{-1})X))\varphi - \varphi}{t} \right).
\end{aligned}$$

The limit exists if φ is C^1 , and this implies that $\pi(g)\varphi$ is C^1 . In other words we have an equality

$$X \circ \pi(g) = \pi(g) \circ \operatorname{Ad}(g^{-1})(X) \quad (4.4)$$

of operators on C^1 vectors in V . Using this equality and induction one shows that if φ is C^j , then $\pi(g)\varphi$ is C^j for all j . $\square$

Lemma 4.2.6 *The action of $\mathfrak{g}$ on V_{sm} defined above is a Lie algebra representation of $\mathfrak{g}$ on V_{sm} .*

Here a representation of $\mathfrak{g}$ on V_{sm} is just a Lie algebra morphism $\mathfrak{g} \rightarrow \operatorname{End}(V_{\text{sm}})$; there is no continuity condition imposed.

Proof. Recall that $C^\infty(G(F))$ is a representation of the Lie algebra $\mathfrak{g}$ where the action is given by sending $X \in \mathfrak{g}$ to $\tilde{X}$, defined as in (4.1). The strategy of the proof is to reduce to this case.

Let $\lambda : V \rightarrow \mathbb{C}$ be a continuous linear functional. We claim that one has a $\mathfrak{g}$ -equivariant map

$$\begin{aligned}
I_\lambda : V_{\text{sm}} &\longrightarrow C^\infty(G(F)) \\
\varphi &\longmapsto (g \mapsto \lambda(\pi(g)\varphi)).
\end{aligned}$$

Theorems 4.2.3 and 4.2.4 imply that I_λ is well-defined. For all $(X, \varphi) \in \mathfrak{g} \times V_{\text{sm}}$ we have

$$\begin{aligned}
\left. \frac{d}{dt} I_\lambda(\varphi)(g \exp(tX)) \right|_{t=0} &= \left. \frac{d}{dt} \lambda(\pi(g)\pi(\exp(tX))\varphi) \right|_{t=0} \\
&= \lambda(\pi(g)X\varphi) \\
&= I_\lambda(X\varphi)(g).
\end{aligned}$$

Thus

$$(\tilde{X} \circ I_\lambda)(\varphi) = (I_\lambda \circ X)(\varphi). \quad (4.5)$$

In order to verify that V_{sm} is a representation of $\mathfrak{g}$, we must check that

$$X(Y\varphi) - Y(X\varphi) = [X, Y]\varphi \quad (4.6)$$

for all $X, Y \in \mathfrak{g}$ and all $\varphi \in V_{\text{sm}}$. We apply (4.5) twice to see that

$$\begin{aligned}
I_\lambda(X(Y\varphi)) - I_\lambda(Y(X\varphi)) &= ((\tilde{X} \circ \tilde{Y} - \tilde{Y} \circ \tilde{X}) \circ I_\lambda)(\varphi) \\
&= ([\tilde{X}, \tilde{Y}] \circ I_\lambda)(\varphi) \\
&= I_\lambda([X, Y]\varphi)
\end{aligned}$$

for all continuous linear functionals $\lambda : V \rightarrow \mathbb{C}$. This implies (4.6). $\square$

Note that, so far, we do not even know if V_{sm} is nonzero; it ought to be large in order for it to be useful. Fortunately, it is indeed large. To make this precise, for $f \in C_c^\infty(G(F))$ and $\varphi \in V$, define

$$\pi(f)\varphi = \int_{G(F)} f(g)\pi(g)\varphi d_r g$$

where $d_r g$ is a right Haar measure. This is an instance of the Gelfand-Pettis integral discussed in §3.3. Let $d_\ell g = \delta_{G(F)}^{-1}(g)d_r g$. Then $d_\ell g$ is a left Haar measure on $G(F)$ by Exercise 3.3.

Proposition 4.2.7 *If $f \in C_c^\infty(G(F))$ and $\varphi \in V$, then $\pi(f)\varphi \in V_{\text{sm}}$. Moreover, the space V_{sm} is dense in V .*

Proof. For $X \in \mathfrak{g}$ let

$$f_X(g) = \left. \frac{d}{dt} f(\exp(-tX)g) \right|_{t=0}. \quad (4.7)$$

Then $f_X \in C_c^\infty(G(F))$. We apply the Leibniz integral rule to obtain

$$\begin{aligned}
\int_{G(F)} \frac{f_X(g)}{\delta_{G(F)}(g)} \pi(g)\varphi d_r g &= \left. \frac{d}{dt} \int_{G(F)} f(\exp(-tX)g)\pi(g)\varphi d_\ell g \right|_{t=0} \\
&= \left. \frac{d}{dt} \int_{G(F)} f(g)\pi(\exp(tX)g)\varphi d_\ell g \right|_{t=0} \\
&= \left. \frac{d}{dt} \pi(\exp(tX))\pi(f\delta_{G(F)}^{-1})\varphi \right|_{t=0} \\
&= X\pi(f\delta_{G(F)}^{-1})\varphi.
\end{aligned} \quad (4.8)$$

Since f is arbitrary, this implies that $\pi(f)\varphi \in C^1$. By induction we see that $\pi(f)\varphi \in V_{\text{sm}}$.

To prove the second claim, we observe that one can choose an approximate identity $\{f_n\}$ of $C_c(G(F))$ such that $f_n \in C_c^\infty(G(F))$ for all n . Thus, the second claim follows from Corollary 3.4.2. $\square$

The subspace of V_{sm} spanned by vectors of the form $\pi(f)\varphi$ for some $f \in C_c^\infty(G(F))$ is known as the **Gårding subspace**. Note that what we actually proved was that the Gårding subspace is contained in V_{sm} and is dense in V . In fact, one has the following [DM78]:

Theorem 4.2.8 (Dixmier-Malliavin theorem) *The Gårding subspace is equal to V_{sm} .* $\square$

Another result from the same paper, known as the Dixmier-Malliavin lemma, also comes up often in practice:

Theorem 4.2.9 (Dixmier-Malliavin lemma) *Let $f \in C_c^\infty(G(F))$. Then it can be written as a finite sum of convolutions:*

$$f = \sum_i h_{i1} * h_{i2}$$

where $h_{ij} \in C_c^\infty(G(F))$. $\square$

Thus far, we have regarded V_{sm} topologically as a subspace of V . It is sometimes necessary to endow V_{sm} with the structure of a Fréchet space. In particular, if V is a Hilbert space with norm $\|\cdot\|_2$, then we give V_{sm} the structure of a Fréchet space as follows: For every finite subset $\{X_1, \dots, X_n\} \subset \mathfrak{g}$ we define a seminorm $\nu_{X_1, \dots, X_n}$ on V_{sm} via

$$\nu_{X_1, \dots, X_n}(\varphi) := \|\pi(X_1) \circ \dots \circ \pi(X_n)\varphi\|_2. \quad (4.9)$$

We endow V_{sm} with the structure of a locally convex topological vector space using these seminorms in the usual manner. We can choose a countable subset of this set of seminorms inducing the same topology. Moreover, the seminorm $\nu_\emptyset$ is just the norm on V , and hence this family of seminorms is separating. Thus, V_{sm} is a Fréchet space representation of $G(F)$. This example is in fact our primary reason for working with Fréchet spaces above and not merely Hilbert spaces.

We warn the reader that the topology on V_{sm} just introduced is in general not the subspace topology induced by the inclusion $V_{\text{sm}} \leq V$. Indeed, V_{sm} is complete with respect to the topology induced by the seminorms (4.9), but we proved in Proposition 4.2.7 that V_{sm} is dense in V with respect to the topology on V .

4.3 Restriction to compact subgroups

The representation theory of compact groups is much simpler than the representation theory of noncompact groups. For example, any irreducible Hilbert space representation of a compact group is unitarizable and finite dimensional (see Theorem 4.3.4 below). Because of this, a profitable strategy in representation theory is to analyze the restriction of a given representation to compact subgroups. We will apply this strategy to the real points of reductive groups in the following section. We complete some preparation in the current section.

Later we will also require results on compact subgroups of groups that are not necessarily Lie groups. Consequently in this section we return to the setting of Chapter 3. Thus, let G be a locally compact (Hausdorff) topological group and let $K \leq G$ be a compact subgroup.

Lemma 4.3.1. *Let π be a representation of G on a Hilbert space V . There exists a (Hermitian) inner product $(\cdot, \cdot) : V \times V \rightarrow \mathbb{C}$ which gives the same topology as the given inner product on V but with respect to which $\pi|_K$ is unitary:*

$$(\pi(k)\varphi_1, \pi(k)\varphi_2) = (\varphi_1, \varphi_2)$$

for all $k \in K$ and $\varphi_1, \varphi_2 \in V$.

Proof. Let $\langle \cdot, \cdot \rangle$ denote the original inner product on the Hilbert space and $\|\cdot\|_2$ the original norm. Define

$$(\varphi_1, \varphi_2) = \int_K \langle \pi(k)\varphi_1, \pi(k)\varphi_2 \rangle dk.$$

It is K -invariant by construction so we need only check the claim about the topology. For $\varphi \in V$, the map $K \rightarrow \mathbb{C}$ given by

$$k \mapsto \langle \pi(k)\varphi, \pi(k)\varphi \rangle = \|\pi(k)\varphi\|_2^2$$

is continuous on K , hence bounded. Thus, the family of continuous operators $\{\pi(k) : k \in K\}$ is pointwise bounded on V , hence uniformly bounded in operator norm by the uniform boundedness principle. In particular, there is an $\alpha_0 \in \mathbb{R}_{>1}$ such that $\|\pi(k)\varphi\|_2 \leq \alpha_0 \|\varphi\|_2$ for all $k \in K$. By an analogous argument the family of operators $\{\pi(k^{-1}) : k \in K\}$ is also uniformly bounded in operator norm. Thus, there is an $\alpha \in \mathbb{R}_{>1}$ such that $\alpha^{-1} \|\varphi\|_2 \leq \|\pi(k)\varphi\|_2 \leq \alpha \|\varphi\|_2$ for all $k \in K$. We deduce that

$$\|\varphi\|_{2,\text{new}}^2 := \int_K \langle \pi(k)\varphi, \pi(k)\varphi \rangle dk$$

satisfies

$$\alpha^{-2} \text{meas}(K) \|\varphi\|_2^2 \leq \|\varphi\|_{2,\text{new}}^2 \leq \alpha^2 \text{meas}(K) \|\varphi\|_2^2.$$

This implies the result. $\square$

We have a representation

$$\begin{aligned} K \times K \times L^2(K) &\longrightarrow L^2(K) \\ ((k_1, k_2), f) &\longmapsto (k \mapsto f(k_1^{-1} k k_2)). \end{aligned} \tag{4.10}$$

It turns out that every irreducible representation of K can be embedded into the restriction of this representation to one copy of K (see Theorem 4.3.4). To prove this, one uses the following definition:

Definition 4.3.2. Let (π, V) be a Hilbert space representation of G with inner product $(\cdot, \cdot)$. A **matrix coefficient** of π is a function of the form

$$\begin{aligned} m : G &\longrightarrow \mathbb{C} \\ g &\longmapsto (\pi(g)\varphi_1, \varphi_2) \end{aligned}$$

for some $\varphi_1, \varphi_2 \in V$.

We point out that matrix coefficients are continuous.

Irreducible representations of K can be recovered from their matrix coefficients:

Proposition 4.3.3 *Suppose (π_1, V_1) and (π_2, V_2) are two representations of K with π_2 unitary. If there exist matrix coefficients m_1, m_2 for π_1, π_2 respectively that are not orthogonal in $L^2(K)$, then there exists a nontrivial intertwining operator $I : V_1 \rightarrow V_2$.*

Proof. Write $(\cdot, \cdot)_i$ for the inner product on V_i . By our assumption that there exists matrix coefficients that are not orthogonal in $L^2(K)$ we can choose $x_1, y_1 \in V_1$ and $x_2, y_2 \in V_2$ such that

$$\int_K (\pi_1(k)x_1, y_1)_1 \overline{(\pi_2(k)x_2, y_2)_2} dk \neq 0. \quad (4.11)$$

Let

$$I(\varphi) = \int_K (\pi_1(k)\varphi, y_1)_1 \pi_2(k^{-1})y_2 dk.$$

We claim that I is an intertwining operator

$$I : V_1 \longrightarrow V_2.$$

Indeed, for all $g \in K$ and $\varphi \in V_1$,

$$\pi_2(g)I(\varphi) = \int_K (\pi_1(k)\varphi, y_1)_1 \pi_2(gk^{-1})y_2 dk.$$

If we change variables $k \mapsto kg$ this becomes $I(\pi_1(g)\varphi)$ and so I is an intertwining operator. We now verify that I is nonzero. One has that

$$\begin{aligned} (I(x_1), x_2)_2 &= \left(\int_K (\pi_1(k)x_1, y_1)_1 \pi_2(k^{-1})y_2 dk, x_2 \right)_2 \\ &= \int_K (\pi_1(k)x_1, y_1)_1 (\pi_2(k^{-1})y_2, x_2)_2 dk \\ &= \int_K (\pi_1(k)x_1, y_1)_1 \overline{(\pi_2(k)x_2, y_2)_2} dk \quad (V_2 \text{ is unitary}) \\ &\neq 0 \end{aligned}$$

by (4.11). Thus, I is nonzero. □

Let $\widehat{K}$ be the set of equivalence classes of irreducible Hilbert representations of K . Since all irreducible Hilbert representations of K are unitarizable by Lemma 4.3.1, Exercise 4.4 below implies this is consistent with the notation of §3.9. For $\pi \in \widehat{K}$ we write (π, V_π) for a representative of the equivalence class which we assume is unitary. This is no loss of generality by Lemma 4.3.1. It gives rise to a representation

$$\text{End}(V_\pi) \cong \pi^\vee \otimes \pi$$

of $K \times K$. There is an obvious inner product on $\text{End}(V_\pi)$ given by

$$\begin{aligned} \text{End}(V_\pi) \times \text{End}(V_\pi) &\longrightarrow \mathbb{C} \\ (S, T) &\longmapsto \langle S, T \rangle := \text{tr}(ST^*) \end{aligned}$$

where the $*$ denotes the adjoint. Here we are using the fact that V_π is necessarily finite dimensional, which is part (a) of the following theorem:

Theorem 4.3.4 *Let K be a compact Hausdorff topological group. Then*

- (a) *Any irreducible representation of K is finite dimensional.*
- (b) *If (π, V) is a unitary representation of K , then V decomposes into a (completed) Hilbert space direct sum of irreducible unitary subrepresentations.*
- (c) *(Peter-Weyl) There is an isomorphism*

$$\begin{aligned} L^2(K) &\xrightarrow{\sim} \widehat{\bigoplus_{\pi \in \widehat{K}} \text{End}(V_\pi)} \\ f &\longmapsto (\pi(f) : V_\pi \rightarrow V_\pi) \end{aligned}$$

that is a unitary equivalence of representations of $K \times K$.

Here the $\widehat{\bigoplus}$ in (c) denotes the (completed) Hilbert space direct sum.

Proof. Assertion (a), (b) and (c) are [DE09, Theorem 7.2.4], [DE09, Theorem 7.3.2] and [DE09, Theorem 7.2.3], respectively. $\square$

We observe that the Peter-Weyl theorem (part (c) of Theorem 4.3.4 above) implies that the span of the matrix coefficients of π in $L^2(K)$ is the inverse image of $\text{End}(V_\pi)$ in $L^2(K)$. For more details we refer to [DE09, §7.2].

If we assume that K is a Lie group and admits a faithful representation, then the proof of Theorem 4.3.4 is not difficult (see Exercise 4.3). This case suffices for the purposes of this chapter. However, when K is a Lie group, the usual proof that K admits a faithful representation uses the Peter-Weyl theorem [Kna02, Corollary 4.22].

4.4 $(\mathfrak{g}, K)$ -modules

Let G be an affine algebraic group over an Archimedean local field F and let $K \leq G(F)$ be a maximal compact subgroup. Our goals in this section are to introduce the notion of admissibility, define $(\mathfrak{g}, K)$ -modules, and attach a $(\mathfrak{g}, K)$ -module to each admissible representation. With these concepts in hand, at the end of the section we will describe the content of the remainder of the chapter.

Before we begin in earnest, let us state two related structural results:

Theorem 4.4.1 *If G is an affine algebraic group over an Archimedean local field F , then $G(F)$ has finitely many components in the Hausdorff topology. Any maximal compact subgroup of $G(F)$ meets all components of $G(F)$ in the Hausdorff topology.*

Proof. When $F = \mathbb{R}$ the first assertion is a special case of the first assertion of Lemma 4.2.1. If $F = \mathbb{C}$ we replace G by $\text{Res}_{\mathbb{C}/\mathbb{R}} G$ to reduce to the case when $F = \mathbb{R}$. For the second assertion, see [Mos55, Theorem 3.1]. $\square$

The second structural result implies in particular that the choice of K is not too serious:

Theorem 4.4.2 *If G is an affine algebraic group over an Archimedean local field F , then any compact subgroup of $G(F)$ is contained in a maximal compact subgroup and any two maximal compact subgroups are conjugate (even by an element of the neutral component of $G(F)$ in the Hausdorff topology).*

Proof. In view of Theorem 4.4.1 this is [Bor98a, Chapter VII, Theorem 1.2]. $\square$

Example 4.2. Let

$$K_n := \{g \in \text{GL}_n(F) : g\bar{g}^t = I_n\}.$$

Here the bar denotes complex conjugation when F is complex and is the identity automorphism when F is real. The group K_n is a maximal compact subgroup of $\text{GL}_n(F)$. It is an orthogonal group when F is real and a unitary group when F is complex. We refer to K_n as the **standard maximal compact** subgroup of $\text{GL}_n(F)$.

Let (π, V) be a representation of K . For each irreducible representation σ of K , let $V(\sigma) \leq V$ be the sum of all (finite dimensional) subrepresentations of V that are equivalent to σ :

$$V(\sigma) = \langle \varphi \in V : \langle \pi(k)\varphi : k \in K \rangle \cong \sigma \rangle. \quad (4.12)$$

Here the brackets denote the $\mathbb{C}$ -span. This is the **σ -isotypic subspace** of V . A vector in V is said to have **K -type σ** if it is an element of $V(\sigma)$. The **K -types** of V are the $\sigma \in \widehat{K}$ such that $V(\sigma) \neq 0$.

Definition 4.4.3. Consider the algebraic direct sum

$$V_{\text{fin}} := \bigoplus_{\sigma \in \widehat{K}} V(\sigma).$$

An element of V_{fin} is a **K -finite vector** and V_{fin} is the **space of K -finite vectors**.

Here, as in the previous section, $\widehat{K}$ is the set of all equivalence classes of irreducible representations of K . We will later show in Proposition 4.4.7 that, if V is a Hilbert space, then V_{fin} is dense in V , and hence V is the Hilbert space direct sum of the $V(\sigma)$.

Definition 4.4.4. A representation V of $G(F)$ is **admissible** if for each $\sigma \in \widehat{K}$, the dimension of $V(\sigma)$ is finite.

Here to define $V(\sigma)$ we are regarding V as a representation of K via restriction. The notion of admissibility isolates a subcategory of the category of representations of $G(F)$ that is closed under many natural operators in representation theory (for example parabolic induction, see §4.9). If G is reductive this category properly contains the subcategory of unitary representations [Wal88, Theorem 3.4.10]:

Theorem 4.4.5 (Harish-Chandra) *If G is a reductive group over an Archimedean field F , then irreducible unitary representations of $G(F)$ are admissible.* $\square$

Unitary representations of nonreductive groups are not in general admissible. A precise discussion of this phenomenon together with examples that are relevant to classical automorphic forms is contained in [BS98, §3.2].

Let $\mathfrak{k} = \text{Lie } K$ be the Lie algebra of the real Lie group K . To analyze admissible representations we introduce the fundamental concept of a $(\mathfrak{g}, K)$ -module:

Definition 4.4.6. A **$(\mathfrak{g}, K)$ -module** is a (complex) vector space V equipped with homomorphisms

$$\mathfrak{g} \longrightarrow \text{End}(V) \quad \text{and} \quad K \longrightarrow \text{GL}(V), \quad (4.13)$$

both denoted by π , satisfying the following:

- (a) The space V is a countable algebraic direct sum $V = \bigoplus_i V_i$ with each V_i a finite dimensional K -invariant vector space.
- (b) For $X \in \mathfrak{k}$ and $\varphi \in V$, we have that

$$\pi(X)\varphi = X\varphi = \left. \frac{d}{dt} \pi(\exp(tX))\varphi \right|_{t=0} = \lim_{t \rightarrow 0} \frac{\pi(\exp(tX))\varphi - \varphi}{t}.$$

In particular, the limit on the right exists.

- (c) For $k \in K$ and $X \in \mathfrak{g}$, we have $\pi(k)\pi(X)\pi(k^{-1})\varphi = \pi(\text{Ad}(k)X)\varphi$.

We say that the $(\mathfrak{g}, K)$ -module is **admissible** if $V(\sigma)$ is finite dimensional for all $\sigma \in \widehat{K}$.

We have not endowed V with any topology so there is no continuity requirement for the homomorphisms (4.13). Thus, to make sense of (b), we proceed as follows. By (a), any $\varphi \in V$ lies in a finite dimensional K -invariant vector subspace $W \leq V$. Finite dimensional complex vector spaces have a canonical topology. With respect to this topology π induces a continuous homomorphism

$$\pi : K \longrightarrow \mathrm{GL}(W).$$

Thus, the expression in (b) is well-defined.

A morphism of $(\mathfrak{g}, K)$ -modules is simply a vector space morphism equivariant with respect to the actions of $\mathfrak{g}$ and K , and an isomorphism is a morphism that induces an isomorphism of the underlying vector spaces. A submodule of a $(\mathfrak{g}, K)$ -module V is a vector subspace of V invariant under the actions of $\mathfrak{g}$ and K . Since we have not required V to be topological, we do not require the subspace to be closed. A $(\mathfrak{g}, K)$ -module V is **irreducible** if it admits no nonzero proper subspace invariant under the actions of $\mathfrak{g}$ and K .

The following proposition states that a Fréchet space representation of $G(F)$ yields a $(\mathfrak{g}, K)$ -module:

Proposition 4.4.7 *Let (π, V) be a Fréchet space representation of $G(F)$. Let $V_{\mathrm{fin}} \leq V$ be the space of K -finite vectors. Then $V_{\mathrm{fin}} \cap V_{\mathrm{sm}}$ is dense in V and invariant under the action of $\mathfrak{g}$. If V is an admissible Hilbert space representation, then $V_{\mathrm{fin}} \leq V_{\mathrm{sm}}$.*

Before giving the proof of Proposition 4.4.7 we state a lemma:

Lemma 4.4.8 *The following are equivalent:*

- (a) *The vector $\varphi \in V$ is K -finite.*
- (b) *The space $\langle \pi(k)\varphi \mid k \in K \rangle$ is finite dimensional.*

□

We leave the proof of Lemma 4.4.8 as an exercise (see Exercise 4.5). Assuming the lemma, we prove Proposition 4.4.7:

Proof of Proposition 4.4.7: Write

$$V_0 := V_{\mathrm{sm}} \cap V_{\mathrm{fin}}.$$

We first prove that V_0 is dense in V . Let U be a neighborhood of 1 in $G(F)$ and let $\varepsilon > 0$. Suppose that f is a nonnegative smooth function on $G(F)$ with support in KU such that

$$\int_{G(F)} f(g) d_r g = 1 \quad \text{and} \quad \int_{G(F)-U} f(g) d_r g < \varepsilon. \quad (4.14)$$

By making U and ε sufficiently small, we can make $\pi(f)\varphi$ as close as we like to φ for all φ in V by Theorem 3.3.1.

It therefore suffices to show that for arbitrary U and $\varepsilon > 0$, we can choose an f with support in KU satisfying (4.14) such that $\pi(f)\varphi$ is K -finite. To construct such an f , let $U_1 \subset G(F)$ and $W \subset K$ be neighborhoods of 1 such that $WU_1 \subset U$, and let f_1 be a nonnegative smooth function supported in U_1 such that $\int_{G(F)} f_1(g) d_r g = 1$. By the Peter-Weyl theorem (Theorem 4.3.4) there exists a matrix coefficient f_0 of a finite dimensional representation of K that is nonnegative such that $\int_K f_0(k) dk = 1$ and $\int_{K-W} f_0(k) dk < \varepsilon$. Let

$$f(g) := \int_K f_0(k) f_1(k^{-1}g) dk.$$

Since K is compact, $\delta_{G(F)}(k) = 1$ for $k \in K$. We deduce that $\int_{G(F)} f(g) d_r g = 1$. Clearly f has support contained in $KU_1 \subset KU$. Moreover, if

$$(k, g) \in K \times (G(F) - U)$$

satisfies $f_1(k^{-1}g) \neq 0$, then $k \notin W$. Indeed, otherwise we would have

$$g = k(k^{-1}g) \in WU_1 \subset U.$$

Therefore

$$\begin{aligned} \int_{G(F)-U} f(g) d_r g &= \int_{G(F)-U} \int_K f_0(k) f_1(k^{-1}g) dk d_r g \\ &= \int_{G(F)-U} \int_{K-W} f_0(k) f_1(k^{-1}g) dk d_r g \\ &\leq \int_{K-W} f_0(k) \int_{G(F)} f_1(k^{-1}g) d_r g dk \\ &= \int_{K-W} f_0(k) dk \\ &< \varepsilon. \end{aligned}$$

Thus, f satisfies (4.14). Here we have again used the fact that $\delta_{G(F)}|_K$ is trivial because K is compact.

We now show that $\pi(f)\varphi$ is K -finite. There is a finite dimensional unitary representation (ρ, V_ρ) of K and a finite set of elements $\{\xi_i, \zeta_i\} \subset V_\rho$ such that

$$f_0(k) = \sum_i (\rho(k)\xi_i, \zeta_i)$$

where $(,)$ is the pairing on the space of V_ρ . For $k_1 \in K$, one has that

$$f(k_1^{-1}g) = \int_K f_0(k) f_1(k^{-1}k_1^{-1}g) dk$$

$$\begin{aligned}
&= \sum_i \int_K (\rho(k)\xi_i, \zeta_i) f_1(k^{-1}k_1^{-1}g) dk \\
&= \sum_i \int_K (\rho(k_1^{-1})\rho(k)\xi_i, \zeta) f_1(k^{-1}g) dk \\
&= \sum_i \int_K (\rho(k)\xi_i, \rho(k_1)\zeta_i) f_1(k^{-1}g) dk.
\end{aligned}$$

Therefore, as k_1 varies over K , the linear span of the functions $g \mapsto f(k_1^{-1}g)$ is contained in the linear span of the functions

$$g \mapsto \int_K (\rho(k)\xi, \zeta) f_1(k^{-1}g) dk$$

as ξ, ζ vary over V_ρ . Hence, the space spanned by the vectors

$$\pi(k_1)\pi(f)\varphi = \int_{G(F)} f(g)\pi(k_1g)\varphi d_r g = \int_{G(F)} f(k_1^{-1}g)\pi(g)\varphi d_r g$$

as k_1 varies over K is finite dimensional; in other words $\pi(f)\varphi \in V_{\text{fin}}$. Moreover, $\pi(f)\varphi$ is smooth for any vector φ by Proposition 4.2.7. It follows that V_0 is dense in V_{sm} , which is dense in V .

Assume that V is an admissible Hilbert space representation. We assume without loss of generality that $\pi|_K$ is unitary. We now prove $V_{\text{fin}} \leq V_{\text{sm}}$. By Lemma 4.2.5 V_0 is K -invariant. Let σ be an irreducible unitary representation of K . Then $V_0(\sigma) \leq V(\sigma)$. Since V_{fin} is the algebraic direct sum of the $V(\sigma)$, it suffices to show that $V_0(\sigma) = V(\sigma)$.

Since $V(\sigma)$ is finite dimensional by admissibility, $V_0(\sigma)$ admits a well-defined orthogonal complement in $V(\sigma)$ (this is the only part of the proof where admissibility is used). If φ is in this orthogonal complement, then φ is orthogonal to all of V_0 , because it is orthogonal to $V(\tau)$ for every $\tau \neq \sigma$. Therefore, $\varphi = 0$, since V_0 is dense. This establishes that $V_0(\sigma) = V(\sigma)$ and hence $V_0 = V_{\text{fin}} \leq V_{\text{sm}}$.

Finally, returning to the case of a general Fréchet space representation, we show that V_0 is invariant under $\mathfrak{g}$. Let $\varphi \in V_0$. The space

$$\{\pi(X)\pi(k)\varphi : (X, k) \in \mathfrak{g} \times K\} \quad (4.15)$$

is finite dimensional. Since $\pi(\text{Ad}(k)X)\pi(k)\varphi = \pi(k)\pi(X)\varphi$ by part (c) of the definition of a $(\mathfrak{g}, K)$ -module, we deduce that (4.15) is equal to

$$\{\pi(k)\pi(X)\varphi : (X, k) \in \mathfrak{g} \times K\}. \quad (4.16)$$

Thus, (4.16) is finite dimensional, which implies by Lemma 4.4.8 that $\pi(X)\varphi$ is K -finite.

We can detect irreducibility of admissible Hilbert space representations using $(\mathfrak{g}, K)$ -modules [Wal88, Theorem 3.4.12]:

Theorem 4.4.9 *Assume G is a reductive group over an Archimedean local field F . An admissible Hilbert space representation (π, V) of $G(F)$ is irreducible if and only if the underlying $(\mathfrak{g}, K)$ -module is irreducible. $\square$*

Two admissible representations are **infinitesimally equivalent** if their underlying $(\mathfrak{g}, K)$ -modules are isomorphic. Certainly equivalent representations are infinitesimally equivalent, but the converse is not true in general [Bum97, Exercise 2.6.1]. However, the situation is better if we assume that both representations are unitary [Wal88, Theorem 3.4.11]:

Theorem 4.4.10 *Assume G is a reductive group over an Archimedean local field F . A unitary representation of $G(F)$ is irreducible if and only if the underlying $(\mathfrak{g}, K)$ -module is irreducible. Two infinitesimally equivalent irreducible unitary representations of $G(F)$ are unitarily equivalent. $\square$*

It is a bit inconvenient that infinitesimal equivalence is strictly weaker than equivalence. Casselman and Wallach found an elegant resolution of this difficulty. We recall some of their results after setting notation. Choose a faithful representation $\iota' : G \rightarrow \mathrm{GL}_n$ and let $\iota : G \rightarrow \mathrm{SL}_{2n}$ be the embedding defined by

$$\iota(g) := \begin{pmatrix} \iota'(g) & \\ & \iota'(g^{-1})^t \end{pmatrix}.$$

Define

$$\|g\| := \|g\|_\iota := \sup_{1 \leq i, j \leq 2n} |\iota(g)_{ij}|. \quad (4.17)$$

Definition 4.4.11. A Fréchet space representation (π, V) of $G(F)$ is **smooth** if the maps

$$\begin{aligned} G(F) &\longrightarrow V \\ g &\longmapsto \pi(g)\varphi \end{aligned}$$

lie in $C^\infty(G(F), V)$ for all $\varphi \in V$.

In particular, for any Hilbert space V , endow the space V_{sm} with the Fréchet space structure described at the end of §4.2. Then V_{sm} is smooth [Wal88, Lemma 1.6.4].

Definition 4.4.12. A smooth Fréchet space representation (π, V) is **of moderate growth** if for every continuous seminorm λ on V there is a continuous seminorm ν_λ on V and $d_\lambda \in \mathbb{R}$ such that

$$\lambda(\pi(g)\varphi) \leq \|g\|^{d_\lambda} \nu_\lambda(\varphi).$$

Here $\|g\|$ is defined as in (4.17).

A morphism of representations of moderate growth is just a morphism of (continuous) representations of $G(F)$.

The following is a special case of [Wal92, Lemma 11.5.1]:

Lemma 4.4.13 *If (π, V) is a Hilbert space representation of $G(F)$, then (π, V_{sm}) is of moderate growth.* $\square$

A key consequence of Casselman and Wallach's work is the following theorem [Cas89] [Wal92, Chapter 11]:

Theorem 4.4.14 *Let (π_1, V_1) and (π_2, V_2) be irreducible Hilbert space representations. They are infinitesimally equivalent if and only if $(\pi_1, V_{1\text{sm}})$ and $(\pi_2, V_{2\text{sm}})$ are isomorphic in the category of representations of moderate growth.* $\square$

In fact for each irreducible admissible $(\mathfrak{g}, K)$ -module (π, V) , Casselman and Wallach define a representation $(\pi, \overline{V})$ of moderate growth with the property that the space of K -finite vectors $\overline{V}_{\text{fin}}$ in $\overline{V}$ is equal to V . The representation $\overline{V}$ is referred to as the globalization of V . In the special case where $V = V'_{\text{fin}}$ is the space of smooth vectors in an irreducible admissible Hilbert space representation (π, V') , the completion $\overline{V}$ is isomorphic to V'_{sm} [Cas89] [Wal92, Chapter 11].

The following technical result is occasionally useful:

Lemma 4.4.15 *If (π, V) is an irreducible admissible Hilbert space representation of $G(F)$, then the Fréchet space V_{sm} is nuclear.*

Proof. As mentioned above, V_{sm} is the globalization of the $(\mathfrak{g}, K)$ -module V_{fin} . Hence, the result is part of [BK14, Corollary 5.6]. $\square$

Now that we have introduced the notions of admissibility and the $(\mathfrak{g}, K)$ -module of an admissible representation, we can begin to make use of them. One manifestation of the power of admissibility is that it allows us to understand the infinite dimensional space V in terms of finite dimensional subspaces. An example of this is given in §4.5.

There is a useful invariant attached to any irreducible $(\mathfrak{g}, K)$ -module called an **infinitesimal character**; its construction will be discussed in §4.6. We then give an example of how $(\mathfrak{g}, K)$ -modules can be computed and classified explicitly in the simplest nontrivial case in §4.7.

The real power of admissibility is that one can classify admissible representations. Thus, the description of the unitary dual $\widehat{G(F)}$ is reduced to the question of which admissible representations are unitary. This is still very difficult, and at the present time, the unitary dual has not been computed for general reductive groups.

The classification of admissible representation (which has several forms) is known as the **Langlands classification**. Langlands used it to establish the local Langlands correspondence for Archimedean local fields. Historically, this was the first case of the local Langlands conjecture proven (see Chapter 12 for

more on the local Langlands correspondence). We will review the Langlands classification in §4.9 after stating some prerequisites involving tempered and discrete series representations in §4.8.

4.5 Hecke algebras with K -types

As above $K \leq G(F)$ is a maximal compact subgroup. Let σ be an irreducible representation of K . For $k \in K$, let

$$e_\sigma(k) := \dim(\sigma)^{-1} \text{tr}(\sigma)(k^{-1}). \quad (4.18)$$

This is a smooth function on K . For any Fréchet space representation (π, V) of $G(F)$, the operator $\pi(e_\sigma)$ is defined:

$$\begin{aligned} \pi(e_\sigma) : V &\longrightarrow V(\sigma) \\ \varphi &\longmapsto \int_K e_\sigma(k) \pi(k) \varphi dk. \end{aligned} \quad (4.19)$$

Here we normalize the Haar measure dk so that the measure of K is 1. These operators are idempotents:

$$\pi(e_\sigma) \circ \pi(e_{\sigma'}) = \begin{cases} \text{Id} & \text{if } \sigma \cong \sigma', \\ 0 & \text{otherwise.} \end{cases} \quad (4.20)$$

One also can define the left and right convolutions of e_σ with any element $f \in C_c^\infty(G(F))$:

$$\begin{aligned} (f * e_\sigma)(g) &:= \int_K f(gk^{-1}) e_\sigma(k) dk, \\ (e_\sigma * f)(g) &:= \int_K e_\sigma(k^{-1}) f(kg) dk. \end{aligned} \quad (4.21)$$

These are again elements of $C_c^\infty(G(F))$.

For any finite subset Ξ of the unitary dual $\hat{K}$ of K , we define

$$e_\Xi := \sum_{\sigma \in \Xi} e_\sigma \quad (4.22)$$

and the algebraic direct sum

$$V(\Xi) := \bigoplus_{\sigma \in \Xi} V(\sigma) \leq V. \quad (4.23)$$

Definition 4.5.1. The **Hecke algebra** of K -type Ξ is

$$C_c^\infty(G(F), \Xi) := e_\Xi * C_c^\infty(G(F)) * e_\Xi. \quad (4.24)$$

We observe that $V(\Xi)$ is naturally a representation of the subalgebra $C_c^\infty(G(F), \Xi)$ of $C_c^\infty(G(F))$. We invite the reader to compare this with the non-Archimedean setting discussed in §5.3. The algebra $C_c^\infty(G(F), \Xi)$ is often called the spherical Hecke algebra of type Ξ , but we will avoid this terminology because it is usually reserved for the special case where Ξ is 1-element set consisting of the trivial representation.

If (π, V) is admissible, then $V(\Xi)$ is finite dimensional for all Ξ . This observation is quite useful in practice as it allows us to reduce certain questions about the infinite dimensional representation V of $G(F)$ to the family of finite dimensional representations of the algebras $C_c^\infty(G(F), \Xi)$. We give some examples in the remainder of this section.

First we state a very general version of Schur's lemma. Let V be a vector space and let $\Lambda \subseteq \text{End}(V)$ be a subset of its set of endomorphisms. Here elements of $\text{End}(V)$ are not required to be continuous because we have not endowed V with a topology. We say that Λ **acts irreducibly** if for any subspace W of V such that $\Lambda W \leq W$, either $W = 0$ or $W = V$. For a proof of the following version of Schur's lemma, see [Wal88, §0.5.2]:

Lemma 4.5.2 *Assume that V is a vector space of countable dimension and that $\Lambda \subseteq \text{End}(V)$ acts irreducibly. If $L \in \text{End}(V)$ commutes with every element of Λ , then L is a scalar multiple of the identity operator.* $\square$

We also require the following version of the Jacobson density theorem:

Theorem 4.5.3 *Let V be a vector space of countable dimension and let $A \leq \text{End}(V)$ be a $\mathbb{C}$ -subalgebra (possibly without identity) that acts irreducibly. Then for any $\varphi_1, \dots, \varphi_n, \varphi'_1, \dots, \varphi'_n \in V$ with $\varphi_1, \dots, \varphi_n$ linearly independent, there is an $a \in A$ such that $a\varphi_i = \varphi'_i$ for all i .* $\square$

We warn the reader that the **Jacobson density theorem** is usually stated for modules over rings with identity. For a version that avoids the identity assumption, we refer to [Jac64, §II.2].

Proposition 4.5.4 *An admissible Fréchet space representation (π, V) of $G(F)$ is irreducible if and only if $V(\Xi)$ is an irreducible $C_c^\infty(G(F), \Xi)$ -module for all Ξ .*

Proof. Suppose V is reducible, that is, $V = V_1 \oplus V_2$, where the V_i are some nonzero closed subrepresentations of $G(F)$. Then

$$V(\Xi) = V_1(\Xi) \oplus V_2(\Xi)$$

as $C_c^\infty(G(F), \Xi)$ -modules for all finite subsets $\Xi \subset \widehat{K}$. By Proposition 4.4.7 $V_1(\Xi)$ and $V_2(\Xi)$ are nonzero for Ξ large enough.

Conversely, suppose V is irreducible, and suppose that

$$V_1 \leq V(\Xi)$$

is a nonzero irreducible $C_c^\infty(G(F), \Xi)$ -submodule for some $\Xi \subset \widehat{K}$. We have

$$\begin{aligned} V_1 &= C_c^\infty(G(F), \Xi)V_1 \\ &= e_\Xi * C_c^\infty(G(F)) * e_\Xi V_1 \\ &= e_\Xi * C_c^\infty(G(F))V_1, \end{aligned} \tag{4.25}$$

where we have used Theorem 4.5.3 for the first equality. It is clear that $e_\Xi * C_c^\infty(G(F))V_1$ is a closed subspace of $V(\Xi)$. Hence, its inverse image under

$$e_\Xi : V \longrightarrow V(\Xi)$$

is a closed nontrivial $G(F)$ -invariant subspace of V . Since V is irreducible this inverse image is all of V and thus $e_\Xi * C_c^\infty(G(F))V_1 = V(\Xi)$. We deduce from this and (4.25) that $V_1 = V(\Xi)$. $\square$

Combining Lemma 4.5.2 and Proposition 4.5.4 we obtain the following corollary:

Corollary 4.5.5 *Assume (π, V) is an irreducible admissible Fréchet space representation of $G(F)$. If $L \in \text{End}(V_{\text{fin}})$ commutes with the action of $C_c^\infty(G(F), \Xi)$ for all finite subsets $\Xi \subseteq \widehat{K}$, then it is a scalar multiple of the identity operator.* $\square$

Here we impose no topology on V_{fin} .

The following proposition will be used in the proof of Proposition 16.2.5. It is an immediate consequence of Theorem 4.5.3 and Proposition 4.5.4:

Proposition 4.5.6 *Let (π, V) be an irreducible admissible Fréchet space representation of $G(F)$ and let $\Xi \subseteq \widehat{K}$ be finite. Let $\varphi_1, \dots, \varphi_n, \varphi'_1, \dots, \varphi'_n \in \bigoplus_{\sigma \in \Xi} V(\sigma)$ with $\varphi_1, \dots, \varphi_n$ linearly independent. Then there exists an $f \in C_c^\infty(G(F), \Xi)$ such that $\pi(f)\varphi_i = \varphi'_i$ for all i .* $\square$

4.6 Infinitesimal characters

Let $\mathfrak{g}$ be a Lie algebra over an Archimedean local field F . We let

$$\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}.$$

We use this odd notation as a reminder that, if F is complex, then $\mathfrak{g}^{\mathbb{C}} \not\cong \mathfrak{g}$, whereas $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}$ is isomorphic to $\mathfrak{g}$ in this case.

We denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$. This is a unital associative algebra. Any associative algebra becomes a Lie algebra with bracket $[X, Y] = XY - YX$, and thus we can regard $U(\mathfrak{g})$ as a Lie algebra. The universal enveloping algebra is equipped with an injective Lie algebra morphism

$$\mathfrak{g}^{\mathbb{C}} \longrightarrow U(\mathfrak{g})$$

by the Poincaré-Birkoff-Witt theorem. This morphism is universal in that given any Lie algebra homomorphism $\mathfrak{g}^{\mathbb{C}} \rightarrow A$, where A is a unital associative algebra (equipped with the Lie algebra structure mentioned above), there is a unique morphism $U(\mathfrak{g}) \rightarrow A$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g}^{\mathbb{C}} & \longrightarrow & U(\mathfrak{g}) \\ & \searrow & \downarrow \\ & & A \end{array}$$

We denote by $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$.

We now present the explicit description of $Z(\mathfrak{g})$ for reductive $\mathfrak{g}$ afforded by the Harish-Chandra isomorphism. Assume that $\mathfrak{g}$ is reductive and let $\mathfrak{t} \leq \mathfrak{g}$ be a Cartan subalgebra. Choose a reductive algebraic group G over F with Lie algebra $\mathfrak{g}$ and a maximal torus $T \leq G$ with Lie algebra $\mathfrak{t}$. We then obtain a root system

$$\Phi := \Phi((\text{Res}_{F/\mathbb{R}} G)_{\mathbb{C}}, (\text{Res}_{F/\mathbb{R}} T)_{\mathbb{C}})$$

as in §1.8. The root system depends only on $\mathfrak{g}$ and $\mathfrak{t}$. Let $\Delta \subset \Phi$ be a base and let $\Phi^+ \subset \Phi$ be the associated set of positive roots. Let $\mathfrak{n}^+ < \mathfrak{g}^{\mathbb{C}}$ be the subalgebra spanned by the roots $\alpha \in \Phi^+$ and let $\mathfrak{n}^- < \mathfrak{g}^{\mathbb{C}}$ be the subalgebra spanned by the roots $\alpha \in \Phi^- = \Phi - \Phi^+$. By the Poincaré-Birkoff-Witt theorem, we have a decomposition

$$U(\mathfrak{g}) = U(\mathfrak{t}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g}) \mathfrak{n}^+)$$

and hence a quotient map of associative algebras $Z(\mathfrak{g}) \rightarrow U(\mathfrak{t})$ [Wal88, §3.2.1]. There is a morphism $U(\mathfrak{t}) \rightarrow U(\mathfrak{t})$ induced by the universal property of universal enveloping algebras from the map

$$\begin{aligned} \mathfrak{t}^{\mathbb{C}} &\longrightarrow U(\mathfrak{t}) \\ X &\longmapsto X - \rho(X), \end{aligned}$$

where

$$\rho := \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \tag{4.26}$$

is the half-sum of positive roots. Composing the maps described above, we obtain the **Harish-Chandra homomorphism**

$$\gamma : Z(\mathfrak{g}) \longrightarrow U(\mathfrak{t}) \longrightarrow U(\mathfrak{t}).$$

This is our first time explicitly mentioning ρ , the ubiquitous half-sum of positive roots. It is incorporated into the definition to make it compatible with parabolic induction (see §4.9).

Let $W := W(\mathrm{Res}_{F/\mathbb{R}}G, \mathrm{Res}_{F/\mathbb{R}}T)(\mathbb{C})$ denote the Weyl group of $\mathrm{Res}_{F/\mathbb{R}}T$ in $\mathrm{Res}_{F/\mathbb{R}}G$; it acts on $\mathfrak{t}^{\mathbb{C}}$ and hence $U(\mathfrak{t})$. For the proof of the following theorem of Harish-Chandra, see [Wal88, Theorem 3.2.3]:

Theorem 4.6.1 *The linear map γ is an algebra homomorphism, is independent of the choice of base Δ , has image in $U(\mathfrak{t})^W$, and induces an isomorphism*

$$\gamma : Z(\mathfrak{g}) \longrightarrow U(\mathfrak{t})^W.$$

□

The theorem allows us, in particular, to explicitly describe the characters of $Z(\mathfrak{g})$ as we now explain. Suppose we are given $\lambda \in (\mathfrak{t}^{\mathbb{C}})^{\vee}$, the $\mathbb{C}$ -linear dual of $\mathfrak{t}^{\mathbb{C}}$. The linear map $\lambda : \mathfrak{t}^{\mathbb{C}} \rightarrow \mathbb{C}$ is tautologically a Lie algebra map because $\mathfrak{t}^{\mathbb{C}}$ is commutative. By the universal property of universal enveloping algebras, this extends uniquely to an algebra morphism $\lambda : U(\mathfrak{t}) \rightarrow \mathbb{C}$. We let

$$\chi_{\lambda} := \lambda|_{U(\mathfrak{t})^W} \circ \gamma : Z(\mathfrak{g}) \longrightarrow \mathbb{C}.$$

This is an algebra morphism (i.e. a character) from $Z(\mathfrak{g})$ to $\mathbb{C}$. For the proof of the following proposition, see [Wal88, Theorem 3.2.4]:

Proposition 4.6.2 *Every algebra morphism from $Z(\mathfrak{g})$ to $\mathbb{C}$ is of the form χ_{λ} for some $\lambda \in (\mathfrak{t}^{\mathbb{C}})^{\vee}$. Two λ define the same χ_{λ} if and only if they are in the same orbit under W .*

□

A $\mathfrak{g}$ -module V is said to admit an **infinitesimal character** if $Z(\mathfrak{g})$ acts via a character on V . Infinitesimal characters are a very useful invariant that is surprisingly closely tied to arithmetic (see §12.8). By Theorem 4.6.1 and Proposition 4.6.2 such a character can be identified with an element of $(\mathfrak{t}^{\mathbb{C}})^{\vee}$.

If V is a $(\mathfrak{g}, K)$ -module, then it is a module under $U(\mathfrak{g})$. It is not hard to see that the induced action of $Z(\mathfrak{g})$ on V commutes with the actions of $\mathfrak{g}$ and K [Kna86, Proposition 3.8]. Thus, using Lemma 4.5.2, we obtain the following corollary:

Corollary 4.6.3 *Let V be an irreducible admissible $(\mathfrak{g}, K)$ -module and let $L : V \rightarrow V$ be a $\mathbb{C}$ -linear map commuting with the actions of $\mathfrak{g}$ and K . Then L is a scalar multiple of identity operator. In particular, any irreducible admissible $(\mathfrak{g}, K)$ -module admits an infinitesimal character.*

□

We mention that there is software available that will compute the set of irreducible admissible $(\mathfrak{g}, K)$ -modules with a given infinitesimal character [Atl].

4.7 Classification of $(\mathfrak{g}, K)$ -modules for $\mathrm{GL}_2\mathbb{R}$

In this section, we state the classification of irreducible admissible $(\mathfrak{g}, K)$ -modules when

$$\mathfrak{g} := \mathfrak{gl}_2 \quad \text{and} \quad K := \mathrm{O}_2(\mathbb{R}).$$

Thus, K is the standard maximal compact subgroup of Example 4.2. We refer to [Bum97, §2.5] for proofs and many more details.

Define

$$H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad Z := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4.27)$$

These elements form a basis for $\mathfrak{g}$. If we set

$$\Delta = -\frac{1}{4}(H^2 + 2XY + 2YX)$$

then $Z(\mathfrak{g}) = \langle Z, \Delta \rangle$ (see Exercise 4.10). The reason for the normalization of Δ is due to its connection with the Laplace-Beltrami operator on the complex upper half plane (see [Bum97, Proposition 2.2.5]). Finally we set

$$w_\theta := \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

For every $(s, \mu) \in \mathbb{C}^2$, consider the character

$$\begin{aligned} \chi : Z(\mathfrak{g}) &\longrightarrow \mathbb{C} \\ Z &\longmapsto \mu, \\ \Delta &\longmapsto s(1-s). \end{aligned}$$

Then for each $\varepsilon \in \{0, 1\}$ one has a $(\mathfrak{g}, K)$ -module

$$(\pi, V) = (\pi_{s, \mu, \varepsilon}, V_{s, \mu, \varepsilon})$$

with infinitesimal character χ constructed as follows. The space is

$$V = \bigoplus_{\ell \equiv \varepsilon \pmod{2}} \mathbb{C} v_\ell.$$

The action is given by

- (a) $\pi(w_\theta)v_\ell = e^{i\ell\theta}v_\ell$ and $\pi\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}v_\ell = v_{-\ell}$,
- (b) $Hv_\ell = \ell v_\ell$,
- (c) $Xv_\ell = \left(s + \frac{\ell}{2}\right)v_{\ell+2}$,
- (d) $Yv_\ell = \left(s - \frac{\ell}{2}\right)v_{\ell-2}$,
- (e) $\Delta v_\ell = s(1-s)v_\ell$,
- (f) $Zv_\ell = \mu v_\ell$.

This is an admissible $(\mathfrak{g}, K)$ -module. Notice that in (a) we are regarding $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as an element of K , but in (b) the element $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is regarded as an element of $\mathfrak{g}$. The representation $\pi_{s,\mu,\varepsilon}$ is irreducible unless $s = \frac{k}{2}$ where k is an integer congruent to ε modulo 2. If $k \geq 1$ and $k \equiv \varepsilon \pmod{2}$, then $\pi_{\frac{k}{2},\mu,\varepsilon}$ has a unique irreducible infinite dimensional subrepresentation denoted by $\pi_{k,\mu}$. The set of K -types of $\pi_{k,\mu}$ consists of the v_ℓ with $|\ell| \geq k$. The quotient of $\pi_{\frac{k}{2},\mu,\varepsilon}$ by $\pi_{k,\mu}$ is a finite dimensional representation, and all irreducible finite dimensional representations are obtained in this manner. In the special case $k = 1$ the finite dimensional quotient is trivial and

$$\pi_{\frac{1}{2},\mu,\varepsilon} = \pi_{1,\mu}$$

is irreducible. If $k > 1$, then $\pi_{k,\mu}$ is a proper subrepresentation of $\pi_{\frac{k}{2},\mu,\varepsilon}$.

If $\pi_{s,\mu,\varepsilon}$ is irreducible it is known as an irreducible **principal series** representation. We warn the reader that often authors take the more restrictive convention that principal series representations are the representations $\pi_{s,\mu,\varepsilon}$ with $\operatorname{Re}(s) = \frac{1}{2}$. In this case the representation is the underlying $(\mathfrak{g}, K)$ -module of a unitary representation (see Example 4.3 below). The representations $\pi_{k,\mu}$ for $k > 1$ are known as **discrete series**. The representation $\pi_{1,\mu}$ is known as a **limit of discrete series**. This terminology should make more sense after the next two sections. It is clear (upon considering K -types and infinitesimal characters, for example) that, if $\pi_{k_1,\mu_1} \cong \pi_{k_2,\mu_2}$, then $k_1 = k_2$ and $\mu_1 = \mu_2$, and that an irreducible representation cannot be both in the principal series and in the discrete series. Moreover, it is easy to see that, if

$$\pi_{s_1,\mu_1,\varepsilon_1} \cong \pi_{s_2,\mu_2,\varepsilon_2}$$

then $\mu_1 = \mu_2$, $\varepsilon_1 = \varepsilon_2$, and either $s_1 = s_2$ or $s_1 = 1 - s_2$.

The following is the classification result mentioned in the title of this section:

Theorem 4.7.1 *Every infinite dimensional irreducible admissible $(\mathfrak{g}, K)$ -module is isomorphic to either $\pi_{s,\mu,\varepsilon}$ for some s, μ, ε such that $s \neq \frac{k}{2}$ for some integer $k \equiv \varepsilon \pmod{2}$ or isomorphic to $\pi_{k,\mu}$ for some integer $k \geq 1$ and μ . $\square$*

4.8 Matrix coefficients

Let G be a reductive group over a local field F . For the moment, we do not assume that F is Archimedean. Let (π, V) be a Hilbert space representation of $G(F)$. Recall the notion of a matrix coefficient of π from Definition 4.3.2 above. In this section we explain how matrix coefficients can be used to isolate important classes of representations of $G(F)$.

We have already seen in §2.6 that the center of G often causes analytic problems for somewhat trivial reasons. One often circumvents this difficulty by defining classes of representations based on their restriction to appropriate subgroups. In analogy with (2.19), we let

$$G(F)^1 := \bigcap_{\chi \in X^*(G)} \ker(|\cdot| \circ \chi : G(F) \longrightarrow \mathbb{R}_{>0}). \quad (4.28)$$

Definition 4.8.1 An irreducible representation (π, V) of $G(F)$ is **square integrable** (resp. **tempered**) if its matrix coefficients lie in $L^2(G(F))$ (resp. $L^{2+\varepsilon}(G(F))$ for any $\varepsilon > 0$). It is **essentially square integrable** (resp. **essentially tempered**) if the matrix coefficients of $\pi|_{G(F)^1}$ lie in $L^2(G(F)^1)$ (resp. $L^{2+\varepsilon}(G(F)^1)$ for any $\varepsilon > 0$).

(Essentially) square integrable representations are also known as (essentially) **discrete series representations**. We concentrate on square integrable and tempered representations in the following discussion, but similar results hold for essentially square integrable and essentially tempered representations.

Discrete series representations can be realized as subrepresentations of $L^2(G(F))$ and hence are always unitarizable (see [Dix77, Chapter 14] for details). When F is Archimedean, a theorem of Harish-Chandra (Theorem 12.3.1) characterizes when $G(F)$ admits essentially square integrable representations. Consider the non-Archimedean case. In this setting, a **supercuspidal representation** is a representation whose matrix coefficients are compactly supported modulo the center (see Definition 8.1.1). Such representations are clearly essentially square integrable. When the characteristic of F is zero, then supercuspidal representations of $G(F)$ exist for every reductive group G [BP16b].

Irreducible admissible tempered representations are infinitesimally equivalent to unitary representations in the Archimedean case [Kna86, Theorem 8.53] and equivalent to unitary representations in the non-Archimedean case [Sil79, Corollary 4.5.13]. We note that, in many references including [Sil79, §4.5], one finds a more technical definition of temperedness. Once one knows that tempered representations are unitary, one can prove that the more technical definition (which we have not given and will not give) is equivalent to the easily stated one above using the arguments of [CHH88].

Now that we have defined these classes of representations, we ought to comment briefly on their significance. We concentrate on the notion of temperedness. Let π be an irreducible admissible representation of $G(F)$ and let $\mathcal{B}_\pi$ be an orthonormal basis of the space V of π with respect to the pairing $\langle \cdot, \cdot \rangle$ giving it its Hilbert space structure. Then for $f \in C_c^\infty(G(F))$, the sum

$$\Theta_\pi(f) := \sum_{\varphi \in \mathcal{B}_\pi} \langle \pi(f)\varphi, \varphi \rangle$$

converges absolutely (this is obvious in the non-Archimedean case, see §8.5, and in the Archimedean case it is proven in [Kna86, Theorem 10.2]). Technically speaking we have not defined admissibility in the non-Archimedean case; it is defined in §5.3. The linear functional

$$\begin{aligned}\Theta_\pi : C_c^\infty(G(F)) &\longrightarrow \mathbb{C} \\ f &\longmapsto \Theta_\pi(f)\end{aligned}$$

is known as the **character** of π .

In the context of Fourier theory on $\mathbb{R}$, tempered distributions are precisely those which admit a Fourier transform; in other words, they are analytically fairly well-behaved. In the Archimedean case, tempered representations are precisely those whose characters define a tempered distribution on a certain Schwartz space containing $C_c^\infty(G(F))$ (see [Kna86, Theorem 12.23] and [Sil79, §4.5]). There is also a version of the Plancherel theorem for $G(F)$ relating spaces of functions on $G(F)$ and tempered representations of $G(F)$. See [Wal92, Theorem 13.4.1] in the Archimedean case and [Wal03] in the non-Archimedean case.

Finally the condition of temperedness is also a representation theoretic encapsulation of how well-behaved automorphic L -functions are. This will be discussed in the context of the Ramanujan conjecture in Conjecture 10.7.4.

4.9 The Langlands classification

The Langlands classification gives a description of representations of reductive groups over Archimedean local fields in terms of tempered representations of Levi subgroups. We state it in this section. There is a refinement where the tempered representations are replaced by discrete series representations and limits of discrete series representations. We will touch on this in §10.6 in the case $G = \mathrm{GL}_n$. For the general case, we refer to [Kna86, §XIV.17].

Let G be a reductive group over an Archimedean local field F and let $P \leq G$ be a parabolic subgroup. It admits a Levi decomposition as in Theorem 1.5.8:

$$P = MN,$$

where M is a Levi subgroup of P and N is its unipotent radical. We choose a maximal compact subgroup $K \leq G(F)$ such that the Iwasawa decomposition $G(F) = P(F)K$ holds (see Appendix B).

We let A_M be the identity component in the real topology of the largest $\mathbb{R}$ -split torus in the center of $\mathrm{Res}_{F/\mathbb{R}} M$ and we let

$$M(F)^1 := \bigcap_{\chi \in X^*(M)} \ker(|\cdot| \circ \chi : M(F) \longrightarrow \mathbb{R}_{>0}),$$

where $X^*(M)$ is the group of characters of M as in §1.7. Then $A_M M(F)^1 = M(F)$ and the product is direct. The **Langlands decomposition** of $P(F)$ is

$$P(F) = A_M M(F)^1 N(F).$$

Let $\mathfrak{a}_M := \text{Hom}(X^*(M), \mathbb{R})$. Then $\mathfrak{a}_M$ can be identified with $\text{Lie } A_M$, explaining the notation. We have a map

$$H_M : M(F) \longrightarrow \mathfrak{a}_M \quad (4.29)$$

defined by

$$\langle H_M(m), \lambda \rangle = \log |\lambda(m)|$$

for $\lambda \in X^*(M)$ and $M(F)^1 = \ker H_M$. If we identify M with the maximal reductive quotient of P , then pullback along the quotient map $P \rightarrow M$ induces an isomorphism

$$X^*(M) \longrightarrow X^*(P).$$

This allows us to extend H_M to a map $H_P : P(F) \rightarrow \mathfrak{a}_M$.

For each $\lambda \in \mathfrak{a}_{M\mathbb{C}}^* := \text{Hom}(\mathfrak{a}_M, \mathbb{C})$, we obtain a quasi-character

$$\begin{aligned} P(F) &\longrightarrow \mathbb{C}^\times \\ p &\longmapsto e^{\langle H_P(p), \lambda \rangle}. \end{aligned} \quad (4.30)$$

Identify $\mathfrak{a}_M^* = X^*(M) \otimes_{\mathbb{Z}} \mathbb{R}$. We have projections

$$\text{Re} : \mathfrak{a}_{M\mathbb{C}}^* \longrightarrow \mathfrak{a}_M^* \quad \text{and} \quad \text{Im} : \mathfrak{a}_{M\mathbb{C}}^* \longrightarrow i\mathfrak{a}_M^* \quad (4.31)$$

defined in the obvious manner. We let ρ denote the half-sum of positive roots defined by the parabolic subgroup P (see §1.9 and (4.26)). For each Hilbert space representation (σ, V) of $M(F)$ we obtain a representation (σ, λ) of $P(F)$ by extending σ trivially to $P(F)$ and twisting by the character $e^{\langle H_P(\cdot), \lambda \rangle}$. We can then form the (normalized) **induced representation**

$$I(\sigma, \lambda)$$

of $G(F)$. A dense subspace of the space of $I(\sigma, \lambda)$ consists of continuous functions $\varphi : G(F) \rightarrow V$ such that

$$\varphi(nmg) = e^{\langle H_P(m), \lambda + \rho \rangle} \sigma(m) \varphi(g) \quad (4.32)$$

for $(n, m, g) \in N(F) \times M(F) \times G(F)$. This space is equipped with an inner product

$$\langle \varphi_1, \varphi_2 \rangle = \int_K \varphi_1(k) \overline{\varphi_2(k)} dk.$$

The whole space of $I(\sigma, \lambda)$ is the completion with respect to this inner product. Finally the action of $G(F)$ is given by

$$I(\sigma, \lambda)(g)\varphi(x) := \varphi(xg).$$

The ρ is incorporated so that, if $\sigma \otimes \langle H_P(\cdot), \lambda \rangle$ is unitary, then so is $I(\sigma, \lambda)$ [Kna86, §VII.1]. This also explains the parenthetic “normalized” mentioned above.

The following is known as the **subquotient theorem** (see [Kna86, Theorem 7.24]):

Theorem 4.9.1 *If σ is unitary and tempered and $\operatorname{Re}(\lambda)$ lies in the positive Weyl chamber, then $I(\sigma, \lambda)$ admits a unique irreducible quotient $J(\sigma, \lambda)$. $\square$*

The quotient $J(\sigma, \lambda)$ is known as the **Langlands quotient** and σ, λ are known as **Langlands data**.

Fix a minimal parabolic subgroup $P_0 \leq G$. This allows us to speak about standard parabolic subgroups. We can now give the Langlands classification [Kna86, Theorem 8.54]:

Theorem 4.9.2 *Every irreducible admissible representation of $G(F)$ is isomorphic to some $J(\sigma, \lambda)$. Moreover, if we insist that the parabolic subgroup defining $J(\sigma, \lambda)$ is standard, fix a Levi decomposition $P = MN$ of each standard parabolic subgroup, assume σ is trivial on A_M , and stipulate that $\operatorname{Re}(\lambda)$ is in the positive Weyl chamber, then every irreducible admissible representation of $G(F)$ is isomorphic to a $J(\sigma, \lambda)$ that is unique up to replacing σ by another representation of $M(F)$ equivalent to σ . $\square$*

We observe that, in both Theorem 4.9.1 and Theorem 4.9.2, the assumption that $\operatorname{Re}(\lambda)$ is in the positive Weyl chamber is automatically satisfied when $P = G$. There is also a version of the Langlands classification in the non-Archimedean setting. For this we refer to §8.4.

Example 4.3 Let $P = B = TN$ be the Borel subgroup of upper triangular matrices in GL_2 . Here T is the torus of diagonal matrices and $N \leq B$ is the unipotent radical. Then

$$T(\mathbb{R})^1 := \left\{ \begin{pmatrix} (-1)^{\delta_1} & \\ & (-1)^{\delta_2} \end{pmatrix} : \delta_1, \delta_2 \in \{0, 1\} \right\}.$$

Thus, the irreducible representations of $T(\mathbb{R})$ are the characters given by

$$\sigma_{m_1, m_2} \begin{pmatrix} a & \\ & b \end{pmatrix} := a^{m_1} b^{m_2} \quad (4.33)$$

with $m_1, m_2 \in \{0, 1\}$. These representations are obviously unitary and tempered. The map H_T is given by

$$H_T : A_T \xrightarrow{\sim} \mathfrak{a}_T$$

$$\begin{pmatrix} a & \\ & b \end{pmatrix} \longmapsto \begin{pmatrix} \log a & \\ & \log b \end{pmatrix}.$$

Identify $\mathfrak{a}_{T\mathbb{C}}^*$ with $\mathbb{C}^2$ via

$$(\lambda_1, \lambda_2) \begin{pmatrix} c \\ d \end{pmatrix} = \lambda_1 c + \lambda_2 d.$$

For $(s, m) \in \mathbb{C} \times \{0, 1\}$ let

$$\chi_{s,m}(a) := |a|^s \left(\frac{a}{|a|} \right)^m.$$

Thus, an element of $I(\sigma_{m_1, m_2}, (\lambda_1, \lambda_2))$ satisfies

$$\varphi(n \begin{pmatrix} a & \\ & b \end{pmatrix} g) = \chi_{\lambda_1 + \frac{1}{2}, m_1}(a) \chi_{\lambda_2 - \frac{1}{2}, m_2}(b) \varphi(g) \quad (4.34)$$

for $(n, a, b, g) \in N(F) \times F^\times \times F^\times \times \mathrm{GL}_2(F)$. Assume that $\mathrm{Re}(\lambda)$ lies in the positive Weyl chamber, which is to say that

$$\mathrm{Re}(\lambda_1) \geq \mathrm{Re}(\lambda_2).$$

Assume that $\lambda_1 - \lambda_2 + 1$ is not an integer $k \geq 2$ of the same parity as $m_1 - m_2 \pmod{2}$. Then $I(\sigma_{m_1, m_2}, (\lambda_1, \lambda_2)) = J(\sigma_{m_1, m_2}, (\lambda_1, \lambda_2))$ is irreducible and its underlying $(\mathfrak{gl}_2, \mathrm{O}_2(\mathbb{R}))$ -module is

$$\pi_{\frac{1}{2}(\lambda_1 - \lambda_2 + 1), \lambda_1 + \lambda_2, \varepsilon}$$

in the notation of §4.7. Here $m_1 + m_2 \equiv \varepsilon \pmod{2}$. If $\lambda_1 - \lambda_2 + 1 = k \in \mathbb{Z}_{\geq 2}$ is an integer of the same parity as $m_1 + m_2 \pmod{2}$, then the underlying $(\mathfrak{gl}_2, \mathrm{O}_2(\mathbb{R}))$ -module of $J(\sigma_{m_1, m_2}, (\lambda_1, \lambda_2))$ is $\pi_{k, -\lambda_1 - \lambda_2}$ in the notation of §4.7. Here the parameter is $-\lambda_1 - \lambda_2$ instead of $\lambda_1 + \lambda_2$ because the Langlands quotient is a quotient, not a subrepresentation. These statements may be deduced from [Bum97, §2.5].

Exercises

4.1. Let $G = \mathbb{G}_m$ and consider the action of $G(\mathbb{R}) = \mathbb{R}^\times$ on $C^\infty(\mathbb{R}^\times)$ defined in (4.2). Prove that, if $X = 1 \in \mathbb{R} = \mathfrak{g}$, then $Xf(t) := \tilde{X}f(t) = t(\frac{d}{dt}f)(t)$.

4.2. Let T be a torus over $\mathbb{R}$ and let $K \leq T(\mathbb{R})$ be a maximal compact subgroup. Show that there exists an isomorphism between the $\mathbb{C}$ -vector space of K -finite functions in $C_c^\infty(T(\mathbb{R}))$ and $C_c^\infty(A_T) \otimes_{\mathbb{C}} \mathbb{C}(\hat{K})$ where A_T is defined as in (2.20). Here by $\mathbb{C}(\hat{K})$ we mean the free vector space on the isomorphism classes of irreducible unitary representations of K .

4.3. Let K be a compact Lie group that admits a faithful representation $K \rightarrow \mathrm{GL}_n(\mathbb{C})$. We can then view K as a subset of $\mathfrak{gl}_n(\mathbb{C})$ which in turn can be viewed as a real vector space. Call a function on K a polynomial if it is the restriction to K of a polynomial on this real vector space. Show that every polynomial function on K is the matrix coefficient of a finite dimensional representation of K , and deduce that the matrix coefficients of

finite dimensional unitary representations of K are dense in $C(K)$ and $L^p(K)$ for all $1 \leq p \leq \infty$ (use the Stone-Weierstrass theorem). Deduce the Peter-Weyl theorem (part (c) of Theorem 4.3.4).

4.4. Let (π_1, V_1) and (π_2, V_2) be two irreducible unitary representations of a compact (Hausdorff) group K . Prove that, if V_1 and V_2 are equivalent, then they are unitarily equivalent.

4.5. Prove Lemma 4.4.8.

4.6. Let (π, V) be an irreducible unitary representation of $G(F)$ where G is a reductive group over an Archimedean local field F and V is a Hilbert space. Show that there is a quasi-character $\omega_\pi : Z_G(F) \rightarrow \mathbb{C}^\times$ such that $\pi(z)$ acts via $\omega_\pi(z)$ on V for all $z \in Z_G(F)$. Here, as usual, Z_G is the center of G . The quasi-character ω_π is called the **central quasi-character** of π . Show that, if $\chi : G(F) \rightarrow \mathbb{C}^\times$ is a quasi-character, then the central quasi-character of $\pi \otimes \chi$ is $\chi|_{Z_G(F)}\omega_\pi$.

4.7. Give an example of a reductive group G over $\mathbb{R}$ and an irreducible admissible representation of $G(\mathbb{R})$ that is not unitary.

4.8. Let $\mathfrak{g}$ be a semisimple Lie algebra over an Archimedean local field F and let $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Let B be the Killing form on $\mathfrak{g}$ and denote also by B its extension to $\mathfrak{g}^{\mathbb{C}}$. It is nondegenerate. Let $\{X_i\}$ denote a basis for $\mathfrak{g}^{\mathbb{C}}$ and denote by X^i the dual basis with respect to B . The element

$$\Delta := \sum_i X_i X^i \in U(\mathfrak{g})$$

is known as the **Casimir element**. Prove that Δ is independent of the choice of basis and that $\Delta \in Z(\mathfrak{g})$.

4.9. Prove that the image of e_σ is contained in $V(\sigma)$ as asserted in (4.19).

4.10. Prove that $Z(\mathfrak{gl}_2) = \langle Z, \Delta \rangle$ in the notation of §4.7.

4.11. Prove that the space $V_{s,\mu,\varepsilon}$ of §4.7 is indeed a $(\mathfrak{g}, K)$ -module with the given action.

4.12. In the notation of §4.7, prove that if $\pi_{k_1,\mu_1} \cong \pi_{k_2,\mu_2}$, then $k_1 = k_2$ and $\mu_1 = \mu_2$, and that, if

$$\pi_{s_1,\mu_1,\varepsilon_1} \cong \pi_{s_2,\mu_2,\varepsilon_2}$$

then $\mu_1 = \mu_2$, $\varepsilon_1 = \varepsilon_2$, and either $s_1 = s_2$ or $s_1 = 1 - s_2$.

4.13. Prove that all finite dimensional $(\mathfrak{gl}_2, \mathrm{O}_2(\mathbb{R}))$ -modules are quotients (resp. submodules) of $V_{s,\mu,\varepsilon}$ for some s, μ, ε , and identify the $V_{s,\mu,\varepsilon}$ of which they are quotients (resp. submodules).

4.14. Let G_1 and G_2 be reductive groups over an Archimedean local field F with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. For each i , let $K_i \leq G_i(F)$ be a maximal compact subgroup. Suppose that for each i we are given irreducible admissible $(\mathfrak{g}_i, K_i)$ -modules (π_i, V_i) . Prove that the algebraic tensor product

$$(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$$

is an irreducible admissible $(\mathfrak{g}_1 \times \mathfrak{g}_2, K_1 \times K_2)$ -module. Prove that any irreducible admissible $(\mathfrak{g}_1 \times \mathfrak{g}_2, K_1 \times K_2)$ -module is isomorphic to

$$(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$$

for some irreducible admissible $(\mathfrak{g}_i, K_i)$ -modules (π_i, V_i) and that the (π_i, V_i) are uniquely determined up to isomorphism by $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$.

Chapter 5

Representations of Totally Disconnected Groups



I will tell you a false proof. But
like every fairy tale, it has a
kernel of truth.

D. Kazhdan

In this chapter, our goal is to develop enough of the representation theory of locally compact totally disconnected groups (or td-groups for short) to state the general definition of an automorphic representation in the next chapter. At the end of the chapter, we prove Flath's theorem. This fundamental result implies that automorphic representations of a global field can be factored, in a suitable sense, along the places of the field.

5.1 Totally disconnected groups

We start with the following definition, following [Car79]:

Definition 5.1.1. A Hausdorff topological group G is of **td-type** or simply a **td-group** if every neighborhood of the identity contains a compact open subgroup.

Here the letters **td** stand for totally disconnected. A td-group is indeed totally disconnected, which explains the terminology (see Exercise 5.1). A td-group is locally compact (see Exercise 5.1), so the theory of §3.1 is applicable.

Our basic examples of td-groups are given by the next lemma. It follows readily from the definition of the topology on the points of an affine algebraic group in §2.2:

Lemma 5.1.2 *Let G be an affine algebraic group over a non-Archimedean local field F . Then $G(F)$ is of td-type. Similarly, if F is a global field and S is*

a finite set of places of F (including the Archimedean places if F is a number field), then $G(\mathbb{A}_F^S)$ is of *td-type*. $\square$

Thus the topology on a td-group is very different from that of a connected Lie group. In fact, in stark contrast to the defining property of td-group, a connected Lie group has no small subgroups in a sense made precise in the following lemma:

Lemma 5.1.3 *Let G be a connected Lie group. Then there is a neighborhood U of the identity 1 so that for all $g \in U - \{1\}$ there is an integer N depending on g such that $g^N \notin U$.*

Proof. It is a standard result that there are neighborhoods V_0 of 0 in $\text{Lie } G$ and U_1 of 1 in G such that the exponential map $\exp : V_0 \rightarrow U_1$ is a diffeomorphism. Let $U = \exp(\frac{1}{2}V_0)$. Then if $g = \exp(\frac{1}{2}v) \in U - \{1\}$ for some $v \in V_0$ one has

$$g^n = \overbrace{\exp(\frac{1}{2}v) \cdots \exp(\frac{1}{2}v)}^{n \text{ times}}.$$

Choosing n large enough that $\frac{n}{2}v \notin V_0$ we deduce the result. $\square$

Our aim in this chapter is to use the special topological properties of td-groups to refine our understanding of their representation theory. We deal with the basic theory in this chapter, including the local Hecke algebra (see §5.2), the definition of admissible representations in this context (see §5.3), and Flath's theorem on decomposition of admissible representations (Theorem 5.7.1). The reader will notice that in this section we essentially make no use of the fact that the groups in question arise as the points of reductive groups in non-Archimedean local fields. The (deeper) part of the theory that requires this structure is relegated to Chapter 8.

Unless otherwise specified, for the remainder of this chapter, we let G be of td-type.

5.2 Smooth functions on td-groups

We begin our treatment of the representation theory of td-groups by defining what we mean by a smooth function in this context:

Definition 5.2.1. Let V be a complex vector space. A function $f : G \rightarrow V$ is **smooth** if it is locally constant. We denote by $C^\infty(G, V)$ the space of smooth functions $f : G \rightarrow V$ and by $C_c^\infty(G, V) \leq C^\infty(G, V)$ the subspace of compactly supported smooth functions.

When $V = \mathbb{C}$ we omit it from notation, writing, e.g. $C_c^\infty(G) := C_c^\infty(G, \mathbb{C})$. A function $f : G \rightarrow \mathbb{C}$ is locally constant if and only if it is continuous with respect to the natural topology on G and the usual topology (or even the discrete topology) on $\mathbb{C}$.

It follows from Lemma 5.2.2 and (5.1) that the complex vector subspace $C_c^\infty(G) \leq C_c(G)$ is preserved under convolution, defined as in §3.4. Thus $C_c^\infty(G)$ is an algebra under convolution. It is known as the **Hecke algebra** of G . If $K \leq G$ is a compact open subgroup, then we let

$$C_c^\infty(G // K) \leq C_c^\infty(G)$$

denote the subalgebra of functions that are right and left K -invariant.

Lemma 5.2.2 *Any element $f \in C_c^\infty(G)$ is in $C_c^\infty(G // K)$ for some compact open subgroup K . If $f \in C_c^\infty(G // K)$, then f is a finite $\mathbb{C}$ -linear combination of elements of the form*

$$\mathbb{1}_{KgK}$$

for $g \in G$. Here $\mathbb{1}_X$ denotes the characteristic function of a set X .

Proof. By local constancy of f , for every g in the support $\text{supp}(f)$ of f , we can choose a compact open subgroup $K(g) \leq G$ so that f is constant on $gK(g)$. Then

$$\{gK(g) : g \in \text{supp}(f)\}$$

is an open cover of the compact set $\text{supp}(f)$. Hence it admits a finite subcover

$$\{g_iK(g_i) : 1 \leq i \leq n\}.$$

Let

$$K_r := \bigcap_{i=1}^n K(g_i);$$

it is a finite intersection of compact open subgroups so it is itself a compact open subgroup. We see that f is right K_r -invariant. Similarly we can find a compact open subgroup $K_\ell \leq G$ such that f is left K_ℓ -invariant, and letting $K := K_\ell \cap K_r$ we see that $f \in C_c^\infty(G // K)$.

Assume $f \in C_c^\infty(G // K)$. Then

$$\{KgK : g \in \text{supp}(f)\}$$

is an open cover of $\text{supp}(f)$, which therefore admits a finite subcover, say $\{Kg_iK : 1 \leq i \leq n\}$. Since f is constant on Kg_iK for each $g \in G$, we deduce that

$$f = \sum_{i=1}^n f(g_i) \mathbb{1}_{Kg_iK}.$$

□

Note that $C_c^\infty(G // K) \leq C_c^\infty(G // K')$ if $K \geq K'$. Thus, in view of the lemma, to explicitly describe the convolution operation on $C_c^\infty(G)$ it suffices to describe

$$\mathbb{1}_{K\alpha K} * \mathbb{1}_{K\beta K}$$

for a fixed compact open subgroup $K \leq G$ and $\alpha, \beta \in G$. We can write

$$K\alpha K = \coprod_i K\alpha_i \quad \text{and} \quad K\beta K = \coprod_j \beta_j K,$$

where both disjoint unions are finite and $\alpha_i, \beta_j \in G$. We then have

$$\mathbb{1}_{K\alpha K} * \mathbb{1}_{K\beta K} = \sum_{i,j} \text{meas}_{d_r g}(\beta_j K) \mathbb{1}_{K\alpha_i \beta_j K}, \quad (5.1)$$

where $d_r g$ is the right Haar measure on G used to define the convolution.

Example 5.1. All compact open subgroups of $\text{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ are of the form

$$K_S \prod_{p \notin S} \text{GL}_n(\mathbb{Z}_p)$$

for a finite set S of finite primes and a compact open subgroup K_S of $\text{GL}_n(\mathbb{Q}_S)$. The subgroup $\text{GL}_n(\widehat{\mathbb{Z}}) = \prod_p \text{GL}_n(\mathbb{Z}_p) \leq \text{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$ is a maximal compact open subgroup, and all maximal compact open subgroups are conjugate to this maximal compact open subgroup [Ser06, Chapter IV, Appendix 1]. Examples of nonmaximal compact open subgroups are given by the kernel of the reduction map

$$\text{GL}_n(\widehat{\mathbb{Z}}) \longrightarrow \text{GL}_n(\mathbb{Z}/m)$$

for integers m .

Let G be an affine algebraic group over a global field F and assume that we have chosen a model for G over $\mathcal{O}_F$ which we denote by the same letter by abuse of notation (see §2.4 for our conventions regarding models). If $g \in G(\mathbb{A}_F^{\infty})$, then $g = g_S g^S$ where $g_S \in G(F_S^{\infty})$ and $g^S \in G(\widehat{\mathcal{O}}_F^S)$ for some finite set S of places of F including the infinite places. If $K^{\infty} \leq G(\mathbb{A}_F^{\infty})$ is a compact open subgroup, then upon enlarging S we can assume that $K^S = G(\widehat{\mathcal{O}}_F^S)$. For such a choice of S we have

$$\mathbb{1}_{K^{\infty} g K^{\infty}} = \mathbb{1}_{K_S g_S K_S} \otimes \mathbb{1}_{K^S}.$$

As we will explain in §5.7, this reduces the study of $C_c^{\infty}(G(\mathbb{A}_F^{\infty}))$ to the study of the local Hecke algebras

$$C_c^{\infty}(G(F_v))$$

as v varies over non-Archimedean places of F .

5.3 Smooth and admissible representations

For the following definition, we temporarily lift our running assumption that representations are continuous:

Definition 5.3.1. A representation (π, V) of G on a complex vector space V is **smooth** if the stabilizer of any vector in V is open in G .

Equivalently, (π, V) is smooth if and only if

$$V = \bigcup_{K \leq G} V^K,$$

where the superscript denotes the subspace of fixed vectors and where the union is over all compact open subgroups $K \leq G$.

In Definition 5.3.1, we have not imposed any continuity assumption or placed any topology on V . However, we can rephrase the condition of smoothness topologically (see Exercise 5.3):

Lemma 5.3.2 *A representation (π, V) of G on a complex vector space V is smooth if and only if the action map*

$$G \times V \longrightarrow V$$

is continuous when V is given the discrete topology. □

With this in mind, we again enforce our running assumption that representations are continuous.

We note that the smooth representations of a given td-group form a category where morphisms are simply G -equivariant $\mathbb{C}$ -linear maps. The category is even abelian.

It is sometimes useful to rephrase the condition of smoothness in terms of the associated $C_c^\infty(G)$ -module. For this, we recall that a module M for an algebra A is **nondegenerate** if every element of M can be written as a finite sum

$$a_1 m_1 + \cdots + a_n m_n$$

for some $(a_i, m_i) \in A \times M$. Of course, this is trivially true if A contains an identity, but we will be applying this concept when $A = C_c^\infty(G)$, which has an identity if and only if G is compact. However, $C_c^\infty(G)$ always has approximate identities in the following sense. For each compact open subgroup $K \leq G$, let

$$e_K := \frac{1}{\text{meas}_{d_r, g}(K)} \mathbb{1}_K, \tag{5.2}$$

where $\mathbb{1}_K$ is the characteristic function of K . Then e_K is the identity element of the algebra

$$C_c^\infty(G // K) = e_K * C_c^\infty(G) * e_K.$$

If (π, V) is a smooth representation of G , then it is naturally a nondegenerate $C_c^\infty(G)$ -module and $C_c^\infty(G // K)$ preserves

$$V^K = e_K V.$$

This observation is in fact the key to the proof of the following lemma, which we leave as an exercise (see Exercise 5.11):

Lemma 5.3.3 *There is an equivalence of categories between nondegenerate $C_c^\infty(G)$ -modules and smooth representations of G .* $\square$

Using this equivalence, we prove the following irreducibility criterion:

Proposition 5.3.4 *A smooth representation (π, V) of G is irreducible if and only if V^K is an irreducible $C_c^\infty(G // K)$ -module for all compact open subgroups $K \leq G$.*

Proof. Suppose V is not irreducible. Let $V' < V$ be a proper subrepresentation. Choose $\varphi \in V - V'$. Since V is smooth, $\varphi \in V^K$ for some K and $\varphi \notin V'^K$. Hence V^K is not irreducible as a $C_c^\infty(G // K)$ -module. Conversely, suppose V is irreducible, and suppose that

$$V_1 \leq V^K$$

is a $C_c^\infty(G // K)$ -submodule for some compact open subgroup $K \leq G$. If $V_1 \neq 0$ then by irreducibility of V one has that

$$V_1 = C_c^\infty(G // K)V_1 = e_K * C_c^\infty(G) * e_K V_1 = e_K * C_c^\infty(G)V_1 = e_K V = V^K$$

which proves the claim. $\square$

We now come to the most important definition of the chapter:

Definition 5.3.5. A representation (π, V) of G is **admissible** if it is smooth and V^K is finite dimensional for every compact open subgroup $K \leq G$. A $C_c^\infty(G)$ -module V is **admissible** if it is nondegenerate and $e_K V$ is finite dimensional for all compact open subgroups $K \leq G$.

It is immediate that a representation (π, V) of G is admissible if and only if its associated $C_c^\infty(G)$ -module is admissible.

Jacquet and Langlands introduced this definition in their classic work [JL70], though some indication of the definition in the real case appeared in work of Harish-Chandra [HC53]. The importance of the definition is that it isolates exactly the correct category of representations to study, and at the same time eliminates extraneous topological assumptions such as the presence of a Hilbert or pre-Hilbert space structure on V . One indication that this is the correct category is the following theorem, which is proven as part of Theorem 8.3.5:

Theorem 5.3.6 *Let F be a non-Archimedean local field. Assume that G is the F -points of a reductive group over F . Then an irreducible smooth representation of G is admissible.* $\square$

Another indication that admissible representations are the correct category to study is the fact that unitary representations of a group G satisfying the assumptions of Theorem 5.3.6 give rise to admissible representations. To explain this, we require some preparation. We keep our running assumption that G is a td-group (we do not yet assume that G is the F -points of a reductive group over a non-Archimedean local field F). Recall that in the Archimedean setting of §4.4 we passed from Hilbert space representations to their subspace of K -finite vectors (K a maximal compact subgroup of the relevant group) in order to define a notion of admissibility for them. We require a similar process here, but it is slightly simpler. Given a Hilbert space representation V of G , we let

$$V_{\text{sm}} := \bigcup_{K \leq G} V^K \leq V,$$

where the union is over all compact open subgroups $K \leq G$. Then $V_{\text{sm}} \leq V$ is evidently a smooth subrepresentation of G . We say that the Hilbert space representation (π, V) is **admissible** if V_{sm} is admissible. We note that in this case V_{sm} is a pre-Hilbert space representation. As a warning, if V is an infinite dimensional Hilbert space representation of G then $V_{\text{sm}} \neq V$, so V itself is not even smooth, let alone admissible (see Exercise 5.5).

Lemma 5.3.7 *Let (π, V) be a Hilbert space representation of G . The subspace $V_{\text{sm}} \leq V$ is dense and G -invariant.* $\square$

We leave the proof as an exercise (see Exercise 5.12). We also have the following analogue of Theorem 4.4.10:

Proposition 5.3.8 *Let G be the F -points of a reductive group over a non-Archimedean local field F . If (π, V) is a unitary representation of G then V is irreducible if and only if V_{sm} is irreducible. If (π_1, V_1) and (π_2, V_2) are irreducible unitary representations of G and their spaces of smooth vectors $V_{1\text{sm}}$ and $V_{2\text{sm}}$ are equivalent, then (π_1, V_1) and (π_2, V_2) are unitarily equivalent.*

We give the proof of Proposition 5.3.8 in §5.4. Finally, we have the following non-Archimedean analogue of Theorem 4.4.5:

Theorem 5.3.9 (Harish-Chandra, Bernstein) *Let G be the F -points of a reductive group over a non-Archimedean local field F . Then all irreducible unitary representations of G are admissible.* $\square$

The proof was reduced to the special case of square integrable representations in [HC70b] (which invoked earlier work of Godement, see loc. cit.). The square integrable case was treated in [Ber74]. In fact, if G is as in Theorem 5.3.9,

Bernstein proves that the dimension of V^K is bounded by a constant that depends only on G and K (in particular, it is independent of the irreducible unitary representation V).

We end the section by discussing the relationship between the definitions of an admissible representation in the Archimedean and non-Archimedean case. For this purpose we adapt the definition of the isotypic subspaces (4.12) to the current setting. Let $K \leq G$ be a compact open subgroup. For an irreducible representation σ of K and a representation (π, V) of G , let

$$V(\sigma) = \langle \varphi \in V : \langle \pi(k)\varphi : k \in K \rangle \cong \sigma \rangle$$

be the σ -isotypic subspace. Here we are regarding V as a representation of K by restriction. One interesting fact about these isotypic subspaces in the non-Archimedean case is that each representation σ factors through a finite index normal subgroup of K . This is due to the following lemma (see Exercise 5.13):

Lemma 5.3.10 *Let K be a compact totally disconnected group and let H be a connected real or complex Lie group. The kernel of any continuous homomorphism $K \rightarrow H$ is a finite index normal subgroup of K . $\square$*

Let us return to the discussion of admissibility. Clearly the non-Archimedean definition of admissibility makes no sense in the Archimedean setting because of the no small subgroups phenomenon (Lemma 5.1.3). However, the definition in the Archimedean case (Definition 4.4.4) does make sense in the non-Archimedean case and is equivalent to the definition we gave above:

Proposition 5.3.11 *Let $K \leq G$ be a compact open subgroup. A representation (π, V) of G is smooth if and only if*

$$V = \bigoplus_{\sigma \in \widehat{K}} V(\sigma). \quad (5.3)$$

It is admissible if and only if $\dim_{\mathbb{C}} V(\sigma) < \infty$ for all σ .

Here, as in §4.3, $\widehat{K}$ is the set of equivalence classes of irreducible representations of K .

Proof. As mentioned before Lemma 5.3.10 each representation $\sigma \in \widehat{K}$ factors through a finite index normal subgroup of K . Thus if (5.3) holds then V is smooth. For any V we have a canonical map

$$\bigoplus_{\sigma \in \widehat{K}} V(\sigma) \longrightarrow V. \quad (5.4)$$

It is injective by orthogonality of characters of representations of K . In other words the internal sum of the subspaces $V(\sigma)$ in V is direct. If $\varphi \in V$ is

smooth then let $K_0 \leq K$ be an open finite index normal subgroup fixing φ . Then

$$\varphi \in \bigoplus_{\sigma \in \widehat{K/K_0}} V(\sigma).$$

Thus (5.4) is surjective if V is smooth. The first assertion of the proposition follows.

As for the second assertion, every element of $V(\sigma)$ is fixed by the kernel of σ , which is a compact open subgroup of K . Thus if V is admissible then $\dim_{\mathbb{C}} V(\sigma) < \infty$ for all σ . Suppose conversely that $\dim_{\mathbb{C}} V(\sigma) < \infty$ for all σ . It suffices to show that $\dim_{\mathbb{C}} V^{K_0} < \infty$ for all normal compact open subgroups $K_0 \leq K$. But K/K_0 is finite, which implies $\widehat{K/K_0}$ is finite, and hence

$$V^{K_0} = \bigoplus_{\sigma \in \widehat{K/K_0}} V(\sigma)$$

is finite dimensional. □

There is an important difference between Proposition 5.3.11 and the decomposition of the space of K -finite vectors in the Archimedean case. Assume for this paragraph that G is a reductive group over an Archimedean local field F and that (π, V) is a Hilbert space representation of $G(F)$. In this setting it is only the proper subspace of finite vectors $V_{\text{fin}} < V_{\text{sm}}$ that admits an algebraic direct sum decomposition as in Proposition 5.3.11.

Given Proposition 5.3.11, one might ask why we did not define admissibility in the non-Archimedean case using the obvious analogue of the definition in the Archimedean case. The reason is that the definition we gave in the non-Archimedean case is simply more convenient; there is no need to keep track of isotypic components.

5.4 Contragredients

In this section we use the proof of Proposition 5.3.8 as a convenient excuse for introducing the useful concept of the contragredient of a smooth representation.

Let (π, V) be a smooth representation of G . Then there is an action of G on the space $\text{Hom}(V, \mathbb{C})$ of $\mathbb{C}$ -linear functionals on V given by

$$\begin{aligned} G \times \text{Hom}(V, \mathbb{C}) &\longrightarrow \text{Hom}(V, \mathbb{C}) \\ (g, \lambda) &\longmapsto \lambda \circ \pi(g^{-1}). \end{aligned} \tag{5.5}$$

A linear functional $\lambda : V \rightarrow \mathbb{C}$ is **smooth** if λ is fixed by some compact open subgroup $K \leq G$. The **smooth dual** $V^{\vee} \subseteq \text{Hom}(V, \mathbb{C})$ is the subspace

of smooth linear functionals. The action (5.5) preserves $V^\vee$ and affords a smooth representation $(\pi^\vee, V^\vee)$ of G .

Definition 5.4.1. The **contragredient representation** $(\pi^\vee, V^\vee)$ is the representation of G on the smooth dual $V^\vee$ given above.

We note that $\vee$ is a contravariant functor from the category of admissible representations of G to itself. Indeed, Proposition 5.3.11 implies that

$$V^\vee = \bigoplus_{\sigma \in \widehat{K}} V(\sigma)^\vee. \quad (5.6)$$

We now define a variant of the contragredient representation. For any $\mathbb{C}$ -vector space V , let

$$V^c := \mathbb{C} \otimes_{(a \mapsto \bar{a}), \mathbb{C}} V.$$

This is the tensor product with respect to the identity map from $\mathbb{C}$ to $\mathbb{C}$ on the left and the complex conjugate of the scaling action of $\mathbb{C}$ on V on the right. Thus

$$ab \otimes v = b \otimes \bar{a}v$$

for $a, b \in \mathbb{C}$ and $v \in V$. The representation

$$(\pi^*, V^*) := (\pi^\vee, (V^\vee)^c)$$

is known as the **Hermitian contragredient**. The map $*$ again defines a contravariant functor from the category of admissible representations of G to itself. If V is admissible then there are natural isomorphisms

$$V \xrightarrow{\sim} (V^\vee)^\vee \quad \text{and} \quad V \xrightarrow{\sim} (V^*)^*$$

(see Exercise 5.14). In particular, suppose we are given a morphism $\iota : V \rightarrow V^*$. Then using the canonical isomorphism $V \xrightarrow{\sim} (V^*)^*$ we can identify ι^* with the following morphism:

$$\begin{aligned} \iota^* : V &\longrightarrow V^* \\ \varphi &\longmapsto (\varphi_0 \mapsto \overline{\iota(\varphi_0)(\varphi)}). \end{aligned}$$

This observation is used in the statement of the following lemma, a variant of [Lau96, Lemma D.6.3]:

Lemma 5.4.2 *Let (π, V) be an admissible representation of G . There is a canonical bijection between the set of G -invariant nondegenerate Hermitian forms on V and the set of isomorphisms*

$$\iota : V \longrightarrow V^*$$

of smooth representations such that $\iota^* = \iota$. If G is second countable and (π, V) is irreducible, then any two G -invariant nondegenerate Hermitian forms on V are $\mathbb{R}^\times$ multiples of each other.

Proof. The last claim follows from Schur's lemma (in the form of Exercise 5.6) and the first claim. Suppose that we have a G -invariant, nondegenerate Hermitian form $(\cdot, \cdot)$ on V that is $\mathbb{C}$ -linear in the first variable and $\mathbb{C}$ -antilinear in the second. We then obtain a G -equivariant morphism

$$\begin{aligned} \iota : V &\longrightarrow V^* \\ \varphi_0 &\longmapsto (\varphi \mapsto (\varphi, \varphi_0)). \end{aligned} \quad (5.7)$$

It is clearly injective. Now if $K \leq G$ is a compact open subgroup, then $\dim_{\mathbb{C}}(V^*)^K = \dim_{\mathbb{C}}(V^\vee)^K$. By (5.6) this is equal to $\dim_{\mathbb{C}} V^K < \infty$. Thus for all compact open subgroups $K \leq G$, we have that (5.7) induces an isomorphism

$$V^K \xrightarrow{\sim} (V^*)^K.$$

Hence by the smoothness of V and V^* , the injection (5.7) is an isomorphism. Since $\overline{(\varphi_1, \varphi_2)} = (\varphi_2, \varphi_1)$ for $\varphi_1, \varphi_2 \in V$ we deduce that $\iota = \iota^*$. Conversely if $\iota : V \rightarrow V^*$ is an isomorphism of admissible representations satisfying $\iota = \iota^*$, we can associate to ι the G -invariant nondegenerate Hermitian form

$$(\varphi_1, \varphi_2) := \iota(\varphi_2)(\varphi_1).$$

□

We use this lemma to prove Proposition 5.3.8:

Proof of Proposition 5.3.8: Assume first that (π, V) is a unitary representation of G . Suppose that V is reducible. Since V is unitary this implies that V is a direct sum of closed G -invariant subspaces $V = V_1 \oplus V_2$. The space of smooth vectors in any Hilbert space representation is dense by Lemma 5.3.7 and we deduce that V_{sm} is reducible.

Conversely suppose (π, V) is irreducible. Let $\varphi \in V_{\text{sm}} - \{0\}$ and let

$$W = \langle \pi(g)\varphi : g \in G \rangle.$$

Let $\overline{W}$ denote the closure of W in V . Since V is unitary, W is preunitary and hence the action of G on W extends to $\overline{W}$. Thus $\overline{W} = V$ by irreducibility of V . We claim that

$$\overline{W}_{\text{sm}} = W. \quad (5.8)$$

Since $\overline{W}$ is unitary it is admissible by Theorem 5.3.9, so it suffices to verify that $\overline{W}^K = W^K$ for all compact open subgroups $K \leq G$. For $w \in \overline{W}^K$ choose a sequence $\{w_n\} \subset W$ such that $w_n \rightarrow w$ as $n \rightarrow \infty$. Then $e_K w_n \rightarrow e_K w = w$

as $n \rightarrow \infty$ which implies that W^K is dense in $\overline{W}^K$. Since W^K and $\overline{W}^K$ are both finite dimensional $\mathbb{C}$ -vector spaces by admissibility, we deduce that $W^K = \overline{W}^K$ which in turn implies (5.8). Thus

$$V_{\text{sm}} = \overline{W}_{\text{sm}} = W = \langle \pi(g)\varphi : g \in G \rangle.$$

Since φ was an arbitrary nonzero element of V_{sm} we deduce that V_{sm} is irreducible.

Suppose that (π_1, V_1) and (π_2, V_2) are irreducible unitary representations of G with invariant pairings $(\cdot, \cdot)_1, (\cdot, \cdot)_2$ and that we are given a G -equivariant isomorphism $\Phi : V_{1\text{sm}} \xrightarrow{\sim} V_{2\text{sm}}$. We have a diagram

$$\begin{array}{ccc} V_{1\text{sm}} & \xrightarrow{\iota_1} & V_{1\text{sm}}^* \\ \downarrow \Phi & & \uparrow \Phi^* \\ V_{2\text{sm}} & \xrightarrow{\iota_2} & V_{2\text{sm}}^* \end{array}$$

where ι_i is induced by $(\cdot, \cdot)_i$ as in Lemma 5.4.2. Since

$$\Phi^* \circ \iota_2 \circ \Phi = \Phi^* \circ \iota_2^* \circ \Phi = (\Phi^* \circ \iota_2 \circ \Phi)^*,$$

the isomorphisms $\Phi^* \circ \iota_2 \circ \Phi$ and ι_1 both correspond to G -invariant nondegenerate Hermitian forms on $V_{1\text{sm}}$. Hence by Lemma 5.4.2 the diagram commutes after possibly multiplying ι_1 by an element of $\mathbb{R}^\times$. After this normalization, the commutativity of the diagram implies

$$(x, y)_1 = \iota_1(y)(x) = \Phi^* \circ \iota_2 \circ \Phi(y)(x) = \iota_2 \circ \Phi(y)(\Phi(x)) = (\Phi(x), \Phi(y))_2.$$

In other words after our normalization, $\Phi : V_{1\text{sm}} \rightarrow V_{2\text{sm}}$ is an isometric isomorphism. Since $V_{i\text{sm}}$ is dense in V_i , this implies that Φ extends by continuity to an isometric isomorphism $\Phi : V_1 \rightarrow V_2$. $\square$

5.5 The unramified Hecke algebra

Let G be a reductive group over a non-Archimedean local field F . Recall from §2.4 that G is **unramified** if G is quasi-split and split over an unramified extension of F . In this case, there exist hyperspecial subgroups of $G(F)$, unique up to the action of conjugation by $(G/Z_G)(F)$ (see §2.4).

Definition 5.5.1. If G is an unramified reductive group and $K \leq G(F)$ is a hyperspecial subgroup, then

$$C_c^\infty(G(F) // K)$$

is known as the **unramified Hecke algebra** or **spherical Hecke algebra**.

Here, as before, $C_c^\infty(G(F) // K)$ is the subalgebra of functions invariant on the left and right under K . It would be more correct to call this the K -unramified Hecke algebra, but the K is usually fixed and omitted from the notation. Similar remarks apply to the definitions below.

The following is arguably the most important fact about this algebra:

Theorem 5.5.2 *The unramified Hecke algebra $C_c^\infty(G(F) // K)$ is commutative.* $\square$

One way to prove this is via the Satake isomorphism (see Theorem 7.2.1). In special cases, this can also be proven using a trick due to Gelfand (see Exercise 5.15). Let us describe the algebra in more detail in a special case:

Example 5.2. The group $\mathrm{GL}_n(\mathbb{Z}_p)$ is a hyperspecial subgroup of $\mathrm{GL}_n(\mathbb{Q}_p)$. The unramified Hecke algebra in this case is

$$C_c^\infty(\mathrm{GL}_n(\mathbb{Q}_p) // \mathrm{GL}_n(\mathbb{Z}_p)).$$

Let

$$\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$$

and let T be the maximal torus of diagonal matrices in GL_n . Then λ defines a cocharacter $\lambda : \mathbb{G}_m \rightarrow T$ given on points by

$$\lambda(x) = \begin{pmatrix} x^{\lambda_1} & & \\ & \ddots & \\ & & x^{\lambda_n} \end{pmatrix}.$$

Consider

$$\{ \mathbb{1}_{\mathrm{GL}_n(\mathbb{Z}_p)\lambda(p)\mathrm{GL}_n(\mathbb{Z}_p)} : \lambda_1 \geq \dots \geq \lambda_n \}.$$

The Smith normal form for matrices over $\mathbb{Q}_p$ from the theory of elementary divisors implies that the set above is a $\mathbb{C}$ -vector space basis for the unramified Hecke algebra.

One can profitably generalize the previous example using the Cartan decomposition. We state a somewhat imprecise version now, and refer to Theorem B.5.4 for a more precise statement and references:

Theorem 5.5.3 (The Cartan Decomposition) *Assume that G is an unramified reductive group over a non-Archimedean local field F . There exists a maximal split torus T of G and a hyperspecial subgroup $K \leq G(F)$ such that for any Borel subgroup $B \geq T$ one has that*

$$G(F) = \coprod_{\lambda} K\lambda(\varpi)K,$$

where the disjoint union is over all cocharacters $\lambda \in X_*(T)$ that are dominant with respect to B . $\square$

Corollary 5.5.4 *Under the assumptions of Theorem 5.5.3,*

$$\{\mathbb{1}_{K\lambda(\varpi)K} : \lambda \in X_*(T) \text{ is dominant with respect to } B\}$$

is a basis of $C_c^\infty(G(F) // K)$ as a vector space over $\mathbb{C}$. $\square$

Though Corollary 5.5.4 gives us a concrete description of the Hecke algebra as a $\mathbb{C}$ -vector space, it does not give us complete information about its structure as an algebra. For more on this algebra structure, see §7.2 and §7.5.

We now return to representation theory:

Definition 5.5.5. Let G be an unramified reductive group over a non-Archimedean local field F and let $K \leq G(F)$ be a hyperspecial subgroup. An irreducible smooth representation (π, V) of $G(F)$ is **K -unramified**, or **K -spherical**, if $V^K \neq 0$.

By Theorem 5.3.6 an irreducible smooth representation is admissible, so K -unramified representations are admissible.

As mentioned earlier, hyperspecial subgroups $K \leq G(F)$ are only unique up to the action of conjugation by $(G/Z_G)(F)$, not $G(F)$. Thus the property of a representation being K -unramified depends on the choice of K in general. For better or worse, the relevant K is often omitted from notation, and one speaks simply of **unramified representations** or **spherical representations**. We also point out that using the terminology “unramified” for Hecke algebras and representations is preferred. This is to avoid confusion when applying the theory of spherical varieties (see §14.4).

An important consequence of Theorem 5.5.2 is that the unramified line in an unramified representation is unique. Let $K \leq G(F)$ be a hyperspecial subgroup.

Corollary 5.5.6 *Let (π, V) be an irreducible admissible K -unramified representation of $G(F)$. Then $\dim_{\mathbb{C}} V^K = 1$.*

Proof. Since V is irreducible, if $V^K \neq 0$ then it is an irreducible $C_c^\infty(G(F) // K)$ -module by Proposition 5.3.4. Since $C_c^\infty(G(F) // K)$ is commutative by Theorem 5.5.2 we deduce the corollary. $\square$

For more information on unramified representations, see Chapter 7.

5.6 Restricted tensor products of modules

In the representation theory of compact or affine algebraic groups, one quickly reduces the representation theory of products of groups to representation theory of the individual factors, thus simplifying problems significantly. We now explain how to accomplish this in the context of automorphic representations. The problem is that $G(\mathbb{A}_F)$ is a restricted direct product

$$G(\mathbb{A}_F) \cong \prod'_v G(F_v)$$

not a direct product. Thus we should not expect a representation of $G(\mathbb{A}_F)$ to decompose into a tensor product of representations in the usual sense. Instead one defines the notion of restricted tensor products of representations. We make this precise in the current section, and then prove in Theorem 5.7.1 that automorphic representations indeed factor into restricted tensor products indexed by the places in F .

We start by defining a restricted tensor product of vector spaces. Let Ξ be a countable set, let $\Xi_0 \subset \Xi$ be a finite subset, let

$$\{W_v : v \in \Xi\} \quad (5.9)$$

be a family of $\mathbb{C}$ -vector spaces and for each $v \in \Xi - \Xi_0$ let $\varphi_{0v} \in W_v - 0$. For all sets

$$\Xi_0 \subseteq S \subseteq \Xi$$

of finite cardinality, set $W_S := \otimes_{v \in S} W_v$. If $S \subseteq S'$ there is a map

$$\begin{aligned} W_S &\longrightarrow W_{S'} \\ \otimes_{v \in S} \varphi_v &\longmapsto \otimes_{v \in S} \varphi_v \otimes (\otimes_{v \in S' - S} \varphi_{0v}). \end{aligned} \quad (5.10)$$

Consider the vector space

$$W := \otimes'_v W_v := \varinjlim_S W_S,$$

where the transition maps are given by (5.10). This is the **restricted tensor product** of the W_v with respect to the φ_{0v} . Thus W is the set of

$$\otimes_v \varphi_v \in \otimes_v W_v$$

such that $\varphi_v = \varphi_{0v}$ for all but finitely many $v \in \Xi$. We note that if we are given, for each $v \in \Xi$, a $\mathbb{C}$ -linear map

$$B_v : W_v \longrightarrow W_v$$

such that $B_v(\varphi_{0v}) = \varphi_{0v}$ for all but finitely many $v \in \Xi$ then this gives a $\mathbb{C}$ -linear map

$$\begin{aligned} B = \otimes'_v B_v : W &\longrightarrow W \\ \otimes'_v \varphi_v &\longmapsto \otimes'_v B_v(\varphi_v). \end{aligned}$$

We now define a restricted tensor product of algebras. Suppose we are given $\mathbb{C}$ -algebras (not necessarily with unit) $\{A_v : v \in \Xi\}$ and idempotents $a_{0v} \in A_v$ for all $v \in \Xi - \Xi_0$. If $S \subseteq S'$ there is a $\mathbb{C}$ -linear map

$$\begin{aligned}
A_S &\longrightarrow A_{S'} \\
\otimes_{v \in S} a_v &\longmapsto \otimes_{v \in S} a_v \otimes (\otimes_{v \in S' - S} a_{0v}).
\end{aligned}
\tag{5.11}$$

It is moreover a morphism of algebras because the a_{0v} are idempotents. Consider the algebra

$$A := \otimes'_v A_v := \varinjlim_S A_S,$$

where the transition maps are given by (5.11). This is the **restricted tensor product** of the A_v with respect to the a_{0v} . Assume the complex vector spaces W_v in (5.9) above are A_v -modules for all $v \in \Xi$. If $a_{0v}\varphi_{0v} = \varphi_{0v}$ for all but finitely many v , then W is an A -module. The isomorphism class of W as an A -module in general depends on the choice of $\{\varphi_{0v}\}$. However, if we replace the φ_{0v} by nonzero scalar multiples we obtain isomorphic A -modules.

An easy example of this construction is the ring of polynomials in infinitely many variables. It can be given the structure of a restricted tensor product of algebras

$$\mathbb{C}[X_1, X_2, \dots] = \otimes'_i \mathbb{C}[X_i],$$

where we take a_{0i} to be the identity in $\mathbb{C}[X_i]$.

Let G be an affine algebraic group over a global field F and let ∞ be the set of infinite places. Let S be a finite set of places of F including the infinite places. Choose a model of G over $\mathcal{O}_F^S$ which we again denote by G by abuse of notation (see §2.4 for our conventions regarding models). For $v \notin S$ write $K_v := G(\mathcal{O}_{F_v})$; it is a compact open subgroup of $G(F_v)$. Let Ξ_0 be a finite set of places of F including S . If G is reductive, then upon enlarging Ξ_0 if necessary, we can and do assume that if $v \notin \Xi_0$ then K_v is hyperspecial (see Proposition 2.4.8). By Proposition 2.4.10 we have an isomorphism

$$G(\mathbb{A}_F^\infty) \xrightarrow{\sim} \prod'_{v \nmid \infty} G(F_v), \tag{5.12}$$

where the restricted direct product is with respect to K_v for $v \notin \Xi_0$. We recall from Proposition 2.4.10 that the isomorphism and the group on the right in (5.12) are in fact independent of the choice of model.

Choose right Haar measures $d_r g_v$ on $G(F_v)$ for all $v \nmid \infty$ and assume that they are normalized so that

$$\prod_{v \nmid \infty} \int_{K_v} d_r g_v < \infty.$$

For example, we could assume that $\int_{K_v} d_r g_v = 1$ for all but finitely many v as in (3.11). Then one can define the Haar measure $\otimes'_{v \nmid \infty} d_r g_v$ on $\prod'_{v \nmid \infty} G(F_v)$ in the obvious manner: it is the unique Haar measure such that all relatively compact measurable sets

$$\prod_{v \nmid \infty} \Omega_v \subseteq \prod'_{v \nmid \infty} G(F_v)$$

have measure

$$\prod_{v \nmid \infty} \int_{\Omega_v} d_r g_v.$$

We write $d_r g$ for the Haar measure on $G(\mathbb{A}_F^\infty)$ obtained from $\otimes'_{v \nmid \infty} d_r g_v$ using the isomorphism (5.12).

Given that we have normalized the Haar measures in this manner, we obtain an isomorphism

$$C_c^\infty(G(\mathbb{A}_F^\infty)) \xrightarrow{\sim} \otimes'_{v \nmid \infty} C_c^\infty(G(F_v)), \quad (5.13)$$

where the restricted tensor product on the right is defined with respect to the idempotents

$$e_{K_v} := \frac{1}{\text{meas}_{d_r g_v}(K_v)} \mathbb{1}_{K_v}. \quad (5.14)$$

Every compact open subgroup of $G(\mathbb{A}_F^\infty)$ contains a finite index subgroup of the form $K' = \prod_{v \nmid \infty} K_v$ where $K_v = G(\mathcal{O}_{F_v})$ for all but finitely many v . The isomorphism (5.13) is the unique $\mathbb{C}$ -linear map sending $\mathbb{1}_{K' g K'}$ to $\prod_{v \nmid \infty} \mathbb{1}_{K_v g_v K_v}$ for all $g \in G(\mathbb{A}_F^\infty)$ and all K' . Since changing the model of G_F affects only finitely many K_v (see the proof of Proposition 2.4.10), the isomorphism (5.13) and the algebra $\otimes'_{v \nmid \infty} C_c^\infty(G(F_v))$ are independent of the choice of model.

5.7 Flath's theorem

Flath's theorem implies that every automorphic representation can be factored into components indexed by the places of the global field over which the representation is defined. This fact is fundamental to the study of automorphic representations. In this section, we state and prove Flath's result.

Let G be a reductive group over a global field F . As in §5.6, we choose a model of G over $\mathcal{O}_F^S$ for a sufficiently large finite set S of places of F including the infinite places and let $K_v := G(\mathcal{O}_{F_v})$ for $v \notin S$. A $C_c^\infty(G(\mathbb{A}_F^\infty))$ -module W is **factorizable** if

- (a) there are admissible representations W_v of $G(F_v)$ for all $v \nmid \infty$ such that $\dim_{\mathbb{C}} W_v^{K_v} = 1$ for all but finitely many v ;
- (b) there is a $\mathbb{C}$ -linear isomorphism

$$W \xrightarrow{\sim} \otimes'_{v \nmid \infty} W_v, \quad (5.15)$$

- where the restricted tensor product is with respect to nonzero elements $\varphi_{0v} \in W_v^{K_v}$ for all but finitely many v ; and
(c) the diagram

$$\begin{array}{ccc}
C_c^\infty(G(\mathbb{A}_F^\infty)) \times W & \longrightarrow & W \\
\downarrow \sim & & \downarrow \sim \\
\otimes'_{v \nmid \infty} C_c^\infty(G(F_v)) \times \otimes'_{v \nmid \infty} W_v & \longrightarrow & \otimes'_{v \nmid \infty} W_v
\end{array}$$

commutes.

Here the left vertical arrow is given by the product of (5.13) and (5.15), the right vertical arrow is given by (5.15), and the horizontal arrows are the action maps. In view of the assumption that $\dim_{\mathbb{C}} W_v^{K_v} = 1$ for all but finitely many v , the module $\otimes'_{v \nmid \infty} W_v$ depends only on the choice of the φ_{0v} up to isomorphism. We observe that W is admissible. It is irreducible if and only if the W_v are irreducible for all v . If assumptions (a), (b), and (c) are valid we write

$$W \cong \otimes'_{v \nmid \infty} W_v.$$

Theorem 5.7.1 (Flath) *Every irreducible admissible $C_c^\infty(G(\mathbb{A}_F^\infty))$ -module W is factorizable. If $W \cong \otimes'_{v \nmid \infty} W_v$ then the $C_c^\infty(G(F_v))$ -modules W_v are uniquely determined by W up to isomorphism.*

Before proving Theorem 5.7.1 we prove a preparatory theorem:

Theorem 5.7.2 *Let G_1 and G_2 be of td-type and let $G = G_1 \times G_2$.*

- (i) *If V_i is an irreducible admissible representation of G_i then $V_1 \otimes V_2$ is an irreducible admissible representation of G .*
- (ii) *If V is an irreducible admissible representation of G then there exists irreducible admissible representations V_i of G_i such that $V \cong V_1 \otimes V_2$. Moreover, the isomorphism classes of the V_i are uniquely determined by V up to isomorphism.*

Proof. We first prove (i). It is easy to see that $V_1 \otimes V_2$ is admissible. We must check that it is irreducible. By the irreducibility criterion in Proposition 5.3.4, for every compact open subgroup $K_1 \times K_2 \leq G_1 \times G_2$, the finite dimensional $C_c^\infty(G_i // K_i)$ -modules $V_i^{K_i}$ are irreducible. Since the $V_i^{K_i}$ are finite dimensional, it is easy to see that $V_1^{K_1} \otimes V_2^{K_2}$ is irreducible as a module under

$$C_c^\infty(G_1 // K_1) \otimes C_c^\infty(G_2 // K_2).$$

But

$$\begin{aligned}
C_c^\infty(G_1 \times G_2 // K_1 \times K_2) &= C_c^\infty(G_1 // K_1) \otimes C_c^\infty(G_2 // K_2), \\
(V_1 \otimes V_2)^{K_1 \times K_2} &= V_1^{K_1} \otimes V_2^{K_2},
\end{aligned}$$

so $V_1 \otimes V_2$ is irreducible by Proposition 5.3.4.

For (ii), let V be an irreducible admissible representation of G . Choose $K = K_1 \times K_2$ such that $V^K \neq 0$ (this is possible by smoothness). Then since V^K is finite dimensional, there exist finite dimensional irreducible $C_c^\infty(G_i // K_i)$ -modules $V_i(K_i)$, unique up to isomorphism, and an isomorphism of $C_c^\infty(G // K)$ modules $V^K \xrightarrow{\sim} V_1(K_1) \otimes V_2(K_2)$. Varying K , we obtain a decomposition

$$V \cong V_1 \otimes V_2$$

as $C_c^\infty(G) \cong C_c^\infty(G_1 \times G_2)$ -modules. Here

$$V_1 := \varinjlim_{K_1} V_1(K_1) \quad \text{and} \quad V_2 := \varinjlim_{K_2} V_2(K_2).$$

The V_i are evidently admissible, and moreover they are irreducible by Proposition 5.3.4. They are uniquely determined by V up to isomorphism since the $V_i(K_i)$ are uniquely determined by V up to isomorphism. $\square$

We now prove Flath's theorem:

Proof of Theorem 5.7.1: Let G be a reductive group over a global field F . Let S be a finite set of places of F including the infinite places. Use (5.13) to identify $C_c^\infty(G(\mathbb{A}_F^\infty))$ and $\otimes'_{v \nmid \infty} C_c^\infty(G(F_v))$ for the proof. We then have a well-defined subalgebra

$$A_S := C_c^\infty(G(F_S^\infty)) \otimes e_{K^S} \leq C_c^\infty(G(\mathbb{A}_F^\infty)),$$

where $e_{K^S} = \otimes_{v \notin S} e_{K_v}$.

By Corollary 5.5.6 and Theorem 5.7.2, as a representation of A_S we have an isomorphism

$$W^{K^S} \cong \otimes_{v \in S-\infty} W_v \otimes W^S,$$

where W^S is a 1-dimensional $\mathbb{C}$ -vector space on which e_{K^S} acts trivially. Hence, by admissibility,

$$W = \bigcup_S W^{K^S} \cong \varinjlim_S \otimes_{v \in S-\infty} W_v \otimes W^S. \quad (5.16)$$

Here the direct limit is taken with respect to the obvious modification of the transition maps (5.10). On the other hand, (5.13) induces an identification

$$C_c^\infty(G(\mathbb{A}_F^\infty)) = \bigcup_S A_S = \varinjlim_S A_S, \quad (5.17)$$

where the direct limit is taken with respect to the obvious modification of the transition maps (5.11). It is clear that (5.16) is equivariant with respect to (5.17). The uniqueness assertion follows from Theorem 5.7.2. $\square$

Exercises

Throughout these exercises G is a group of td-type and (π, V) is a representation of G .

5.1. Prove that a group of td-type is locally compact and totally disconnected.

5.2. Assume that (π, V) is smooth. Prove that (π, V) is admissible if and only if V^U is finite dimensional for any open subgroup $U \leq G$. Here $V^U \leq V$ denotes the subspace of U -fixed vectors.

5.3. Prove Lemma 5.3.2.

5.4. Define

$$V_{\text{sm}} := \bigcup_K V^K,$$

where the union is over all compact open subgroups $K \leq G$. Prove that V_{sm} is preserved by G and is a smooth subrepresentation of G . The subspace $V_{\text{sm}} \leq V$ is the space of smooth vectors in V (in the current setting of td-type groups).

5.5. Suppose that V is an admissible Hilbert space representation. Assume that G is second countable. Define V_{sm} as in Exercise 5.4 above. Show that V_{sm} admits a countable algebraic (also known as **Hamel**) basis. Show that if V is infinite dimensional then any algebraic basis of V is uncountable. Conclude that if V is infinite dimensional then $V_{\text{sm}} \subsetneq V$.

5.6. Assume that G is second countable and V is smooth and irreducible. Show that any linear map $L : V \rightarrow V$ commuting with the action of G is a scalar multiple of the identity map.

5.7. Assume that G is second countable and V is smooth and irreducible. Let $\{\varphi_1, \dots, \varphi_n\}$ and $\{\varphi'_1, \dots, \varphi'_n\}$ be two sets of elements of V with $\varphi_1, \dots, \varphi_n$ linearly independent. Then there is an $f \in C_c^\infty(G(F))$ such that

$$\pi(f)\varphi_i = \varphi'_i$$

for all $1 \leq i \leq n$.

5.8. Assume that G is second countable and V is smooth and irreducible. Let Z_G be the center of G . Show that there is a quasi-character $\omega_\pi : Z_G \rightarrow \mathbb{C}^\times$ such that $\pi(z)$ acts via $\omega_\pi(z)$ on V for all $z \in Z_G$. The quasi-character ω_π is called the **central quasi-character** of π . Show that if $\chi : G \rightarrow \mathbb{C}^\times$ is a quasi-character, then the central quasi-character of $\pi \otimes \chi$ is $\chi|_{Z_G} \omega_\pi$.

5.9. Let p be a prime number in $\mathbb{Z}$. Prove that there are no constants $c_1, c_2 \in \mathbb{R}_{>0}$ such that

$$c_1 |\cdot|_\infty \leq |\cdot|_p \leq c_2 |\cdot|_\infty,$$

where $|\cdot|_\infty$ denotes the usual Archimedean norm on $\mathbb{Q}$ and $|\cdot|_p$ denotes the p -adic norm.

5.10. Prove the equality (5.1).

5.11. Prove Lemma 5.3.3.

5.12. Prove Lemma 5.3.7.

5.13. Prove Lemma 5.3.10.

5.14. Prove that if V is an admissible representation of a td-type group then there are canonical isomorphisms $V \rightarrow (V^\vee)^\vee$ and $V \rightarrow (V^*)^*$.

5.15. Let F be a non-Archimedean local field. Consider the unramified Hecke algebra $C_c^\infty(\mathrm{GL}_n(F) // \mathrm{GL}_n(\mathcal{O}_F))$. For $f \in C_c^\infty(\mathrm{GL}_n(F))$ let $f^\dagger(g) := f(g^t)$. Show that for every $f_1, f_2 \in C_c^\infty(\mathrm{GL}_n(F))$,

$$(f_1 * f_2)^\dagger(g) = (f_2^\dagger * f_1^\dagger)(g).$$

Show, on the other hand, that $f^\dagger = f$ for $f \in C_c^\infty(\mathrm{GL}_n(F) // \mathrm{GL}_n(\mathcal{O}_F))$. Conclude that $C_c^\infty(\mathrm{GL}_n(F) // \mathrm{GL}_n(\mathcal{O}_F))$ is commutative.

Chapter 6

Automorphic Forms



Discovery is the privilege of the child: the child who has no fear of being once again wrong, of looking like an idiot, of not being serious, of not doing things like everyone else.

Alexander Grothendieck

In this chapter, we define automorphic forms and use them to give the general definition of an automorphic representation. We then explain the relationship between discrete automorphic representations and automorphic representations.

6.1 Smooth functions

Our goal in this chapter is to give the general definition of an automorphic representation and explain its relation to the definition of a discrete automorphic representation discussed in Chapter 3. There are at least three reasons for introducing the general definition. The first is that it is easier to relate it to spaces of automorphic forms on locally symmetric spaces (in the number field case). The second is that the general definition allows us to move from the category of Hilbert space representations to the category of admissible representations. The last and most compelling reason is that representations constructed from Eisenstein series are automorphic representations, but are usually not discrete automorphic representations. We refer to Chapter 10 for more information about Eisenstein series.

Let G be a reductive group over a global field F . As preparation for our discussion below, we define precisely what one means by a smooth function on $G(\mathbb{A}_F)$ and $A_G \backslash G(\mathbb{A}_F)$. There is an obvious isomorphism

$$A_G \backslash G(\mathbb{A}_F) \cong A_G \backslash G(F_\infty) \times G(\mathbb{A}_F^\infty).$$

We define

$$\begin{aligned} C^\infty(G(\mathbb{A}_F)) &:= C^\infty(G(F_\infty)) \otimes C^\infty(G(\mathbb{A}_F^\infty)), \\ C_c^\infty(G(\mathbb{A}_F)) &:= C_c^\infty(G(F_\infty)) \otimes C_c^\infty(G(\mathbb{A}_F^\infty)), \\ C^\infty(A_G \backslash G(\mathbb{A}_F)) &:= C^\infty(A_G \backslash G(F_\infty)) \otimes C^\infty(G(\mathbb{A}_F^\infty)), \\ C_c^\infty(A_G \backslash G(\mathbb{A}_F)) &:= C_c^\infty(A_G \backslash G(F_\infty)) \otimes C_c^\infty(G(\mathbb{A}_F^\infty)). \end{aligned} \tag{6.1}$$

Here the tensor products are algebraic. We let $C^\infty(G(\mathbb{A}_F)^1)$ be the space of restrictions to $G(\mathbb{A}_F)^1$ of elements of $C^\infty(G(\mathbb{A}_F))$. Similarly, $C_c^\infty(G(\mathbb{A}_F)^1)$ is the space of restrictions to $G(\mathbb{A}_F)^1$ of elements of $C_c^\infty(G(\mathbb{A}_F))$.

6.2 Classical automorphic forms

For this section, F is a number field, $K_\infty \leq G(F_\infty)$ is a maximal compact subgroup, and $K^\infty \leq G(\mathbb{A}_F^\infty)$ is a compact open subgroup. Recall the homeomorphism

$$G(F) \backslash G(\mathbb{A}_F) / K^\infty \xrightarrow{\sim} \coprod_i \Gamma_i(K^\infty) \backslash G(F_\infty) \tag{6.2}$$

of (2.23). By Theorem 2.6.1, there are finitely many indices in this disjoint union. For the purposes of the current section, the precise definition of $\Gamma_i(K^\infty)$ is not so important, but it is important that they are arithmetic subgroups of $G(F_\infty)$ such that the quotient $A_G \Gamma_i(K^\infty) \backslash G(F_\infty)$ has finite volume (see Theorem 2.6.3). Clearly, automorphic representations are related to functions on the sets appearing in (6.2). In this section and the following, we make this relationship precise. We work only at infinity in the current section and then pass to the adelic setting in the next.

Let

$$\iota' : G \longrightarrow \mathrm{GL}_n$$

be a closed embedding and let $\iota : G \rightarrow \mathrm{SL}_{2n}$ be the embedding defined by

$$\iota(g) := \begin{pmatrix} \iota'(g) & \\ & \iota'(g^{-1})^t \end{pmatrix}. \tag{6.3}$$

For $g \in G(F_\infty)$, we then define the norm

$$\|g\| := \|g\|_\iota = \prod_{v|_\infty} \sup_{1 \leq i, j \leq 2n} |\iota(g)_{ij}|_v. \tag{6.4}$$

To see why we work with ι instead of ι' , consider the case where $G = \mathbb{G}_m$, $F = \mathbb{Q}$, and $\iota' : \mathbb{G}_m \rightarrow \mathrm{GL}_1$ is the identity. Then for $t \in \mathbb{G}_m(\mathbb{R}) = \mathbb{R}^\times$, the norm $\|t\|$ is bounded if and only if $t \in \mathbb{R}^\times$ is bounded and bounded away from zero. In this special case, if we defined $\|g\|$ only using ι' , then the norm would be bounded, but not bounded away from zero.

Let

$$\mathfrak{g} := \oplus_{v|\infty} \mathfrak{g}_v, \quad \mathfrak{g}_v = \mathrm{Lie} \, \mathrm{Res}_{F_v/\mathbb{R}} G_{F_v} \quad (6.5)$$

and let $K_\infty \leq G(F_\infty)$ be a maximal compact subgroup. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}^\mathbb{C}$ and $Z(\mathfrak{g})$ its center (see §4.6).

Definition 6.2.1. A continuous function

$$\varphi : G(F_\infty) \longrightarrow \mathbb{C}$$

is of **moderate growth** or **slowly increasing** if there exists $(c, r) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that

$$|\varphi(g)| \leq c \|g\|^r.$$

It is of **uniform moderate growth** if it is smooth and there exists an $r \in \mathbb{R}$ such that for all $X \in U(\mathfrak{g})$

$$|X\varphi(g)| \leq c_X \|g\|^r$$

for some $c_X \in \mathbb{R}_{>0}$.

It turns out that different choices of ι lead to norms that are equivalent in a sense made precise by Exercise 6.1. In particular, the notion of moderate growth and uniform moderate growth is independent of the choice of ι .

For the definition of automorphic forms, we will require the notion of $Z(\mathfrak{g})$ -finite functions.

Definition 6.2.2. Let V be a $Z(\mathfrak{g})$ -module. A vector $\varphi \in V$ is **$Z(\mathfrak{g})$ -finite** if $Z(\mathfrak{g})\varphi$ is finite dimensional, or equivalently, if there is an ideal $J \leq Z(\mathfrak{g})$ with $\dim_{\mathbb{C}} Z(\mathfrak{g})/J < \infty$ that annihilates φ .

At the moment, we are interested in the case $V = C^\infty(G(F_\infty))$. In more detail, the right action of $G(F_\infty)$ on $G(F_\infty)$ induces a left action of $G(F_\infty)$ on $C^\infty(G(F_\infty))$. Regard $G(F_\infty) = \mathrm{Res}_{F/\mathbb{Q}} G(\mathbb{R})$ as a real Lie group and endow $C^\infty(G(F_\infty))$ with its usual Fréchet topology. Then $C^\infty(G(F_\infty))$ is a (continuous) representation of $G(F_\infty)$. It is even smooth in the sense of Definition 4.4.11. We say that a function $\varphi \in C^\infty(G(F_\infty))$ is **$Z(\mathfrak{g})$ -finite** if it is a $Z(\mathfrak{g})$ -finite vector in $C^\infty(G(F_\infty))$. Similarly, a function $\varphi \in C^\infty(G(F_\infty))$ is **K_∞ -finite** if it is a K_∞ -finite vector in $C^\infty(G(F_\infty))$ in the sense of Definition 4.4.3.

We now finally come to the notion of an automorphic form:

Definition 6.2.3. Let $\Gamma \leq G(F)$ be an arithmetic subgroup. A smooth function $\varphi : G(F_\infty) \rightarrow \mathbb{C}$ of moderate growth is an **automorphic form** for Γ

if it is left Γ -invariant, K_∞ -finite, and $Z(\mathfrak{g})$ -finite. We denote the space of automorphic forms on Γ by $\mathcal{A}(\Gamma)$.

We will now show that automorphic forms are automatically of uniform moderate growth. To prove this, we use the following theorem of Harish-Chandra [HC66, §8, Theorem 1]:

Theorem 6.2.4 *For any function $\varphi : G(F_\infty) \rightarrow \mathbb{C}$ that is K_∞ -finite and $Z(\mathfrak{g})$ -finite, there is an $f \in C_c^\infty(G(F_\infty))$ satisfying $f(kxk^{-1}) = f(x)$ for all $k \in K_\infty$ such that*

$$R(f)\varphi(g) := \int_{G(F_\infty)} f(h)\varphi(gh)dh = \varphi(g).$$

The support of f can be taken to lie in any neighborhood of 1 in $G(F_\infty)$. $\square$

Corollary 6.2.5 *Any automorphic form is of uniform moderate growth.*

Proof. Let $\varphi : G(F_\infty) \rightarrow \mathbb{C}$ be an automorphic form. Choose f as in Theorem 6.2.4. Then, for $X \in \mathfrak{g}$, we have

$$X\varphi = X(R(f)\varphi) = R(f_X)\varphi$$

where $f_X \in C_c^\infty(G(F_\infty))$ is defined as in (4.7) and we have used (4.8). By induction on the degree of an element of $U(\mathfrak{g})$, we see that for any $X \in U(\mathfrak{g})$ there is an $f_X \in C_c^\infty(G(F_\infty))$ such that

$$X\varphi = R(f_X)\varphi.$$

Assume that there exists $(c, r) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that

$$|\varphi(g)| \leq c\|g\|^r$$

for all g . Then

$$|R(f_X)\varphi(g)| \leq c \int_{G(F_\infty)} |f_X(h)| \cdot \|gh\|^r dh \leq c_X \|g\|^r$$

for some $c_X \in \mathbb{R}_{>0}$ (see Exercise 6.2). $\square$

Often it is convenient to isolate the subspace of $\mathcal{A}(\Gamma)$ that is annihilated by a particular ideal $J \leq Z(\mathfrak{g})$. It is denoted $\mathcal{A}(\Gamma, J)$. This space decomposes still further:

$$\mathcal{A}(\Gamma, J) = \bigoplus_{\sigma \in \widehat{K_\infty}} \mathcal{A}(\Gamma, J, \sigma) \tag{6.6}$$

where $\mathcal{A}(\Gamma, J, \sigma) := \mathcal{A}(\Gamma, J)(\sigma)$ is the σ -isotypic subspace (see (4.12)). We record the following important result of Harish-Chandra [HC68]:

Theorem 6.2.6 (Harish-Chandra) *For each ideal $J \leq Z(\mathfrak{g})$ with*

$$\dim_{\mathbb{C}} Z(\mathfrak{g})/J < \infty,$$

the space $\mathcal{A}(\Gamma, J)$ is an admissible $(\mathfrak{g}, K_{\infty})$ -module. In particular, $\mathcal{A}(\Gamma, J, \sigma)$ is finite dimensional for each σ . $\square$

As pointed out in [BJ79, §1.7], this is stated for semisimple groups in [HC68], but the proof given there is valid for general reductive groups.

One might ask about the relationship between $\mathcal{A}(\Gamma)$ and $L^2(A_G \Gamma \backslash G(F_{\infty}))$. There is no obvious relationship between them except under more stringent assumptions. For example, Eisenstein series (which will be discussed in Chapter 10) provide examples of elements of $\mathcal{A}(\Gamma)$ that need not even be square integrable. It turns out to be more convenient to explain the relationship between the so-called cuspidal subspaces of $\mathcal{A}(\Gamma)$ and $L^2(A_G \Gamma \backslash G(F_{\infty}))$. We defer the discussion of this connection to §6.5.

6.3 Adelic automorphic forms over number fields

Let $\iota : G \rightarrow \mathrm{SL}_{2n}$ be the embedding of (6.3). In this section, we assume that F is a number field. Let $g \in G(\mathbb{A}_F)$. For a place v of F let

$$\|g\|_v := \|g\|_{\iota, v} = \sup_{1 \leq i, j \leq 2n} |\iota(g_v)_{ij}|_v \quad (6.7)$$

and for a set of places S of F (finite or infinite) let

$$\|g\|_S := \prod_{v \in S} \|g\|_v. \quad (6.8)$$

If S is the set of all places of F then we omit it from notation. We observe that, since $\iota(g) \in \mathrm{SL}_{2n}(\mathbb{A}_F)$, we have $\iota(g_v) \in \mathrm{SL}_{2n}(\mathcal{O}_{F_v})$ for all but finitely many v and hence $\|g\|_v = 1$ for all but finitely many v . In particular, $\|g\|$ is well-defined.

As in the Archimedean setting, we have a definition of a function of moderate growth:

Definition 6.3.1. A function

$$\varphi : G(\mathbb{A}_F) \longrightarrow \mathbb{C}$$

is of **moderate growth** or **slowly increasing** if there exists $(c, r) \in \mathbb{R}_{>0} \times \mathbb{R}$ such that

$$|\varphi(g)| \leq c \|g\|^r.$$

The function φ is of **uniform moderate growth** if it is smooth and there exists an $r \in \mathbb{R}$ such that for all $X \in U(\mathfrak{g})$

$$|X\varphi(g)| \leq c_X \|g\|^r$$

for some $c_X \in \mathbb{R}_{>0}$.

Here $X\varphi$ is defined as follows: Any smooth φ can be written as a finite sum of vectors of the form $\varphi_\infty \otimes \varphi^\infty$ with $(\varphi_\infty, \varphi^\infty) \in C^\infty(G(F_\infty)) \times C^\infty(G(\mathbb{A}_F^\infty))$. One defines

$$X(\varphi_\infty \otimes \varphi^\infty) := X\varphi_\infty \otimes \varphi^\infty$$

and extends linearly.

It is sometimes convenient to rephrase the notion of moderate growth in terms of Siegel sets. For this purpose, let $P \leq G$ be a minimal parabolic subgroup and let $K \leq G(\mathbb{A}_F)$ be a maximal compact subgroup such that $G(\mathbb{A}_F) = P(\mathbb{A}_F)K$. Write $P = MN$ where N is the unipotent radical of P and M is a Levi subgroup containing a maximal split torus T_0 of G^{der} . Recall that reduction theory allows us to study the adelic quotient

$$[G] := A_G G(F) \backslash G(\mathbb{A}_F)$$

of (2.21) using Siegel sets

$$\Omega_N \Omega_M A_{T_0}(t) K$$

defined as in (2.25). Here $\Omega_N \times \Omega_M \subset N(\mathbb{A}_F) \times M(\mathbb{A}_F)^1$ is a compact set. Now for all

$$\lambda \in X^*(M)$$

there are real numbers $c, r \in \mathbb{R}_{>0}$ such that

$$\lambda(m) \leq c \|nmk\|^r \quad (6.9)$$

for $(m, n, k) \in M(\mathbb{A}_F) \times N(\mathbb{A}_F) \times K$. Here c and r depend on λ , the choice of embedding ι , K , and P . Moreover, there exist $\lambda \in X^*(M)$ and $c \in \mathbb{R}_{>0}$ such that

$$\|nmsk\| \leq c\lambda(s) \quad (6.10)$$

for all $(n, m, s, k) \in \Omega_N \times \Omega_M \times A_{T_0}(t) \times K$. The real number c and character λ depend on Ω_N , Ω_M , t , K and ι . Let Δ be the set of simple roots defined by T_0 and P as explained above (2.25). Using (6.9) and (6.10), one deduces the following lemma [Bor07, §5.4]:

Lemma 6.3.2 *A function*

$$\varphi : [G] \longrightarrow \mathbb{C}$$

is of moderate growth if and only if for any Siegel set as above, there exists $(c, r) \in \mathbb{R}_{>0} \times \mathbb{R}$ (depending on Ω_N , Ω_M , t , K , and ι) such that

$$|\varphi(nmsk)| \leq c\alpha(s)^r$$

for all $(n, m, s, k) \in \Omega_N \times \Omega_M \times A_{T_0}(t) \times K$ and $\alpha \in \Delta$.

□

Let $K = K_\infty K^\infty \leq G(\mathbb{A}_F)$ where $K_\infty \leq G(F_\infty)$ is a maximal compact subgroup and $K^\infty \leq G(\mathbb{A}_F^\infty)$ is a compact open subgroup. A $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -**module** (π, V) is a complex vector space V equipped with homomorphisms

$$\begin{aligned}\pi : \mathfrak{g} &\longrightarrow \text{End}(V), \\ \pi : K_\infty &\longrightarrow \text{Aut}(V)\end{aligned}\tag{6.11}$$

giving V the structure of a $(\mathfrak{g}, K_\infty)$ -module together with a homomorphism

$$\pi : G(\mathbb{A}_F^\infty) \longrightarrow \text{Aut}(V)\tag{6.12}$$

such that $\pi(\mathfrak{g})$ (resp. $\pi(K_\infty)$) commutes with $\pi(G(\mathbb{A}_F^\infty))$.

A **morphism** of $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -modules is a complex linear map commuting with the actions of $\mathfrak{g}, K_\infty$, and $G(\mathbb{A}_F^\infty)$. A $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -module is **irreducible** if it does not contain any nonzero proper invariant submodule. We do not assume that V is endowed with a topology, so in the definition of a $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -module, elements of $\text{End}(V)$ and $\text{Aut}(V)$ are not required to be continuous, morphisms of $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -modules are not required to be continuous, and submodules of a $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -module are not required to be closed. A $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -module is **admissible** if for each compact open subgroup $K^\infty \leq G(\mathbb{A}_F^\infty)$, the space of fixed vectors V^{K^∞} is admissible as a $(\mathfrak{g}, K_\infty)$ -module.

We say that a function $\varphi : G(\mathbb{A}_F) \rightarrow \mathbb{C}$ is **K -finite** if the $\mathbb{C}$ -span of the translates

$$\{x \longmapsto \varphi(xk) : k \in K\}$$

is finite dimensional. An alternate way of phrasing this condition is given in Exercise 6.4.

Definition 6.3.3. A smooth function

$$\varphi : G(\mathbb{A}_F) \longrightarrow \mathbb{C}$$

of moderate growth is an **automorphic form** on G with respect to K if it is left $G(F)$ -invariant, K -finite, and $Z(\mathfrak{g})$ -finite. The $\mathbb{C}$ -vector space of automorphic forms on G with respect to K is denoted by $\mathcal{A}(G, K)$.

The definition of an automorphic form depends on the choice of maximal compact subgroup K_∞ , but by Exercise 6.4, it does not depend on the choice of compact open subgroup $K^\infty < G(\mathbb{A}_F^\infty)$. The dependence is not so serious. Indeed, any two maximal compact subgroups $K_\infty \leq G(F_\infty)$ are $G(F_\infty)$ -conjugate by Theorem 4.4.2. Moreover, for all $g \in G(F_\infty)$, there is a $Z(\mathfrak{g}) \times G(\mathbb{A}_F^\infty)$ -equivariant isomorphism

$$\begin{aligned}\mathcal{A}(G, K_\infty K^\infty) &\xrightarrow{\sim} \mathcal{A}(G, (gK_\infty g^{-1})K^\infty) \\ \varphi &\longmapsto (x \mapsto \varphi(xg)).\end{aligned}$$

We will often omit G and/or K from notation if they are understood, writing

$$\mathcal{A} := \mathcal{A}(G) := \mathcal{A}(G, K). \quad (6.13)$$

The space $\mathcal{A}$ is equipped with an action of $(\mathfrak{g}, K_\infty)$ and an action of $G(\mathbb{A}_F^\infty)$ by right translations:

$$\begin{aligned} (\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty) \times \mathcal{A} &\longrightarrow \mathcal{A} \\ (X, k, g, \varphi) &\longmapsto (x \mapsto X\varphi(xkg)). \end{aligned} \quad (6.14)$$

It is important to note that the action of $G(\mathbb{A}_F^\infty)$ does not extend to an action of $G(\mathbb{A}_F)$. Indeed, if we let $\varphi \in \mathcal{A}$ and let $g \in G(F_\infty)$, then $x \mapsto \varphi(xg)$ is smooth, but in general, it is only finite under $gK_\infty g^{-1}$, not K_∞ .

If $J \leq Z(\mathfrak{g})$ is an ideal with $\dim_{\mathbb{C}} Z(\mathfrak{g})/J < \infty$, then the subspace of automorphic forms annihilated by J is denoted by $\mathcal{A}(J)$. If σ is an irreducible representation of K_∞ , we denote by $\mathcal{A}(\sigma)$ (resp. $\mathcal{A}(J, \sigma)$) the σ -isotypic subspace of $\mathcal{A}$ (resp. $\mathcal{A}(J)$). We have

$$\mathcal{A} := \bigcup_J \mathcal{A}(J) \quad (6.15)$$

where the sum is over ideals J of $Z(\mathfrak{g})$ such that $\dim_{\mathbb{C}} Z(\mathfrak{g})/J < \infty$. Moreover

$$\mathcal{A}(J) = \bigoplus_{\sigma \in \widehat{K_\infty}} \mathcal{A}(J, \sigma) \quad (6.16)$$

where the algebraic direct sum is over equivalence classes of irreducible representations of K_∞ . The action of $G(\mathbb{A}_F^\infty)$ preserves these spaces and the action of $(\mathfrak{g}, K_\infty)$ preserves $\mathcal{A}(J)$ (but of course not $\mathcal{A}(J, \sigma)$ in general).

The following is the adelic analogue of Theorem 6.2.6:

Theorem 6.3.4 *Let $J \leq Z(\mathfrak{g})$ be an ideal with $\dim_{\mathbb{C}} Z(\mathfrak{g})/J < \infty$. Then $\mathcal{A}(J)$ is an admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -module.*

Proof. We are to show that $\mathcal{A}(J, \sigma)^{K^\infty}$ is finite dimensional for each K_∞ -type σ and each compact open subgroup $K^\infty \leq G(\mathbb{A}_F^\infty)$. We start by recalling the description of the adelic quotient given by (2.23). Letting $t_1, \dots, t_h$ denote a set of representatives for the finite set $G(F) \backslash G(\mathbb{A}_F^\infty)/K^\infty$, one has a homeomorphism

$$\prod_{i=1}^h \Gamma_i(K^\infty) \backslash G(F_\infty) \longrightarrow G(F) \backslash G(\mathbb{A}_F)/K^\infty$$

given on the i th component by $\Gamma_i(K^\infty)g_\infty \mapsto G(F)g_\infty t_i K^\infty$ where $\Gamma_i(K^\infty) = G(F) \cap t_i K^\infty t_i^{-1}$. By pullback we obtain an isomorphism of $\mathbb{C}$ -vector spaces

$$\begin{aligned} \mathcal{A}(J)^{K^\infty} &\xrightarrow{\sim} \bigoplus_{i=1}^h \mathcal{A}(\Gamma_i(K^\infty), J) \\ \varphi &\longmapsto (x_1 \mapsto \varphi(x_1 t_1), \dots, x_h \mapsto \varphi(x_h t_h)) \end{aligned} \quad (6.17)$$

which intertwines the natural action of $(\mathfrak{g}, K_\infty)$ on both sides. We can now deduce the theorem by applying Theorem 6.2.6. $\square$

In view of the previous theorem the following definition is reasonable:

Definition 6.3.5. An **automorphic representation** of $G(\mathbb{A}_F)$ is an irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -module isomorphic to a subquotient of $\mathcal{A}$.

Two automorphic representations are said to be **isomorphic** if they are isomorphic in the category of $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -modules.

We defined discrete automorphic representations in Definition 3.7.1. We will discuss the precise relationship between discrete automorphic representations and general automorphic representations in §6.6 below.

Discrete automorphic representations are defined in terms of irreducible subrepresentations of $L^2([G])$. We warn the reader that there are certainly automorphic representations that are most naturally described as subquotients of $\mathcal{A}$, not submodules (see §10.7, for example). We do not know whether or not every automorphic representation is in fact a submodule (and not merely a subquotient) of $\mathcal{A}$.

Let us end the section by discussing factorization. We leave the proof of the following lemma as an exercise (see Exercise 6.7).

Lemma 6.3.6 *Every irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -module (π, V) is an algebraic tensor product*

$$(\pi, V) = (\pi_\infty \otimes \pi^\infty, V_\infty \otimes V^\infty)$$

where (π_∞, V_∞) is an irreducible admissible $(\mathfrak{g}, K_\infty)$ -module and (π^∞, V^∞) is an irreducible admissible representation of $G(\mathbb{A}_F^\infty)$. The isomorphism classes of (π_∞, V_∞) and (π^∞, V^∞) are uniquely determined by (π, V) . $\square$

Assume that $K_\infty = \prod_{v|\infty} K_v$ is a maximal compact subgroup of $G(F_\infty)$ and $K^\infty = \prod_{v \nmid \infty} K_v$ is a compact open subgroup of $G(\mathbb{A}_F^\infty)$.

Theorem 6.3.7 *Every irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -module (π, V) admits a factorization*

$$(\pi, V) = (\otimes_{v|\infty} \pi_v, \otimes_{v|\infty} V_v) \otimes (\otimes'_{v \nmid \infty} \pi_v, \otimes'_{v \nmid \infty} V_v)$$

where (π_v, V_v) is an irreducible admissible $(\mathfrak{g}_v, K_v)$ -module for $v|\infty$ and is an irreducible admissible representation of $G(F_v)$ for $v \nmid \infty$. The restricted tensor product is taken with respect to a choice of nonzero vectors $\varphi_v \in V_v^{K_v}$ for all

but finitely many places v . The factors (π_v, V_v) are uniquely determined by (π, V) up to isomorphism.

Proof. By Lemma 6.3.6, we have

$$(\pi, V) = (\pi_\infty \otimes \pi^\infty, V_\infty \otimes V^\infty)$$

with notation as in that lemma. One has that

$$(\pi_\infty, V_\infty) \cong (\otimes_{v|\infty} \pi_v, \otimes_{v|\infty} V_v)$$

by Exercise 4.14 and

$$(\pi^\infty, V^\infty) \cong (\otimes'_{v \nmid \infty} \pi_v, \otimes'_{v \nmid \infty} V_v)$$

by Flath's theorem (Theorem 5.7.1). The uniqueness statement follows from the corresponding uniqueness statements in each of the results cited in the proof. $\square$

6.4 Adelic automorphic forms over function fields

We assume throughout this section that F is a function field. The definition of an automorphic form in the function field case is fairly straightforward.

Definition 6.4.1. A function

$$\varphi : G(\mathbb{A}_F) \longrightarrow \mathbb{C}$$

invariant on the left under $G(F)$ and on the right under a compact open subgroup of $G(\mathbb{A}_F)$ is an **automorphic form** on G if the $\mathbb{C}$ -vector space spanned by the functions

$$\{x \longmapsto \varphi(xg) : g \in G(\mathbb{A}_F)\}$$

is an admissible representation of $G(\mathbb{A}_F)$.

As in the number field case, we denote by $\mathcal{A}$ or $\mathcal{A}(G)$ the $\mathbb{C}$ -vector space of automorphic forms. It is naturally equipped with an action of $G(\mathbb{A}_F)$.

Let $K = \prod_v K_v \leq G(\mathbb{A}_F)$ be a compact open subgroup. As in the number field case, a function $\varphi : G(\mathbb{A}_F) \rightarrow \mathbb{C}$ is said to be **K -finite** if the $\mathbb{C}$ -span of the translates

$$\{x \longmapsto \varphi(xk) : k \in K\}$$

is finite dimensional. We observe that any element of $\mathcal{A}$ is K -finite for any compact open subgroup $K \leq G(\mathbb{A}_F)$. This is in sharp contrast to the number field case, in which $\mathcal{A}$ depends on K_∞ .

Definition 6.4.2. An **automorphic representation** of $G(\mathbb{A}_F)$ is an irreducible admissible representation of $G(\mathbb{A}_F)$ that is isomorphic to a subquotient of $\mathcal{A}$.

In analogy with the number field case, two automorphic representations of $G(\mathbb{A}_F)$ are **isomorphic** if they are isomorphic in the category of admissible representations of $G(\mathbb{A}_F)$.

Theorem 6.4.3 *Every irreducible admissible $G(\mathbb{A}_F)$ -module (π, V) admits a factorization*

$$(\pi, V) = (\otimes'_v \pi_v, \otimes'_v V_v)$$

where (π_v, V_v) is an irreducible admissible representation of $G(F_v)$ for each place v . The restricted tensor product is taken with respect to a choice of nonzero vectors $\varphi_v \in V_v^{K_v}$ for all but finitely many places v . The factors (π_v, V_v) are uniquely determined by (π, V) up to isomorphism.

Proof. Combine Flath's theorem (Theorem 5.7.1) with Theorem 5.7.2. $\square$

This is the analogue of Theorem 6.3.7 in the current context.

6.5 The cuspidal subspace

Let $K \leq G(\mathbb{A}_F)$ be a maximal compact subgroup. When F is a number field we use K to define $\mathcal{A} = \mathcal{A}(G, K)$ (no choice of K is necessary when F is a function field). We now isolate what is arguably the most important subspace of $\mathcal{A}$:

Definition 6.5.1. An automorphic form $\varphi \in \mathcal{A}$ is said to be **cuspidal** or a **cusp form** if for every proper parabolic subgroup $P < G$ with unipotent radical N , one has that

$$\int_{[N]} \varphi(ng)dn = 0 \tag{6.18}$$

for all $g \in G(\mathbb{A}_F)$.

We recall that $[N]$ is compact (see Theorem 2.6.3, for instance) so the integral in the definition is well-defined. The subspace of $\mathcal{A}$ consisting of cuspidal forms will be denoted $\mathcal{A}_{\text{cusp}}(G)$ or $\mathcal{A}_{\text{cusp}}$ if G is understood. By Exercise 6.8, to check that a function is a cuspidal it suffices to fix a minimal parabolic subgroup (and hence a set of standard parabolic subgroups) and check that (6.18) vanishes whenever N is the unipotent radical of a standard parabolic subgroup. In particular, one only needs to check that (6.18) vanishes for a finite set of P .

Before continuing, we motivate the definition of a cusp form following the exposition in [God66]. It is natural to try and reduce the study of $\mathcal{A}(G)$, or subspaces of it, to spaces of functions on simpler groups. For example, we could consider Levi subgroups of G . Given a parabolic subgroup P with Levi subgroup M and unipotent radical N and a $\varphi \in \mathcal{A}(G)$, we can define the **constant term**

$$\varphi_P(m) := \int_{[N]} \varphi(nm) dn. \quad (6.19)$$

Here $m \in M(F) \backslash M(\mathbb{A}_F)$. The constant term gives us a map

$$(\cdot)_P : \mathcal{A}(G) \longrightarrow \mathcal{A}(M).$$

Here in the number field case, we assume that the maximal compact subgroup $K_M \leq M(\mathbb{A}_F)$ implicit in the definition of $\mathcal{A}(M)$ satisfies

$$K_{M\infty} := K_\infty \cap M(F_\infty).$$

This is no essential loss of generality by Theorem B.4.1.

One has that

$$\mathcal{A}_{\text{cusp}}(G) \leq \ker(\cdot)_P$$

for all proper parabolic subgroups $P < G$. Thus, cusp forms cannot be related to automorphic forms on Levi subgroups via taking constant terms. This discussion and refinements of it are the motivation for the philosophy of cusp forms mentioned in §8.1 below.

Definition 6.5.2. A **cuspidal automorphic representation** of $G(\mathbb{A}_F)$ is an automorphic representation equivalent to a subquotient of $\mathcal{A}_{\text{cusp}}$.

One justification for our claim that $\mathcal{A}_{\text{cusp}}$ is the most important subspace of $\mathcal{A}$ is that $L^2([G])$ admits a decomposition in terms of $\mathcal{A}_{\text{cusp}}(M)$ as M runs over the Levi subgroups of parabolic subgroups of G . See §10.5 below.

The **cuspidal subspace**

$$L^2_{\text{cusp}}([G]) \leq L^2([G]) \quad (6.20)$$

is the space of $\varphi \in L^2([G])$ such that for every proper parabolic subgroup $P < G$ with unipotent radical N , one has that

$$\int_{[N]} \varphi(ng) dn = 0 \quad (6.21)$$

for almost every $g \in A_G \backslash G(\mathbb{A}_F)$ with respect to a Haar measure on $A_G \backslash G(\mathbb{A}_F)$. An element of $L^2([G])$ is **cuspidal** if it is contained in $L^2_{\text{cusp}}([G])$. We will show in Lemma 9.2.1 that $L^2_{\text{cusp}}([G])$ is closed, and hence is an $A_G \backslash G(\mathbb{A}_F)$ -subrepresentation of $L^2([G])$. We will show in Corollary 9.1.3 below that one has a decomposition

$$L_{\text{cusp}}^2([G]) = \widehat{\bigoplus_{\pi} L_{\text{cusp}}^2(\pi)}$$

where $L_{\text{cusp}}^2(\pi)$ is the π -isotypic subspace of $L_{\text{cusp}}^2([G])$ and the Hilbert space direct sum is over all isomorphism classes of cuspidal automorphic representations π of $A_G \backslash G(\mathbb{A}_F)$.

Let us say, temporarily, that an irreducible subrepresentation of $L_{\text{cusp}}^2([G])$ is a **discrete cuspidal automorphic representation**. Discrete cuspidal automorphic representations and cuspidal automorphic representations (in the sense of Definition 6.5.2) are closely related. To explain the relationship, we observe that though $G(\mathbb{A}_F)$ does not act on $\mathcal{A}$ in the number field case, for any global field the subgroup A_G does act on $\mathcal{A}$ and the action preserves $\mathcal{A}_{\text{cusp}}$. An **automorphic representation** of $A_G \backslash G(\mathbb{A}_F)$ is an automorphic representation of $G(\mathbb{A}_F)$ that is isomorphic to a subquotient of $\mathcal{A}^{A_G}$. A **cuspidal automorphic representation** of $A_G \backslash G(\mathbb{A}_F)$ is an automorphic representation of $G(\mathbb{A}_F)$ isomorphic to a subquotient of $\mathcal{A}_{\text{cusp}}^{A_G}$. For any cuspidal automorphic representation π , we denote by $\mathcal{A}_{\text{cusp}}^{A_G}(\pi)$ the π -isotypic component of $\mathcal{A}_{\text{cusp}}^{A_G}$. When F is a number field (resp. function field) this is the algebraic direct sum of all irreducible admissible ($\mathfrak{g}, K_{\infty}) \times G(\mathbb{A}_F^{\infty})$ -submodules (resp. smooth $G(\mathbb{A}_F)$ -subrepresentations) of $\mathcal{A}_{\text{cusp}}^{A_G}$ that are equivalent to π . We emphasize that $\mathcal{A}_{\text{cusp}}^{A_G}(\pi)$ is a subspace of $\mathcal{A}_{\text{cusp}}^{A_G}$, not merely a subquotient.

To avoid repetition, in the following theorem, we will refer to a (discrete) cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ simply as a (discrete) cuspidal automorphic representation.

Theorem 6.5.3 *The space of cuspidal automorphic forms $\mathcal{A}_{\text{cusp}}^{A_G}$ is a dense subspace of $L_{\text{cusp}}^2([G])$. If (π, V) is a discrete cuspidal automorphic representation, then the space of K -finite vectors V_{fin} in V is a cuspidal automorphic representation (π, V_{fin}) . One has*

$$\mathcal{A}_{\text{cusp}}^{A_G}(\pi) = L_{\text{cusp}}^2(\pi)_{\text{fin}},$$

the space of K -finite vectors in $L_{\text{cusp}}^2(\pi)$. The multiplicity of (π, V_{fin}) in $\mathcal{A}_{\text{cusp}}^{A_G}(\pi)$ is finite and equal to the multiplicity of (π, V) in $L_{\text{cusp}}^2(\pi)$. One has that

$$\mathcal{A}_{\text{cusp}}^{A_G} = \bigoplus_{\pi} \mathcal{A}_{\text{cusp}}^{A_G}(\pi) \quad (6.22)$$

where the (algebraic) sum is over isomorphism classes of cuspidal automorphic representations. The association of (π, V_{fin}) to (π, V) defines a bijection between isomorphism classes of discrete cuspidal automorphic representations and isomorphism classes of cuspidal automorphic representations.

Because of Theorem 6.5.3 we will drop the adjective “discrete” from discrete cuspidal automorphic representation when there is no danger of confusion.

The decomposition (6.22) implies in particular that a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ is in fact a submodule of $\mathcal{A}_{\text{cusp}}^{A_G}$, though the original definition only required it to be a subquotient. We defer the proof of Theorem 6.5.3 to §9.7, where it is restated as Theorem 9.7.3.

In general, the relationship between $\mathcal{A}$ and subspaces of $L^2([G])$ is more complicated than Theorem 6.5.3 might lead one to believe. See Theorem 10.5.1 for more information.

6.6 Discrete automorphic representations revisited

In §3.7, we defined a discrete automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ to be an irreducible unitary representation of $A_G \backslash G(\mathbb{A}_F)$ equivalent to a subrepresentation of $L^2([G])$. In the number field case, we defined an automorphic representation in terms of irreducible admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -modules in Definition 6.3.5. In the function field case, we defined an automorphic representation in terms of irreducible admissible $G(\mathbb{A}_F)$ -modules in Definition 6.4.2. In this section, we prove Theorem 6.6.4, which describes the precise relationship between these definitions.

We begin with considerations that do not require the space $[G]$. Thus, let (π, V) be an irreducible unitary representation of $G(\mathbb{A}_F)$.

Theorem 6.6.1 *One has that*

$$(\pi, V) \cong (\pi_\infty \otimes \pi^\infty, V_\infty \widehat{\otimes} V^\infty) \quad (6.23)$$

where (π_∞, V_∞) is an irreducible unitary representation of $G(F_\infty)$ and (π^∞, V^∞) is an irreducible unitary representation of $G(\mathbb{A}_F^\infty)$. The representations (π_∞, V_∞) and (π^∞, V^∞) are determined up to unitary equivalence by (π, V) .

In the theorem, $V_\infty \widehat{\otimes} V^\infty$ is the Hilbert space tensor product of V_∞ and V^∞ . The action of $\pi_\infty \otimes \pi^\infty$ on the dense subspace $V_\infty \otimes V^\infty \leq V_\infty \widehat{\otimes} V^\infty$ extends by continuity.

Proof. The first assertion is [Fol95, Theorem 7.25] given Theorem 3.10.2. The second assertion is [Fol95, Corollary 7.22]. $\square$

We let $K = K_\infty K^\infty$ where $K_\infty \leq G(F_\infty)$ is a maximal compact subgroup and $K^\infty \leq G(\mathbb{A}_F^\infty)$ is a compact open subgroup. Let

$$V_{\text{fin}} \leq V, \quad V_{\infty \text{fin}} \leq V_\infty, \quad V_{\text{fin}}^\infty \leq V^\infty \quad (6.24)$$

be the spaces of K -finite, K_∞ -finite, and K^∞ -finite vectors, respectively. We observe that these definitions are independent of the choice of K^∞ , but in the number field case depend on the choice of K_∞ .

Theorem 6.6.2 *The space V_{fin} is admissible and irreducible. One has that*

$$(\pi_\infty, V_{\text{fin}}) \cong (\pi_\infty \otimes \pi^\infty, V_{\infty\text{fin}} \otimes V_{\text{fin}}^\infty).$$

If (π_1, V_1) and (π_2, V_2) are irreducible unitary representations of $G(\mathbb{A}_F)$ and $V_{1\text{fin}}$ is isomorphic to $V_{2\text{fin}}$, then (π_1, V_1) and (π_2, V_2) are unitarily equivalent.

Here when we say V_{fin} is admissible (resp. irreducible), we mean as a $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -module in the number field case and as a smooth representation of $G(\mathbb{A}_F)$ in the function field case. Similarly, when we say $V_{1\text{fin}}$ is isomorphic to $V_{2\text{fin}}$, we mean as $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -modules in the number field case and as smooth $G(\mathbb{A}_F)$ -representations in the function field case. We use the same convention for the remainder of this section.

Proof. Let τ be a K_∞ -type and let $K^\infty \leq G(\mathbb{A}_F^\infty)$ be a compact open subgroup. The isomorphism (6.23) induces an isomorphism

$$V_{\text{fin}}(\tau)^{K^\infty} \cong V_{\infty\text{fin}}(\tau) \widehat{\otimes} V^{\infty K^\infty}. \quad (6.25)$$

The space $V_{\infty\text{fin}}(\tau)$ is finite dimensional by Theorem 4.4.5 and the space $V^{\infty K^\infty}$ is finite dimensional by Theorem 5.3.9 and Corollary 5.5.6. Thus, the tensor product in (6.25), which a priori is a completed tensor product in the sense of Hilbert spaces, is in fact algebraic, and $V_{\text{fin}}(\tau)^{K^\infty}$ is finite dimensional. This implies V_{fin} is admissible. Irreducibility follows from Lemma 4.4.9 and Proposition 5.3.8. Taking the union over K_∞ -types τ and $K^\infty \leq G(\mathbb{A}_F^\infty)$, we deduce the isomorphism asserted in the theorem.

The last assertion follows from Theorem 4.4.10, Proposition 5.3.8, and Theorem 6.6.1. $\square$

In the case where F is a number field, passing to K_∞ -finite vectors is often awkward because a general element of $G(F_\infty)$ will not preserve this subspace. With this in mind, it is often useful to work with smooth vectors. Let (π, V) be an irreducible unitary representation of $G(\mathbb{A}_F)$. If F is a function field, then the space of smooth vectors V_{sm} is just defined to be V_{fin} ; it is stable under $G(\mathbb{A}_F)$. If F is a number field, the space of **smooth vectors** V_{sm} is the space of vectors $\varphi \in V$ that are smooth under the action of $G(F_\infty)$ in the sense of §4.2 and K^∞ -finite under the action of $G(\mathbb{A}_F^\infty)$. Unlike the subspace $V_{\text{fin}} < V$ of K -finite vectors, the space of smooth vectors $V_{\text{sm}} < V$ is preserved under $G(\mathbb{A}_F)$.

Theorem 6.6.3 *One has that*

$$(\pi, V_{\text{sm}}) \cong (\pi_\infty \otimes \pi^\infty, V_{\infty\text{sm}} \otimes V_{\text{fin}}^\infty),$$

where the product is algebraic.

Proof. For any compact open subgroup $K^\infty \leq G(\mathbb{A}_F^\infty)$, by restricting the isomorphism (6.23), we have

$$V^{K^\infty} \cong V_\infty \widehat{\otimes} V^{\infty K^\infty} = V_\infty \otimes V^{\infty K^\infty}$$

since $V^{\infty K^\infty}$ is finite dimensional by admissibility (see Theorem 5.3.9 and Corollary 5.5.6). Thus

$$V_{\text{sm}}^{K^\infty} \cong V_{\infty\text{sm}} \otimes V^{\infty K^\infty}.$$

Taking the union over K^∞ , we deduce the result. □

The discussion above is valid for any unitary representation. In particular, it is valid for discrete automorphic representations. This implies most of the following theorem:

Theorem 6.6.4 *The space of K -finite vectors in a discrete automorphic representation is an automorphic representation of $A_G \backslash G(\mathbb{A}_F)$. Two discrete automorphic representations are unitarily equivalent if and only if the associated automorphic representations are equivalent.*

Proof. Suppose we are given a discrete automorphic representation (π, V) . Thus, we have a continuous $A_G \backslash G(\mathbb{A}_F)$ -equivariant injection of Hilbert spaces

$$V \longrightarrow L^2([G]).$$

In particular, we can and do view elements of V as functions on $[G]$.

The space V_{fin} is admissible and irreducible by Theorem 6.6.2. Thus, it suffices to show that V_{fin} lies in $\mathcal{A}^{A_G}$. In the function field setting, this follows from admissibility. In the number field setting, we can apply Proposition 4.4.7 to deduce that the elements of V_{fin} are smooth. They are $Z(\mathfrak{g})$ -finite since any irreducible admissible $(\mathfrak{g}, K_\infty)$ -module admits an infinitesimal character by Corollary 4.6.3. Finally, they are of moderate growth by Lemma 4.4.13. □

In view of Theorem 6.6.4, when studying irreducible subrepresentations of $L^2([G])$, we can work with either the definition of automorphic representations in terms of admissible representations presented in this chapter, or discrete automorphic representations in the sense of Definition 3.7.1. Both have advantages and drawbacks. The definition of a discrete automorphic representation is more direct. However, in view of Lemma 3.8.1 and Langlands' decomposition of $L^2([G])$ explained in §10.5, this definition has the disadvantage that the full decomposition of $L^2([G])$ cannot in general be described in terms of discrete automorphic representations of $A_G \backslash G(\mathbb{A}_F)$. Moreover, this definition excludes most Eisenstein series (see Chapter 10 for an introduction to Eisenstein series). The definition of automorphic representations in terms of admissible representations has the advantage that one can immediately reduce questions to finite dimensional settings, either by passing to K_∞ -types or vectors fixed under a compact open subgroup. It also has some drawbacks, mostly due to the fact that, in the number field case, automorphic representations are not representations of $G(F_\infty)$. One way around this difficulty is through the use of representations of moderate growth in the sense of Definition 4.4.12.

Assume F is a number field. We warn the reader that later in this book we will often not be specific about whether we are viewing a discrete automorphic representation as a unitary representation of $A_G \backslash G(\mathbb{A}_F)$ or as an admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -module. Similarly, if F is a function field, we will often not be specific about whether we are viewing a discrete automorphic representation as a unitary representation of $A_G \backslash G(\mathbb{A}_F)$ or as an admissible representation of $A_G \backslash G(\mathbb{A}_F)$. Indeed, in practice one often has to switch back and forth between the two perspectives. The particular point of view must be deduced from the context.

6.7 From modular forms to automorphic forms

For readers familiar with modular forms, we make the connection between modular forms and automorphic forms precise in this section.

Let

$$\begin{aligned}\Gamma_0(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) : N|c \right\}, \\ \Gamma_1(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) : N|c, N|(d-1) \right\}.\end{aligned}\tag{6.26}$$

Thus, $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$. Let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ be a character. For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, d is relatively prime to N . Thus, we have a well-defined map

$$\begin{aligned}\Gamma_0(N)/\Gamma_1(N) &\longrightarrow \mathbb{C}^\times \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto \chi(d)\end{aligned}$$

which is easily checked to be a homomorphism. We continue to denote this map by χ .

Let

$$\mathrm{GL}_2(\mathbb{R})^+ := \{g \in \mathrm{GL}_2(\mathbb{R}) : \det g > 0\}.\tag{6.27}$$

Let $\mathfrak{H}$ be the complex upper half plane. One has an action

$$\begin{aligned}\mathrm{GL}_2(\mathbb{R})^+ \times \mathfrak{H} &\longrightarrow \mathfrak{H} \\ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) &\longmapsto \frac{az + b}{cz + d}.\end{aligned}\tag{6.28}$$

The operators attached to a given $g \in \mathrm{GL}_2(\mathbb{R})^+$ by this action are called Möbius transformations or fractional linear transformations. Let

$$\Gamma_0(N)^+ := \Gamma_0(N) \cap \mathrm{GL}_2(\mathbb{R})^+ = \Gamma_0(N) \cap \mathrm{SL}_2(\mathbb{Z}).\tag{6.29}$$

We warn the reader that in most references $\Gamma_0(N)^+$ is denoted $\Gamma_0(N)$.

We recall the definition of a modular form:

Definition 6.7.1. Let $\ell \in \mathbb{Z}_{>0}$. The space of **weight ℓ modular forms of level N and nebentypus χ** is the space $M_\ell(N, \chi)$ of holomorphic functions $f : \mathfrak{H} \rightarrow \mathbb{C}$ satisfying the following conditions:

- (a) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^\ell \chi(d) f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^+$ and for all $z \in \mathfrak{H}$,
- (b) f extends holomorphically to the cusps of $\Gamma_0(N)^+$.

If f additionally vanishes at the cusps of $\Gamma_0(N)^+$, we say that f is a **cusp form**. The space of weight ℓ cusp forms of level N and nebentypus χ is denoted $S_\ell(N, \chi)$.

There are many books on modular forms that the reader can consult for more details, including the definition of the cusps of $\Gamma_0(N)^+$. We mention [DS05, GG12, Kob93, Lan95, Miy06, Ser73, Shi94]. A recent extension of the theory to larger classes of functions that arise naturally in applications is explained in [BFOR17].

We remark that if ℓ is even then $M_\ell(N, \chi)$ can be identified with a certain space of holomorphic differential forms. This is explained in great detail in a slightly more general setting in [GG12, Chapter 6]. Thus, it is reasonable to call elements of $M_\ell(N, \chi)$ modular “forms,” although this is not the origin of the term “modular form” [EvdGM08].

The spaces $M_\ell(N, \chi)$ are finite dimensional [Shi94, Theorem 2.23]. For each $p \nmid N$, define the p th **Hecke operator**

$$T_p f(z) := \sum_{\gamma} \frac{p^{\ell-1}}{(cz+d)^\ell \chi(d)} f\left(\frac{az+b}{cz+d}\right), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Here the sum on γ is over the finite set

$$\Gamma_0(N)^+ \backslash \Gamma_0(N)^+ \begin{pmatrix} p & \\ & 1 \end{pmatrix} \Gamma_0(N)^+. \quad (6.30)$$

One checks that $T_p f \in M_\ell(N, \chi)$ and that T_p preserves $S_\ell(N, \chi)$. A **Hecke eigenform** is an element $f \in M_\ell(N, \chi)$ such that

$$T_p f = \lambda_f(p) f$$

for all $p \nmid N$. Here $\lambda_f(p) \in \mathbb{C}$ are known as the p th **Hecke eigenvalue** of f . There exists a basis of $M_\ell(N, \chi)$ consisting of Hecke eigenforms [Kob93, §III.5]. We point out that the Hecke eigenvalues do not determine f . This is the point of departure for the classical theory of newforms and oldforms.

We now relate the space $S_\ell(N, \chi)$ to a space of automorphic forms. Let

$$\begin{aligned} K_0(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : N|c \right\}, \\ K_1(N) &:= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : N|c, N|(d-1) \right\}. \end{aligned}$$

Then

$$\Gamma_0(N) = K_0(N) \cap \mathrm{GL}_2(\mathbb{Q}) \quad \text{and} \quad \Gamma_1(N) = K_1(N) \cap \mathrm{GL}_2(\mathbb{Q}).$$

We observe that the image of $K_0(N)$ under the determinant homomorphism $\det : \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty}) \rightarrow \mathbb{A}_{\mathbb{Q}}^{\infty \times}$ is

$$\det K_0(N) = \widehat{\mathbb{Z}}^{\times}. \quad (6.31)$$

One also has an isomorphism

$$\begin{aligned} \mathrm{lr} : K_0(N)/K_1(N) &\longrightarrow \prod_{p|N} \mathbb{Z}_p^{\times} / (1 + p\mathbb{Z}_p) = (\mathbb{Z}/N)^{\times} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto d. \end{aligned}$$

The lr stands for “lower right.”

The key consequence of (6.31) is that it implies that the homeomorphism (2.23) is

$$\Gamma_0(N) \backslash \mathrm{GL}_2(\mathbb{R}) \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) / K_0(N). \quad (6.32)$$

In other words, there is only one component on the left of (2.23) in this case. In fact, one can do even better. There exists an element in $\Gamma_0(N)$ of negative determinant. Thus, (6.32) implies that there is a homeomorphism

$$\begin{aligned} \Gamma_0(N)^+ \backslash \mathrm{GL}_2(\mathbb{R})^+ &\longrightarrow \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) / K_0(N) \\ \Gamma_0(N)^+ g &\longmapsto \mathrm{GL}_2(\mathbb{Q})(g, I_2) K_0(N), \end{aligned} \quad (6.33)$$

where I_2 is the identity of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$.

Because of the nebentypus character χ , we slightly modify (6.33) as follows. The group $\Gamma_0(N)$ may be viewed as a subgroup of $K_0(N)$ via the diagonal embedding $\mathrm{GL}_2(\mathbb{Z}) \hookrightarrow \mathrm{GL}_2(\widehat{\mathbb{Z}})$. Thus, the group $\Gamma_0(N)^+$ acts diagonally on the left on $\mathrm{GL}_2(\mathbb{R})^+ \times K_0(N)$. We therefore obtain a map

$$\begin{aligned} \Gamma_0(N)^+ \backslash (\mathrm{GL}_2(\mathbb{R})^+ \times K_0(N)/K_1(N)) &\longrightarrow \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) / K_1(N) \\ \Gamma_0(N)^+(g, kK_1(N)) &\longmapsto \mathrm{GL}_2(\mathbb{Q})(g, k) K_1(N). \end{aligned} \quad (6.34)$$

The fact that (6.33) is a homeomorphism implies that (6.34) is a homeomorphism as well.

The stabilizer of i under (6.28) is $A_{\mathrm{GL}_2} \mathrm{SO}_2(\mathbb{R})$. We therefore obtain a quotient map

$$A_{\mathrm{GL}_2} \backslash \mathrm{GL}_2(\mathbb{R})^+ \longrightarrow A_{\mathrm{GL}_2} \backslash \mathrm{GL}_2(\mathbb{R})^+ / \mathrm{SO}_2(\mathbb{R}) = \mathfrak{H}. \quad (6.35)$$

In this setting,

$$A_{\mathrm{GL}_2} := \left\{ \begin{pmatrix} r & \\ & r \end{pmatrix} : r \in \mathbb{R}_{>0} \right\}.$$

We set

$$j \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) = (ad - bc)^{-1/2} (cz + d) \quad (6.36)$$

for

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \in \mathrm{GL}_2(\mathbb{R})^+ \times \mathfrak{H}.$$

One can check that

$$j(gg', z) = j(g, g'z)j(g', z) \quad \text{and} \quad j \left(r \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, i \right) = e^{-i\theta} \quad (6.37)$$

for all $g, g' \in \mathrm{GL}_2(\mathbb{R})^+$, $z \in \mathfrak{H}$, $(r, \theta) \in \mathbb{R}_{>0} \times \mathbb{R}$. Thus, for each integer ℓ , the function $j(g, z)^{-\ell}$ is an **automorphy factor** for the representation

$$\begin{aligned} A_{\mathbb{G}_m} \mathrm{SO}_2(\mathbb{R}) &\longrightarrow \mathbb{C}^\times \\ r \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} &\longmapsto e^{i\ell\theta} \end{aligned}$$

(see [GG12, §6.3] for details).

For $f \in S_\ell(N, \chi)$ set

$$\begin{aligned} \varphi_f : \mathrm{GL}_2(\mathbb{R})^+ \times K_0(N) &\longrightarrow \mathbb{C} \\ (g, k) &\longmapsto j(g, i)^{-\ell} f(gi) \chi^{-1}(\mathrm{lr}(k)). \end{aligned}$$

We observe that (6.37) implies that $\varphi_f(\gamma g, \gamma k) = \varphi_f(g, k)$ for $\gamma \in \Gamma_0(N)^+$. In view of (6.33) φ_f extends uniquely to a function on

$$\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) / K_0(N).$$

It is an automorphic form, but to specify the type we require further notation.

Consider the character

$$\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \longmapsto e^{i\ell\theta} \quad (6.38)$$

of $\mathrm{SO}_2(\mathbb{R})$. Let σ_ℓ be the induction of this character to a 2-dimensional representation of $\mathrm{O}_2(\mathbb{R})$. Let Δ be the Casimir element of $\mathfrak{gl}_2^{\mathbb{C}}$ (see §4.7). It and the element Z defined as in §4.7 generate $Z(\mathfrak{gl}_2)$. Thus we can consider the space of automorphic forms $\mathcal{A}(\langle \Delta - \frac{\ell}{2}(1 - \frac{\ell}{2}) \rangle, \sigma_\ell)$ as in (6.16), and within this space the subspace

$$\mathcal{A}_{\mathrm{cusp}} \left(\langle \Delta - \frac{\ell}{2}(1 - \frac{\ell}{2}) \rangle, \sigma_\ell \right)$$

of cusp forms. For any quasi-character $\omega : [\mathbb{G}_m] \rightarrow \mathbb{C}^\times$ we let

$$\mathcal{A}_{\mathrm{cusp}} \left(\langle \Delta - \frac{\ell}{2}(1 - \frac{\ell}{2}) \rangle, \sigma_\ell, \omega \right)$$

be the subspace of $\varphi \in \mathcal{A}_{\text{cusp}}(\langle \Delta - \frac{\ell}{2}(1 - \frac{\ell}{2}) \rangle, \sigma_\ell)$ such that $\varphi(zg) = \omega(z)\varphi(g)$ for $(z, g) \in Z_{\text{GL}_2}(\mathbb{A}_{\mathbb{Q}}) \times \text{GL}_2(\mathbb{A}_{\mathbb{Q}})$.

Let

$$K_{\mathbb{G}_m}(N) := \prod_{p|N} (1 + N\mathbb{Z}_p) \times \prod_{p \nmid N} \mathbb{Z}_p^\times. \quad (6.39)$$

This is a compact open subgroup of $\widehat{\mathbb{Z}}^\times$. The character $\chi : (\mathbb{Z}/N)^\times \rightarrow \mathbb{C}^\times$ induces a character $\omega_\chi : [\mathbb{G}_m] \rightarrow \mathbb{C}^\times$ by pushing forward along the isomorphism

$$\begin{aligned} (\mathbb{Z}/N)^\times &\xrightarrow{\sim} \prod_{p|N} \mathbb{Z}_p^\times / (1 + N\mathbb{Z}_p) \xrightarrow{\sim} A_{\mathbb{G}_m} \mathbb{Q}^\times \backslash \mathbb{A}_{\mathbb{Q}}^\times / K_{\mathbb{G}_m}(N) \\ k &\mapsto A_{\mathbb{G}_m} \mathbb{Q}^\times(1, k) K_{\mathbb{G}_m}(N). \end{aligned}$$

Proposition 6.7.2 *For each integer $\ell \geq 1$ one has a $\mathbb{C}$ -linear isomorphism of finite dimensional vector spaces*

$$\begin{aligned} S_\ell(N, \chi) &\longrightarrow \mathcal{A}_{\text{cusp}}(\langle \Delta - \frac{\ell}{2}(1 - \frac{\ell}{2}) \rangle, \sigma_\ell, \omega_\chi^{-1})^{K_1(N)} \\ f &\longmapsto \varphi_f. \end{aligned}$$

Proof. This is [KL08, Proposition 12.5]. We warn the reader that in loc. cit. the space $S_\ell(N, \chi)$ is denoted $S_\ell(N, \chi^{-1})$. $\square$

Let S be the union of the infinite place of $\mathbb{Q}$ and the places dividing N . Temporarily let

$$\mathbb{1}_{p^{-1}}^S := \mathbb{1}_{\text{GL}_2(\widehat{\mathbb{Z}}^S)}(p^{-1} \quad 1)_{\text{GL}_2(\widehat{\mathbb{Z}})} \in C_c^\infty(\text{GL}_2(\mathbb{A}_{\mathbb{Q}}^S) // \text{GL}_2(\widehat{\mathbb{Z}}^S)).$$

Then $R(\mathbb{1}_{p^{-1}}^S)$ is an operator on $\mathcal{A}_{\text{cusp}}(\langle \Delta - \frac{\ell}{2}(1 - \frac{\ell}{2}) \rangle, \sigma_\ell)$. This operator depends on a choice of Haar measure on $\text{GL}_2(\mathbb{A}_{\mathbb{Q}}^S)$. We normalize it so that $\text{GL}_2(\widehat{\mathbb{Z}}^S)$ has measure 1.

Lemma 6.7.3 *For $f \in S_\ell(N, \chi)$ and $p \nmid N$ one has that*

$$\varphi_{T_p f} = p^{\ell/2-1} R(\mathbb{1}_{p^{-1}}^S) \varphi_f.$$

Proof. For $(g, k) \in \text{GL}_2(\mathbb{R})^+ \times K_0(N)$ we have that

$$\varphi_{T_p f}(g, k) = \sum_{\gamma} \frac{p^{\ell/2-1} (j(\gamma, gi) j(g, i))^{-\ell}}{\chi(\text{lr}(\gamma)) \chi(\text{lr}(k))} f(\gamma gi) \quad (6.40)$$

where the sum on γ is over the finite set

$$\Gamma_0(N)^+ \backslash \Gamma_0(N)^+ \begin{pmatrix} p & \\ & 1 \end{pmatrix} \Gamma_0(N)^+.$$

Let $\Gamma_1(N)^+ := \Gamma_1(N) \cap \mathrm{GL}_2(\mathbb{R})^+$. Since $p \nmid N$ the natural map

$$\Gamma_1(N)^+ \backslash \Gamma_1(N)^+ \begin{pmatrix} p & \\ & 1 \end{pmatrix} \Gamma_1(N)^+ \longrightarrow \Gamma_0(N)^+ \backslash \Gamma_0(N)^+ \begin{pmatrix} p & \\ & 1 \end{pmatrix} \Gamma_0(N)^+$$

is a bijection, so we can take our γ to lie in $\Gamma_1(N)^+ \begin{pmatrix} p & \\ & 1 \end{pmatrix} \Gamma_1(N)^+$. Using this assumption and (6.37) we see that (6.40) is equal to

$$\sum_{\gamma} p^{\ell/2-1} \varphi_f(\gamma g, k) = \sum_{\gamma} p^{\ell/2-1} \varphi_f(g, \gamma^{-1} k). \quad (6.41)$$

The natural map

$$\Gamma_1(N)^+ \backslash \Gamma_1(N)^+ \begin{pmatrix} p & \\ & 1 \end{pmatrix} \Gamma_1(N)^+ \longrightarrow K_1(N) \backslash K_1(N) \begin{pmatrix} p & \\ & 1 \end{pmatrix} K_1(N)$$

is a bijection so (6.41) is $p^{\ell/2-1} R(\mathbb{1}_{p^{-1}}^S) \varphi_f$. $\square$

The following theorem is essentially a special case of the multiplicity one theorem (Theorem 11.4.3). We leave the proof as Exercise 11.7:

Theorem 6.7.4 *Let $f \in S_{\ell}(N, \chi)$. Then one has the following:*

- (a) *The automorphic form φ_f generates an irreducible $(\mathfrak{gl}_2, \mathrm{O}_2(\mathbb{R})) \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ -module if and only if f is a Hecke eigenform. Moreover, this $(\mathfrak{gl}_2, \mathrm{O}_2(\mathbb{R})) \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ -module is a cuspidal automorphic representation.*
- (b) *Let $\ell_1, \ell_2 \in \mathbb{Z}_{>0}$ and $f_i \in S_{\ell_i}(N, \chi)$ for $1 \leq i \leq 2$. If φ_{f_1} and φ_{f_2} generate isomorphic $(\mathfrak{gl}_2, \mathrm{O}_2(\mathbb{R})) \times \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}^{\infty})$ -modules then $\ell_1 = \ell_2$, and for all $p \nmid N$ one has that $\lambda_{f_1}(p) = \lambda_{f_2}(p)$.*

$\square$

One might ask if all cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ arise from elements of $S_{\ell}(N, \chi)$ in this fashion. In fact, we have missed many of them. The missing automorphic representations correspond to **Maass forms**. From the automorphic perspective it is very simple to define these objects; they are the automorphic representations of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ whose associated $(\mathfrak{gl}_2, \mathrm{O}_2(\mathbb{R}))$ -module is in the principal series in the sense of §4.7. Defining them classically involves at least as much work as required to formalize the notion of an element of $S_{\ell}(N, \chi)$. We refer the reader to [Bum97, §3.2] for more details. The ease with which we can talk about Maass forms and holomorphic cusp forms on equal footing illustrates the utility of the perspective of automorphic representation theory.

6.8 Ramanujan's Δ -function

We would be remiss not to give an example of a modular form from the classical perspective, and to an expert, it should not be a surprise that we choose Ramanujan's Δ -function

$$\Delta(z) = e^{2\pi iz} \prod_{i=1}^{\infty} (1 - e^{2\pi i n z})^{24}, \quad z \in \mathfrak{H}.$$

This Δ should not be confused with the Casimir operator from the previous section.

Temporarily denote by χ_0 the trivial character of the group with one element $(\mathbb{Z}/1)^\times$. One has the following basic theorem:

Theorem 6.8.1 *The complex vector space $S_{12}(1, \chi_0)$ is 1-dimensional and spanned by $\Delta(z)$. $\square$*

The proof of this fact can be found in any of the standard references [DS05, Lan95, Miy06, Kob93, Ser73, Shi94].

Define $\tau(n) \in \mathbb{C}$ to be the unique integers such that

$$\sum_{n=1}^{\infty} \tau(n) e^{2\pi i n z} = \Delta(z).$$

In other words, the $\tau(n)$ can be viewed as the Fourier coefficients of $\Delta(z)$. Ramanujan conjectured that

$$\sum_{n=1}^{\infty} \frac{\tau(n)}{n^t} = \prod_p \frac{1}{1 - \tau(p)p^{-t} + p^{11-2t}} \quad (6.42)$$

for a sufficiently large real number t [Ram00b, (101)]. Remarkably this is exactly his notation. He also conjectures that

$$|\tau(p)| \leq 2p^{11/2} \quad (6.43)$$

[Ram00b, (104)]. For those who have not seen this before, we pause and explain how fantastically prescient these two assertions are. In (6.42) Ramanujan has written down the L -function of the automorphic representation attached to $\Delta(z)$ over 20 years prior to Hecke's general theory of L -functions for modular forms [Hec37b, Hec37a] (Ramanujan's assertion had been proven some time earlier by Mordell [Mor]). The assertion (6.43) was proven by Deligne as a consequence of his proof of the Weil conjectures [Del73a], however, the natural generalization of (6.43) to automorphic representations of GL_n remains open even in the case $n = 2$; it will be discussed in Conjecture 10.7.4 later.

Exercises

6.1. Let $\iota'_1 : G \rightarrow \mathrm{GL}_{n_1}$, $\iota'_2 : G \rightarrow \mathrm{GL}_{n_2}$ be a pair of faithful representations of a reductive group G over a global field F and let S be a set of places of F

(finite or infinite). Let $\|g\|_{\iota_1 S}, \|g\|_{\iota_2 S}$ be the corresponding norms defined in (6.8). Prove that there are constants $c \in \mathbb{R}_{>0}$ and $r \in \mathbb{R}_{>1}$ such that

$$\|g\|_{\iota_2 S} \leq c \|g\|_{\iota_1 S}^r.$$

Deduce that the notions of moderate growth and uniform moderate growth are independent of the choice of faithful representation $\iota' : G \rightarrow \mathrm{GL}_n$.

6.2. Let v be a place of the global field F and let $\|\cdot\|_v$ be defined as in (6.7). Prove the following:

- (a) There is a $c \in \mathbb{R}_{>0}$ such that $\|gh\|_v \leq c \|g\|_v \|h\|_v$ for all $g, h \in G(F_v)$.
- (b) One has that $\|g^{-1}\|_v = \|g\|_v$ for all $g \in G(F_v)$.
- (c) For any compact subset $\Omega \subset G(F_v)$ there are $c_1(\Omega), c_2(\Omega) \in \mathbb{R}_{>0}$ such that

$$c_1(\Omega) < \frac{\|gh\|}{\|g\|} < c_2(\Omega)$$

for all $(g, h) \in G(F_v) \times \Omega$.

6.3. Let F be a local field, G an affine algebraic group over F , and $K \leq G(F)$ a compact subgroup. For every irreducible finite dimensional complex representation σ of K , let χ_σ be the character of σ and let $d(\sigma)$ be its degree. For any K -module (π, V) , let

$$\begin{aligned} V &\longrightarrow V \\ \varphi &\longmapsto \int_K \pi(k) \varphi \frac{\overline{\chi_\sigma(k)}}{d(\sigma)} dk. \end{aligned}$$

Here we normalize the measure so that $dk(K) = 1$. Show that this linear map is an idempotent that projects V onto the σ -isotypic subspace $V(\sigma)$.

6.4. Let G be a reductive group over a global field F and let $K = K_\infty K^\infty \leq G(\mathbb{A}_F)$ where $K_\infty \leq G(F_\infty)$ is a maximal compact subgroup and $K^\infty \leq G(\mathbb{A}_F^\infty)$ is a compact open subgroup. Show that a continuous function $\varphi : [G] \rightarrow \mathbb{C}$ is K -finite if and only if for all $x^\infty \in G(\mathbb{A}_F^\infty)$, $x_\infty \mapsto \varphi(x_\infty x^\infty)$ is K_∞ -finite and there is a compact open subgroup $K'^\infty \leq G(\mathbb{A}_F^\infty)$ such that $\varphi(xk) = \varphi(x)$ for all $x \in G(\mathbb{A}_F)$ and $k \in K'^\infty$.

6.5. Let F be a local field and let G be a reductive group over F . Let $K_1, K_2 \leq G(F)$ be maximal compact subgroups. If F is non-Archimedean, show that $K_1 \cap K_2$ is a compact open subgroup of $G(F)$ and deduce that a function on $G(F)$ is K_1 -finite if and only if it is K_2 -finite. Construct an example to show that if F is Archimedean then a function that is K_1 -finite need not be K_2 -finite.

6.6. Prove (6.9) and (6.10).

6.7. Prove Lemma 6.3.6.

6.8. Let G be a reductive group over a global field F . Choose a minimal parabolic subgroup $P_0 \leq G$; thus standard parabolic subgroups are defined. Prove that a function φ in $L^2([G])$ or $\mathcal{A}(G)$ is cuspidal if and only if, for all standard maximal parabolic subgroups $P \leq G$ with unipotent radical N , one has that

$$\int_{[N]} \varphi(n g) dn = 0$$

for almost every $g \in G(\mathbb{A}_F)$.

6.9. Let $\Gamma < \mathrm{GL}_2(\mathbb{R})$ be an arithmetic subgroup. Using the classification of §4.7, identify the underlying $(\mathfrak{g}, K)$ -module of $\mathcal{A}_{\mathrm{cusp}}(\Gamma, \langle \Delta - \frac{\ell}{2}(1 - \frac{\ell}{2}), Z \rangle)$.

6.10. Prove Ramanujan's assertion that (6.42) is valid using Theorem 6.8.1.

Chapter 7

Unramified Representations



Demazure nous indique que, derrière cette terminologie [épinglage], il y a l'image du papillon (que lui a fournie Grothendieck): le corps est un tore maximal T , les ailes sont deux sous-groupes de Borel opposées par rapport à T , on déploie le papillon en étalant les ailes, puis on fixe des éléments dans les groupes additifs (des épingle)s...

P. Gille and P. Polo

In this chapter, we describe the classification of unramified representations of reductive groups over non-Archimedean local fields. Along the way, we discuss the Satake isomorphism and the Langlands dual group.

7.1 Unramified representations

Let $K^\infty := \prod_{v \nmid \infty} K_v \leq G(\mathbb{A}_F^\infty)$ be a compact open subgroup. Then K_v is hyperspecial for all but finitely many v by Corollary 2.4.12. By Flath's theorem (Theorem 5.7.1), if (π, V) is an automorphic representation of $G(\mathbb{A}_F)$, then

$$(\pi, V) \cong (\otimes'_v \pi_v, \otimes'_v V_v)$$

where for all but finitely many v , one has that $V_v^{K_v} \neq 0$ (see also theorems 6.3.7 and 6.4.3). In particular, for all but finitely many v , the representation (π_v, V_v) is unramified in the sense of Definition 7.1.1 below. Thus, unramified representations play an essential role in automorphic representation theory.

In this section, we discuss the classification of unramified representations. It turns out that they can be explicitly parametrized in terms of conjugacy classes in the dual group of G by Corollary 7.5.2 below. This fundamental fact will be used in §7.7 to state a version of the Langlands functoriality conjecture.

Throughout this section, we let G be a reductive group over a non-Archimedean local field F . Recall that G is **unramified** if it is quasi-split and split over an unramified extension of F . In this case, there exists a hyperspecial subgroup $K \leq G(F)$. We recall Definition 5.5.5:

Definition 7.1.1. Let G be an unramified reductive group over F and let $K \leq G(F)$ be a hyperspecial subgroup. An irreducible smooth representation (π, V) of $G(F)$ is **K -unramified** if $V^K \neq 0$.

As mentioned below Definition 5.5.5, unramified representations are admissible by Theorem 5.3.6.

Hyperspecial subgroups of $G(F)$ are not necessarily $G(F)$ -conjugate; they are only conjugate under $(G/Z_G)(F)$ (see §2.4). Thus, the property of a representation being K -unramified in general depends on K . We will however follow tradition and usually suppress the dependence on K (but see §12.5).

Let $K \leq G(F)$ be a hyperspecial subgroup. We recall from §5.5 that the subalgebra

$$C_c^\infty(G(F) // K) \leq C_c^\infty(G(F)) \quad (7.1)$$

is known as the **unramified Hecke algebra** of $G(F)$ (with respect to K). It is commutative by Theorem 5.5.2. Let $f \in C_c^\infty(G(F) // K)$ and let (π, V) be unramified. By Corollary 5.5.6, we have $\dim_{\mathbb{C}} V^K = 1$. Thus, $\pi(f)$ acts via a scalar on V^K . It is sensible to denote the scalar by $\mathrm{tr} \pi(f)$ (see also §8.5). The map

$$\begin{aligned} C_c^\infty(G(F) // K) &\longrightarrow \mathbb{C} \\ f &\longmapsto \mathrm{tr} \pi(f) \end{aligned}$$

is called the **Hecke character** of π . Every nontrivial algebra homomorphism from $C_c^\infty(G(F) // K)$ to $\mathbb{C}$ is the Hecke character of some irreducible unramified representation [Lau96, Proposition D.1.8].

Proposition 7.1.2 *Let $K \leq G(F)$ be a compact open subgroup (not necessarily hyperspecial). If (π_1, V_1) and (π_2, V_2) are irreducible representations of $G(F)$ such that V_1^K and V_2^K are nonzero and isomorphic as $C_c^\infty(G(F) // K)$ -modules, then (π_1, V_1) is isomorphic to (π_2, V_2) as a smooth representation. In particular, an unramified representation is determined up to isomorphism by its Hecke character.*

The proof we give is from [Cas73, p. 33]. For a refinement of this result, see [Lau96, Proposition D.1.8].

Proof. Let (π_1, V_1) and (π_2, V_2) be as stated in the proposition, and let

$$I : V_1^K \longrightarrow V_2^K$$

be a $C_c^\infty(G(F) // K)$ -intertwining map. We claim that for $f \in C_c^\infty(G(F))$ and $\varphi \in V_1^K$ the map

$$I(\pi_1(f)\varphi) := \pi_2(f)I(\varphi)$$

is well-defined. Since V_1 is irreducible, if I is well-defined, it is clearly an intertwining map of $C_c^\infty(G(F))$ -modules from V_1 to V_2 . By Lemma 5.3.3, I therefore defines an intertwining map of smooth representations from V_1 to V_2 . Since V_2 is also irreducible, I is an isomorphism.

To prove the claim, it suffices to check that if $\pi_1(f)\varphi = 0$ then $\pi_2(f)I(\varphi) = 0$. Suppose that $\pi_1(f)\varphi = 0$. Then, writing $e_K := \frac{1}{\text{meas}_{dg}(K)} \mathbb{1}_K$ as in (5.2), we have

$$\pi_1(e_K * f_1 * f * e_K)\varphi = 0$$

for all $f_1 \in C_c^\infty(G(F))$, which implies

$$\pi_2(e_K * f_1 * f * e_K)I(\varphi) = 0$$

for all $f_1 \in C_c^\infty(G(F))$ since I is a $C_c^\infty(G(F) // K)$ -intertwining map. But then

$$\pi_2(e_K * f_1)\pi_2(f)I(\varphi) = 0$$

for all $f_1 \in C_c^\infty(G(F))$. Since V_2 is irreducible and V_2^K is nonzero, we deduce that $\pi_2(f)I(\varphi) = 0$. $\square$

7.2 The Satake isomorphism for split groups

In this section and in the remainder of this chapter, we make use of basic properties of actions of algebraic groups on affine schemes. These topics are discussed in detail in §17.2 and §17.3. Fortunately, we will only require the theory over the complex numbers in the current chapter, which leads to certain simplifications. When we encounter a quotient or a related object in this chapter, we explain it in concrete terms. Thus, it is not necessary to read §17.2 and §17.3 prior to reading this chapter. We will however indicate precisely where in §17.2 and §17.3 the reader can find more details to obtain a deeper understanding.

Here is a key example of the constructions we will encounter below. Let G be a reductive group over F and let $\widehat{G}$ be the complex reductive group with root datum dual to that of G (see §1.8). Let $\widehat{T} \leq \widehat{G}$ be a maximal torus. We observe that $\widehat{T}$ is a $\mathbb{C}$ -scheme, so the coordinate ring $\mathcal{O}(\widehat{T})$ is a $\mathbb{C}$ -algebra. The group scheme $W(\widehat{G}, \widehat{T})$ acts on $\widehat{T}$ and we define $\mathcal{O}(\widehat{T})^{W(\widehat{G}, \widehat{T})}$ as in (17.8). Informally, since $W(\widehat{G}, \widehat{T})$ acts on $\widehat{T}$, it acts on regular functions on $\widehat{T}$, and one defines $\mathcal{O}(\widehat{T})^{W(\widehat{G}, \widehat{T})} \leq \mathcal{O}(\widehat{T})$ to be the subalgebra fixed under this action.

Set

$$\widehat{T}/W(\widehat{G}, \widehat{T}) := \operatorname{Spec}(\mathcal{O}(\widehat{T})^{W(\widehat{G}, \widehat{T})}). \quad (7.2)$$

By Proposition 17.3.4, $(\widehat{T}/W(\widehat{G}, \widehat{T}))(\mathbb{C})$ is the set of closed $W(\widehat{G}, \widehat{T})$ -orbits in $\widehat{T}$. Since $W(\widehat{G}, \widehat{T})$ is finite, we deduce that

$$(\widehat{T}/W(\widehat{G}, \widehat{T}))(\mathbb{C}) = \widehat{T}(\mathbb{C})/W(\widehat{G}, \widehat{T})(\mathbb{C}). \quad (7.3)$$

We use analogous notation for quotients by reductive group schemes in the remainder of this chapter, and the analogous descriptions of their points over algebraically closed fields. For a systematic treatment, the reader can consult §17.2 and §17.3.

Moving back to the situation at hand, let G be a split reductive group over a non-Archimedean local field F and let $K \leq G(F)$ be a hyperspecial subgroup. By Proposition 7.1.2, unramified representations of $G(F)$ are determined by the $C_c^\infty(G(F) // K)$ -modules they induce. If we did not know anything about $C_c^\infty(G(F) // K)$, then we could hardly regard this information as useful. Thankfully, it turns out that $C_c^\infty(G(F) // K)$ has a simple description:

Theorem 7.2.1 (Satake) *Assume that G is split. There is an isomorphism of algebras*

$$\mathcal{S} : C_c^\infty(G(F) // K) \longrightarrow \mathcal{O}(\widehat{T})^{W(\widehat{G}, \widehat{T})}.$$

□

The map $\mathcal{S}$ in Theorem 7.2.1 is the **Satake isomorphism**.

Example 7.1. The Hecke algebra $C_c^\infty(\operatorname{GL}_n(F) // \operatorname{GL}_n(\mathcal{O}_F))$, as a $\mathbb{C}$ -module, has a basis given by

$$\mathbb{1}_\lambda := \mathbb{1}_{\operatorname{GL}_n(\mathcal{O}_F)} \left(\begin{pmatrix} \varpi^{\lambda_1} & & \\ & \ddots & \\ & & \varpi^{\lambda_n} \end{pmatrix} \right)_{\operatorname{GL}_n(\mathcal{O}_F)}$$

with $\lambda := (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, $\lambda_i \geq \lambda_{i+1}$ for all $1 \leq i \leq n-1$ (this follows from the theory of elementary divisors). As a $\mathbb{C}$ -algebra it is generated by $\mathbb{1}_\lambda$ with $\lambda = (1^r) := (1, \dots, 1, 0, \dots, 0)$ (r ones and $n-r$ zeros) for $1 \leq r \leq n$ and $\lambda = ((-1)^n) := (-1, \dots, -1)$.

Let $\widehat{T} \leq \operatorname{GL}_{n\mathbb{C}} = \widehat{\operatorname{GL}}_n$ be the maximal torus of diagonal matrices. On the generating set above, the Satake isomorphism is given by

$$\mathcal{S}(\mathbb{1}_{(1^r)}) = q^{r(n-r)/2} \operatorname{tr}(\wedge^r t) \quad \text{and} \quad \mathcal{S}(\mathbb{1}_{(-1)^n}) = \det(t^{-1}). \quad (7.4)$$

Here $t \in \widehat{T}(\mathbb{C})$ and $\operatorname{tr}(\wedge^r t)$ is the polynomial function of t given by the character of the representation $\wedge^r$ of GL_n (see [Gro98, (3.14)]).

The simplicity of this example belies the complexity of describing the Satake isomorphism explicitly in general. See [Li17, Sak18] for more information.

It is instructive to indicate various ways of rephrasing Theorem 7.2.1. By the Chevalley restriction theorem [Ste65, §6], the inclusion $\widehat{T} \hookrightarrow \widehat{G}$ induces an isomorphism

$$\mathrm{Spec} \left(\mathcal{O}(\widehat{T})^{W(\widehat{G}, \widehat{T})} \right) = \widehat{T}/W(\widehat{G}, \widehat{T}) \longrightarrow \widehat{G}/\mathrm{conj} = \mathrm{Spec} \left(\mathcal{O}(\widehat{G})^{\widehat{G}} \right) \quad (7.5)$$

where the quotient on the right is the quotient of $\widehat{G}$ by the action of conjugation (see §17.2). In particular, $(\widehat{G}/\mathrm{conj})(\mathbb{C})$ is just the set of closed conjugacy classes in $\widehat{G}(\mathbb{C})$. The closed conjugacy classes are precisely the conjugacy classes of semisimple elements; a nice reference for this is [Ste65, Corollary 6.13]. Thus, we have a sequence of bijections

$$\mathrm{Hom}(C_c^\infty(G(F) // K), \mathbb{C}) \xrightarrow{(\mathcal{S}^{-1})^*} \mathrm{Hom}(\mathcal{O}(\widehat{T})^{W(\widehat{G}, \widehat{T})}, \mathbb{C}) \rightarrow (\widehat{G}/\mathrm{conj})(\mathbb{C}) \quad (7.6)$$

where the first map is pullback along the inverse of the Satake isomorphism. Here homomorphism means homomorphism of $\mathbb{C}$ -algebras. In Proposition 7.1.2, we saw that unramified representations were in bijection with Hecke characters, which are precisely elements of $\mathrm{Hom}(C_c^\infty(G(F) // K), \mathbb{C})$. We have therefore proven the following corollary of the Satake isomorphism:

Corollary 7.2.2 *Assume that G is split. The composite isomorphism (7.6) induces a bijection between semisimple conjugacy classes in $\widehat{G}(\mathbb{C})$ and isomorphism classes of irreducible unramified representations of $G(F)$. $\square$*

From the point of view of automorphic representation theory, the assumption that G is split in Theorem 7.2.1 and Corollary 7.2.2 is problematic. Indeed, suppose for the moment that G is a reductive group over a global field F . Then, for all but finitely many places v of F , the group G_{F_v} is unramified by Proposition 2.4.8 and hence in particular is quasi-split. However, the group G_{F_v} can be nonsplit for infinitely many v (see Exercise 7.5).

Langlands was able to extend the Satake isomorphism to the quasi-split case. Historically this was important because it gave crucial hints as to the structure of the Langlands dual group, which he introduced at the same time [Lan] (see also [Lan70]). We will define the Langlands dual group in the next section and use it to extend the Satake isomorphism in §7.5.

7.3 The Langlands dual group

For the moment, let G be a reductive group over a field F with separable closure F^{sep} and let $T \leq G$ be a maximal torus. Then T splits over F^{sep} . To $(G_{F^{\mathrm{sep}}}, T_{F^{\mathrm{sep}}})$, we associated in §1.8 a root datum $\Psi(G_{F^{\mathrm{sep}}}, T_{F^{\mathrm{sep}}}) =$

$(X^*(T_{F^{\text{sep}}}), X_*(T_{F^{\text{sep}}}), \Phi, \Phi^\vee)$. We remind the reader that $X^*(T_{F^{\text{sep}}})$ and $X_*(T_{F^{\text{sep}}})$ are the character and cocharacter groups, $\Phi \subset X^*(T_{F^{\text{sep}}})$ is the set of roots of $T_{F^{\text{sep}}}$ in $\mathfrak{g}$ and $\Phi^\vee \subset X_*(T_{F^{\text{sep}}})$ is the set of coroots. We also remind the reader that the root datum characterizes $G_{F^{\text{sep}}}$ up to isomorphism. To ease notation, we let

$$\text{Gal}_F := \text{Gal}(F^{\text{sep}}/F).$$

If G is split, then we set

$${}^L G := \widehat{G}(\mathbb{C}) \times \text{Gal}_F$$

where $\widehat{G}$ is the complex dual group defined as in §1.8; it is the reductive group over $\mathbb{C}$ with root datum $(X_*(T_{F^{\text{sep}}}), X^*(T_{F^{\text{sep}}}), \Phi^\vee, \Phi)$.

If G is not split then ${}^L G$ has a more complicated definition involving a Galois action on $\widehat{G}(\mathbb{C})$ that records the fact that G is a nonsplit group over F . This Galois action is algebraic in the sense that it is given in terms of elements of

$$\text{Aut}(\widehat{G}),$$

the group of automorphisms of the algebraic group $\widehat{G}$. The group $\text{Aut}(\widehat{G})$ is actually the $\mathbb{C}$ -points of another algebraic group, but we will not need this fact in this section. We describe a refinement $\Psi(\widehat{G}, \widehat{B}, \widehat{T})$ of $\Psi(\widehat{G}, \widehat{T})$ called a based root datum and a surjective map

$$\text{Aut}(\widehat{G}) \longrightarrow \text{Aut}(\Psi(\widehat{G}, \widehat{B}, \widehat{T})).$$

We then provide an explicit description of splittings of this map via something known as pinning, or *épinglage*, of $\widehat{G}$. Finally we define a homomorphism $\text{Gal}_F \rightarrow \text{Aut}(\Psi(\widehat{G}, \widehat{B}, \widehat{T}))$. This suffices to construct our desired action of Gal_F on $\widehat{G}(\mathbb{C})$.

We now begin this process. Let $(X, Y, \Phi, \Phi^\vee)$ be a root datum. Let $\Delta \subset \Phi$ be a base for Φ (see (1.27)). The set

$$\Delta^\vee := \{\alpha^\vee : \alpha \in \Delta\}$$

is a base for $\Phi^\vee$ [Con14, Exercise 1.6.17]. The tuple

$$(X, Y, \Phi, \Phi^\vee, \Delta, \Delta^\vee) \tag{7.7}$$

is a **based root datum**. There is an obvious notion of isomorphism of based root data; it is simply a pair of linear isomorphisms on the first two factors preserving the pairing and the bases. We observe that Δ determines $\Delta^\vee$ and conversely.

Assume until otherwise specified that G is a reductive group over a separably closed field k . Thus, G is split. For each $\alpha \in \Phi$ let $N_\alpha \leq G$ be the corresponding root group defined as in (1.29). By Theorem 1.9.3, there is a

bijection

$$\{\text{bases } \Delta \subseteq \Phi(G, T)\} \xrightarrow{\sim} \{\text{Borel subgroups } B \geq T\} \quad (7.8)$$

given by sending a base Δ with associated set of positive roots Φ^+ to $\langle T, \{N_\alpha\}_{\alpha \in \Phi^+} \rangle$. We let

$$\Psi(G, B, T) := (X^*(T), X_*(T), \Phi, \Phi^\vee, \Delta, \Delta^\vee) \quad (7.9)$$

where $B \leq G$ is the Borel subgroup corresponding to Δ under (7.8).

The based root datum $\Psi(G, B, T)$ depends on the pair (B, T) . It is convenient to rigidify the pair (B, T) slightly via the notion of a pinning:

Definition 7.3.1. A **pinning** of G is a tuple

$$(B, T, (X_\alpha)_{\alpha \in \Delta})$$

where T is a maximal torus, B is a Borel subgroup containing T , Δ is the set of simple roots attached to B and T , and $X_\alpha \in \mathfrak{g}_\alpha - \{0\}$ for all α .

We refer to [Mil17, §23.d] and [Con14, §1.5] for more information about pinnings. We reproduce some arguments from these references below.

Let $\text{Aut}(B, T, (X_\alpha)_{\alpha \in \Delta})$ be the group of automorphisms ϕ of G that preserve B , T , and such that the automorphism $\mathfrak{g} \rightarrow \mathfrak{g}$ induced by ϕ preserves each X_α for $\alpha \in \Delta$.

Proposition 7.3.2 *If Δ and B correspond under (7.8), then the natural map*

$$\text{Aut}(B, T, (X_\alpha)_{\alpha \in \Delta}) \longrightarrow \text{Aut}(\Psi(G, B, T))$$

is an isomorphism.

Proof. By Theorem 1.8.8, any automorphism in $\text{Aut}(\Psi(G, B, T))$ is induced by an automorphism $\phi : G \rightarrow G$ preserving T and Δ . The group $T(k)$ acts on $\{(X_\alpha) \in \prod_{\alpha \in \Delta} (\mathfrak{g}_\alpha - \{0\})\}$ by conjugation. Since the set Δ is linearly independent in $X^*(T)_\mathbb{Q}$, the action is surjective. Thus, by replacing ϕ by a $T(k)$ -conjugate, we may assume $\phi \in \text{Aut}(B, T, (X_\alpha)_{\alpha \in \Delta})$. Since replacing ϕ by a $T(k)$ -conjugate does not affect the automorphism of $\Psi(G, B, T)$ induced by ϕ , we deduce that the map in the proposition is surjective.

As for injectivity, Theorem 1.8.8 implies that anything in the kernel of the morphism in the proposition is an inner automorphism by an element $t \in T(k)$. Since this inner automorphism fixes the X_α , it satisfies $\alpha(t) = 1$ for all $\alpha \in \Delta$. The element t therefore centralizes every root group, hence the Bruhat cell $B^{\text{op}}TB$, where B^{op} is the opposite Borel. The Bruhat cell $B^{\text{op}}TB$ is open in G [Mil17, Theorem 21.84], so t centralizes all of G . In particular, the inner automorphism defined by t is trivial. $\square$

Let $\text{Aut}(G)$ be the group (not group scheme) of automorphisms of the algebraic group G and let $\text{Inn}(G)$ be the subgroup of inner automorphisms, that is, automorphisms induced by conjugation by elements of $G(k)$.

Proposition 7.3.3 *There is an exact sequence*

$$1 \longrightarrow \operatorname{Inn}(G) \xrightarrow{a} \operatorname{Aut}(G) \xrightarrow{b} \operatorname{Aut}(\Psi(G, B, T)) \longrightarrow 1. \quad (7.10)$$

Any pinning of G provides a section of b via the isomorphism of Proposition 7.3.2.

Proof. The arrow a is the obvious injection. To describe b , note that for any automorphism $\phi \in \operatorname{Aut}(G)$, there is a $g \in G(k)$ such that $\phi \circ \operatorname{Ad}(g)$ preserves B and T . Indeed, all Borel subgroups are conjugate under $G(k)$ and all (split) maximal tori in B are conjugate under $B(k)$. The fact that $\phi \circ \operatorname{Ad}(g)$ preserves B and T implies that it preserves the set of simple roots Δ corresponding to B under (7.8). We claim this gives a well-defined homomorphism

$$b : \operatorname{Aut}(G) \longrightarrow \operatorname{Aut}(\Psi(G, B, T)). \quad (7.11)$$

To check that it is well-defined, assume that ϕ and $\phi \circ \operatorname{Ad}(g)$ both preserve B and T . Then

$$g \in N_{G(k)}(B) \cap N_{G(k)}(T) = B(k) \cap N_{G(k)}(T) = T(k) \quad (7.12)$$

(see [Mil17, Theorem 17.48 and Proposition 17.53]) and hence $\phi \circ \operatorname{Ad}(g)$ and ϕ induce the same automorphism of $\Psi(G, B, T)$. The map (7.11) is surjective by Theorem 1.8.8. We claim that $\ker b = \operatorname{Inn}(G)$, and hence the sequence of (7.10) is exact. Suppose $\phi \in \operatorname{Aut}(G)$ maps to the identity in $\operatorname{Aut}(\Psi(G, B, T))$ under b . After replacing ϕ by $\phi \circ \operatorname{Ad}(g)$ for some $g \in G(k)$, we may assume ϕ preserves B and T . The fact that $b(\phi)$ is the identity implies that ϕ is an inner automorphism by an element of $T(k)$ by Theorem 1.8.8.

Since $\operatorname{Aut}(B, T, (X_\alpha)_{\alpha \in \Delta})$ is defined as a subset of $\operatorname{Aut}(G)$ the last assertion is clear. $\square$

The content of propositions 7.3.2 and 7.3.3 can be summarized informally as follows. The theorem of Chevalley and Demazure (Theorem 1.8.8) allows us to identify automorphisms of a reductive group over k that fix a maximal torus T with automorphisms of root data, but only up to an inner automorphism by an element of $T(k)$. Fixing a pinning and refining root data to based root data removes this ambiguity.

With this preparation complete, we can now define the Langlands dual group. Assume that G is a reductive group over a local or global field F . Choose a Borel subgroup $B \leq G_{F^{\text{sep}}}$ and a maximal torus $T \leq B \leq G_{F^{\text{sep}}}$. We obtain a based root datum

$$\Psi(G_{F^{\text{sep}}}, B, T).$$

We now define a homomorphism

$$\mu_G : \operatorname{Gal}_F \longrightarrow \operatorname{Aut}(\Psi(G_{F^{\text{sep}}}, B, T)). \quad (7.13)$$

Given $\sigma \in \text{Gal}_F$ and an affine F^{sep} -scheme X of finite type over F^{sep} , let ${}^\sigma X$ be the base change of X along the automorphism $\sigma : F^{\text{sep}} \rightarrow F^{\text{sep}}$. Thus ${}^\sigma X$ may be obtained by applying σ to the coefficients of the polynomials cutting X out of an affine space. We observe that ${}^\sigma T$ is a maximal torus of $G_{F^{\text{sep}}}$ and ${}^\sigma B \leq G_{F^{\text{sep}}}$ is a Borel subgroup of $G_{F^{\text{sep}}}$. Choose $g \in G(F^{\text{sep}})$ such that $g({}^\sigma B)g^{-1} = B$ and $g({}^\sigma T)g^{-1} = T$. Then $\text{Ad}(g) \circ \sigma$ induces an isomorphism of $\Psi(G_{F^{\text{sep}}}, B, T)$. It follows as in the proof of Proposition 7.3.3 that this isomorphism is independent of the choice of g and hence we obtain the homomorphism (7.13).

We made a choice of B and T in the definition of μ_G so we pause and indicate the extent to which the definition of μ_G depends on this choice. Suppose we are given Borel subgroups B and B' of $G_{F^{\text{sep}}}$ and maximal tori $T \leq B$ and $T' \leq B'$. Choose $g \in G(F^{\text{sep}})$ such that $gBg^{-1} = B'$ and $gTg^{-1} = T'$. Conjugation by g induces an isomorphism

$$\text{Aut}(\Psi(G_{F^{\text{sep}}}, B, T)) \longrightarrow \text{Aut}(\Psi(G_{F^{\text{sep}}}, B', T')). \quad (7.14)$$

It follows as in the proof of Proposition 7.3.3 that this isomorphism is independent of the choice of g . Thus, our definition of the homomorphism μ_G is dependent on the choice of B and T , but different choices give rise to homomorphisms that are related by a canonical isomorphism. One can use this observation to remove the choice of B and T from the definition of root data entirely if desired [Mil17, §21.42, 21.43].

Now

$$(X_*(T), X^*(T), \Phi^\vee, \Phi, \Delta^\vee, \Delta)$$

is again a based root datum with associated root datum $\Psi(\widehat{G}, \widehat{T})$, so we have a Borel subgroup $\widehat{B} \leq \widehat{G}$ such that

$$(X_*(T), X^*(T), \Phi^\vee, \Phi, \Delta^\vee, \Delta) = \Psi(\widehat{G}, \widehat{B}, \widehat{T}). \quad (7.15)$$

Let

$$(X_{\alpha^\vee})_{\alpha^\vee \in \Delta^\vee}$$

a set of root vectors such that $(\widehat{B}, \widehat{T}, (X_{\alpha^\vee})_{\alpha^\vee \in \Delta^\vee})$ is a pinning of $\widehat{G}$. By Proposition 7.3.3, this defines a splitting of the surjection

$$\text{Aut}(\widehat{G}) \longrightarrow \text{Aut}(\Psi(\widehat{G}, \widehat{B}, \widehat{T})).$$

We define

$$\widehat{\mu}_G : \text{Gal}_F \longrightarrow \text{Aut}(\Psi(G_{F^{\text{sep}}}, B, T)) \xrightarrow{\sim} \text{Aut}(\Psi(\widehat{G}, \widehat{B}, \widehat{T})) \longrightarrow \text{Aut}(\widehat{G}) \quad (7.16)$$

where the isomorphism is canonical and the last arrow is defined by the pinning $(\widehat{B}, \widehat{T}, (X_{\alpha^\vee})_{\alpha^\vee \in \Delta^\vee})$. By examining the proof of Proposition 7.3.3, we see that any $\widehat{T}(\mathbb{C})$ -conjugate of $\widehat{\mu}_G$ corresponds to another choice of $(X_{\alpha^\vee})_{\alpha^\vee \in \Delta^\vee}$ and conversely. The **Langlands dual group** of G or the **L -group** of G to

be the semidirect product

$${}^L G := \widehat{G}(\mathbb{C}) \rtimes \mathrm{Gal}_F$$

with respect to $\widehat{\mu}_G$.

We give $\widehat{G}(\mathbb{C})$ the Hausdorff topology and ${}^L G$ the product topology. Then the neutral component $({}^L G)^\circ$ of ${}^L G$ is $\widehat{G}(\mathbb{C})$. Thus, any continuous homomorphism ${}^L H \rightarrow {}^L G$ maps $\widehat{H}(\mathbb{C})$ to $\widehat{G}(\mathbb{C})$. A morphism of L -groups

$${}^L H \longrightarrow {}^L G$$

is simply a continuous homomorphism commuting with the projections to Gal_F such that its restriction to the neutral components is induced by a map of algebraic groups $\widehat{H} \rightarrow \widehat{G}$. Morphisms are also sometimes called **L -maps**. Stipulating that $\widehat{H} \rightarrow \widehat{G}$ is algebraic is equivalent to requiring the map $\widehat{H}(\mathbb{C}) \rightarrow \widehat{G}(\mathbb{C})$ to be holomorphic [Con14, Proposition D.2.1].

Assume for the moment that F is a global field. For every place v , upon choosing an embedding $F^{\mathrm{sep}} \rightarrow F_v^{\mathrm{sep}}$, we obtain an injection

$$\mathrm{Gal}_{F_v} \longrightarrow \mathrm{Gal}_F$$

unique up to conjugacy. The embedding also induces an identification of the complex dual groups attached to $(G_{F^{\mathrm{sep}}}, T_{F^{\mathrm{sep}}})$ and $(G_{F_v^{\mathrm{sep}}}, T_{F_v^{\mathrm{sep}}})$ as explained after (1.38). This induces a morphism

$${}^L G_{F_v} \longrightarrow {}^L G_F \tag{7.17}$$

which allows us to relate local and global L -groups.

Occasionally, one works with modifications of the L -group. For example, we can replace Gal_F by its quotient $\mathrm{Gal}(E/F)$ where E/F is a finite degree Galois extension such that G_E splits. One can also replace Gal_F by any topological group Γ admitting a continuous homomorphism $\Gamma \rightarrow \mathrm{Gal}_F$ that surjects onto $\mathrm{Gal}(E/F)$ for any field E such that G_E is split; all of the constructions above have obvious analogues in this level of generality. See [Mok15] for an illuminating example.

Example 7.2. Let $G = \mathrm{GL}_n$ and let E/F be a field extension of finite degree. One has that

$${}^L \mathrm{Res}_{E/F} \mathrm{GL}_n \cong \mathrm{GL}_n(\mathbb{C})^{\mathrm{Hom}_F(E, F^{\mathrm{sep}})} \rtimes \mathrm{Gal}_F \tag{7.18}$$

where Gal_F acts via its action on $\mathrm{Hom}_F(E, F^{\mathrm{sep}})$. Here we denote by $\mathrm{GL}_n(\mathbb{C})^{\mathrm{Hom}_F(E, F^{\mathrm{sep}})}$ the group of maps of sets from $\mathrm{Hom}_F(E, F^{\mathrm{sep}})$ to $\mathrm{GL}_n(\mathbb{C})$, so $\mathrm{GL}_n(\mathbb{C})^{\mathrm{Hom}_F(E, F^{\mathrm{sep}})} \cong \mathrm{GL}_n(\mathbb{C})^{[E:F]}$. More explicitly, for

$$((g_\xi) \rtimes \sigma) \in \mathrm{GL}_n(\mathbb{C})^{\mathrm{Hom}_F(E, F^{\mathrm{sep}})} \rtimes \mathrm{Gal}_F$$

one has

$$\sigma((g_\xi)) = (g_{\sigma^{-1} \circ \xi}).$$

In other words, σ maps the entry corresponding to ξ to the entry corresponding to $\sigma \circ \xi$.

Example 7.3. Let

$$J_n := \begin{pmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Z})$$

be the antidiagonal matrix and let

$$J'_{2n} := \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix} \quad \text{and} \quad J'_{2n+1} := \begin{pmatrix} & J_n \\ & 1 \\ -J_n & \end{pmatrix}. \quad (7.19)$$

Let M/F be a quadratic extension and let $\sigma \in \mathrm{Gal}(M/F)$ be the nontrivial element. For an F -algebra R , consider the quasi-split unitary group attached to the extension M/F :

$$\mathrm{U}_n(R) := \{g \in \mathrm{GL}_n(M \otimes_F R) : J_n'^{-1} \sigma(g)^{-t} J_n' = g\}.$$

One has that

$${}^L\mathrm{U}_n \cong \mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}_F \quad (7.20)$$

where Gal_F acts via its quotient $\mathrm{Gal}(M/F)$. The nontrivial element $\sigma \in \mathrm{Gal}(M/F)$ is sent to the isomorphism $g \mapsto J_n'^{-1} g^{-t} J_n'$ of $\mathrm{GL}_n(\mathbb{C})$. There is an L -map

$$\begin{aligned} {}^L\mathrm{U}_n &\longrightarrow {}^L\mathrm{Res}_{M/F} \mathrm{GL}_n \\ g \rtimes \sigma &\longmapsto (g, J_n'^{-1} g^{-t} J_n') \rtimes \sigma. \end{aligned}$$

7.4 Parabolic subgroups of L -groups

Let G be a reductive group over a local or global field F . In §12.5 and §13.2, we will require the notion of a parabolic subgroup of an L -group. It is most natural to define this concept immediately after discussing the L -group. However, since we will not require this material until much later in the book, the reader should feel free to skip this section and then refer back to it as needed.

We follow the discussion in [Bor79]. Identify $\widehat{G}(\mathbb{C})$ with the subgroup

$$\widehat{G}(\mathbb{C}) \rtimes \{1\} \leq {}^L G.$$

A **parabolic subgroup** of ${}^L G$ is a closed subgroup Q such that $Q \cap \widehat{G}(\mathbb{C})$ is the complex points of a parabolic subgroup of $\widehat{G}$ and the restriction of the canonical map ${}^L G \rightarrow \text{Gal}_F$ to Q is surjective. Clearly

$${}^L B := \widehat{B}(\mathbb{C}) \rtimes \text{Gal}_F \quad (7.21)$$

is a parabolic subgroup, where $\widehat{B}$ is the group of (7.15). A parabolic subgroup of ${}^L G$ is **standard** if it contains ${}^L B$. As in the case of algebraic groups, every parabolic subgroup of ${}^L G$ is conjugate to one and only one standard parabolic subgroup.

We now isolate the subset of relevant parabolic subgroups of ${}^L G$. This will allow us to associate to a parabolic subgroup $P \leq G$ a dual parabolic subgroup ${}^L P \leq {}^L G$. We will also explain how this duality behaves with respect to Levi subgroups.

For every field extension $F \leq E \leq F^{\text{sep}}$, we let $\mathcal{P}(E)$ denote the set of parabolic subgroups of G_E . Let $(\mathcal{P}/G)(F^{\text{sep}})$ denote the set of $G(F^{\text{sep}})$ -conjugacy classes of parabolic subgroups in $G(F^{\text{sep}})$ and let

$$(\mathcal{P}/G)(E) = (\mathcal{P}/G)(F^{\text{sep}})^{\text{Gal}(F^{\text{sep}}/E)}.$$

This is the set of conjugacy classes that are fixed by $\text{Gal}(F^{\text{sep}}/E)$ or in other words are defined over E . There is a natural map

$$\mathcal{P}(E) \longrightarrow (\mathcal{P}/G)(E).$$

Let $B \leq G_{F^{\text{sep}}}$ be a Borel subgroup and $T \leq B$ be a maximal torus. We then obtain a based root datum

$$\Psi(G_{F^{\text{sep}}}, B, T) = (X^*(T), X_*(T), \Phi, \Phi^\vee, \Delta, \Delta^\vee).$$

In §7.3, we explained how to endow the based root datum with an action of Gal_F . One has a bijection

$$\{J \subseteq \Delta\} \longleftrightarrow (\mathcal{P}/G)(F^{\text{sep}}) \quad (7.22)$$

by Theorem 1.9.4 and the discussion preceding it. The bijection is Gal_F -equivariant.

Lemma 7.4.1 *If G is quasi-split, then the map $\mathcal{P}(F) \rightarrow (\mathcal{P}/G)(F)$ is surjective.*

Proof. Let $B' \leq G$ be a Borel subgroup. We can assume $B = B'_{F^{\text{sep}}}$, which implies in particular that B is fixed by Gal_F . Let $P \leq G_{F^{\text{sep}}}$ be a parabolic subgroup whose conjugacy class is Gal_F -fixed. Upon replacing P by a $G(F^{\text{sep}})$ -conjugate if necessary, we can assume $P \geq B$. Now for all $\sigma \in \text{Gal}_F$, we have $\sigma P \geq B$ and $\sigma P = g^{-1}Pg$ for some $g \in G(F^{\text{sep}})$. Since every parabolic subgroup is conjugate to a unique standard parabolic sub-

group, we conclude that $P = {}^\sigma P$ for all $\sigma \in \text{Gal}_F$. Hence, P is defined over F , that is, $P = P'_{F^{\text{sep}}}$ for some parabolic subgroup $P' \leq G$. $\square$

Similarly, let ${}^L\mathcal{P}$ denote the set of parabolic subgroups of ${}^L G$ and let ${}^L\mathcal{P}/{}^L G$ denote the set of ${}^L G$ -conjugacy classes of parabolic subgroups of ${}^L G$. Again by Theorem 1.9.4 and the discussion preceding it, we have a bijection

$$\{J \subseteq \Delta^\vee\}^{\text{Gal}_F} \longleftrightarrow {}^L\mathcal{P}/{}^L G. \quad (7.23)$$

Here we are using the fact that the standard parabolic subgroups of ${}^L G$ (with respect to ${}^L B = \widehat{B}(\mathbb{C}) \rtimes \text{Gal}_F$) are the subgroups of the form $P(\mathbb{C}) \rtimes \text{Gal}_F$ where $P \leq \widehat{G}$ is a parabolic subgroup containing $\widehat{B}$ whose corresponding subset of $\Delta^\vee$ is fixed by Gal_F . Combining (7.22) with (7.23) yields a bijection

$$(\mathcal{P}/G)(F) \longleftrightarrow {}^L\mathcal{P}/{}^L G.$$

We say that a parabolic subgroup of ${}^L G$ is **relevant** if its ${}^L G$ -conjugacy class is in the image of the composite

$$\mathcal{P}(F) \longrightarrow (\mathcal{P}/G)(F) \longrightarrow {}^L\mathcal{P}/{}^L G. \quad (7.24)$$

The image of $P \in \mathcal{P}(F)$ under (7.24) is a ${}^L G$ -conjugacy class of parabolic subgroups of ${}^L G$. Let ${}^L P$ be the unique standard parabolic subgroup in this conjugacy class.

Let Q be a parabolic subgroup of ${}^L G$. The unipotent radical N of Q° is normal in Q and will also be called the unipotent radical of Q . Then Q is the semidirect product of N by the normalizer in Q of any Levi subgroup M° of Q° . These normalizers will be called the **Levi subgroups** of Q . A Levi subgroup of a parabolic subgroup ${}^L P$ of ${}^L G$ is **relevant** if ${}^L P$ is relevant.

For given parabolic subgroup $P \leq G$, let $J(P) \subseteq \Delta$ be the subset corresponding to the image of P in $\mathcal{P}/G(F)$ under (7.22). Let

$$\Phi(P) := \mathbb{Z}J(P) \cap \Phi$$

be the set of elements of Φ that are $\mathbb{Z}$ -linear combinations of elements of $J(P)$. We then let

$$J(P)^\vee := \{\alpha^\vee \in \Delta^\vee : \alpha \in J(P)\} \quad (7.25)$$

and set

$$\Phi(P)^\vee := \mathbb{Z}J(P)^\vee \cap \Phi^\vee.$$

Given a Levi subgroup $M \leq P$, there is a $g' \in P(F^{\text{sep}})$ such that $g'Mg'^{-1}$ has based root datum

$$(X^*(T), X_*(T), \Phi(P), \Phi(P)^\vee, J(P), J(P)^\vee).$$

Hence, there is a Levi subgroup $\widehat{M}$ of $({}^L P)^\circ$ with based root datum

$$(X_*(T), X^*(T), \Phi(P), \Phi(P)^\vee, J(P)^\vee, J(P)).$$

The action of $\text{Gal}(F^{\text{sep}}/F)$ on $({}^L P)^\circ$ preserves $\widehat{M}$. The semidirect product of $\widehat{M}$ and $\text{Gal}(F^{\text{sep}}/F)$ is then a Levi subgroup of ${}^L P$ denoted by ${}^L M$. This is reasonable as ${}^L M$ is the dual group of M .

7.5 The Satake isomorphism for unramified groups

Let G be an unramified reductive group over a local field F and let $K \leq G(F)$ be a hyperspecial subgroup. In this section, we explain Langlands' extension of the Satake isomorphism to this setting. Let $\text{Fr} \in \text{Gal}_F$ be a choice of geometric Frobenius element. In other words, if $\mathfrak{p}$ is the prime ideal of the ring of integers $\mathcal{O}_F$ of F , then $\text{Fr}(x) = x^{|\mathcal{O}_F/\mathfrak{p}|^{-1}}$ for x in the algebraic closure of $\mathcal{O}_F/\mathfrak{p}$. Since G is unramified, it splits over an unramified extension. Hence, we can take the L -group to be

$$\widehat{G}(\mathbb{C}) \rtimes \text{Fr}^{\mathbb{Z}}.$$

In the split case, the Satake transform was phrased in terms of the quotient

$$(\widehat{G}/\text{conj})(\mathbb{C}). \quad (7.26)$$

In the nonsplit case, we replace $\widehat{G}$ by

$$\widehat{G} \rtimes \text{Fr} \subset \widehat{G} \rtimes \text{Fr}^{\mathbb{Z}},$$

the coset of $\widehat{G} \rtimes 1$ in $\widehat{G} \rtimes \text{Fr}^{\mathbb{Z}}$ containing $I \rtimes \text{Fr}$ (I being the identity of $\widehat{G}(\mathbb{C})$). We replace the action of $\widehat{G}$ on itself by conjugation with the action given on points in a $\mathbb{C}$ -algebra R by

$$\begin{aligned} \widehat{G}(R) \times (\widehat{G}(R) \rtimes \text{Fr}) &\longrightarrow \widehat{G}(R) \rtimes \text{Fr} \\ (g, (h \rtimes \text{Fr})) &\longmapsto gh\text{Fr}(g)^{-1} \rtimes \text{Fr}. \end{aligned} \quad (7.27)$$

This action can also be described as the action of $\widehat{G}$ on itself via Fr -conjugacy, a notion that plays a key role in the twisted trace formula (see §18.5 below, where the role of Fr is played by θ). Let

$$(\widehat{G} \rtimes \text{Fr} / \sim) := \text{Spec}(\mathcal{O}[\widehat{G} \rtimes \text{Fr}]^{\widehat{G}}),$$

where the implicit action is (7.27). Then the appropriate analogue of (7.26) in the nonsplit case is

$$(\widehat{G} \rtimes \text{Fr} / \sim)(\mathbb{C}).$$

As explained in Proposition 17.3.4, one can identify $(\widehat{G} \rtimes \text{Fr} / \sim)(\mathbb{C})$ with closed $\widehat{G}$ -orbits in $\widehat{G} \rtimes \text{Fr}$. An element of $\widehat{G}(\mathbb{C}) \rtimes \text{Fr}$ is said to be **Fr-semisimple** if its $\widehat{G}$ -orbit is closed. We can alternately describe $(\widehat{G} \rtimes \text{Fr} / \sim)(\mathbb{C})$ as the set of $\widehat{G}(\mathbb{C})$ -orbits of Fr-semisimple elements of $\widehat{G}(\mathbb{C}) \rtimes \text{Fr}$ under the action (7.27).

Let $T \leq G$ be a maximal torus contained in a Borel subgroup B . Its dual $\widehat{T}$ is a maximal torus of $\widehat{G}$. Consider the isomorphism

$$W(G, T)(F^{\text{sep}}) \xrightarrow{\sim} W(\widehat{G}, \widehat{T})(\mathbb{C}) \quad (7.28)$$

of (1.37). Let

$${}_FW \leq W(\widehat{G}, \widehat{T}) \quad (7.29)$$

be the unique algebraic subgroup of $W(\widehat{G}, \widehat{T})$ such that ${}_FW(\mathbb{C})$ is the image of $W(G, T)(F)$ under (7.28). We then let ${}_FN$ be the inverse image of ${}_FW$ under the homomorphism

$$N_{\widehat{G}}(\widehat{T}) \longrightarrow W(\widehat{G}, \widehat{T}). \quad (7.30)$$

One has an isomorphism of affine schemes

$$(\widehat{T} \rtimes \text{Fr}) / {}_FN \xrightarrow{\sim} \widehat{G} \rtimes \text{Fr} / \sim \quad (7.31)$$

[Bor79, Proposition 6.7]. This is the analogue of the Chevalley restriction theorem in this setting. The isomorphism (7.31) implies that an element of $\widehat{G}(\mathbb{C}) \rtimes \text{Fr}$ is Fr-semisimple if and only if its orbit intersects $\widehat{T} \rtimes \text{Fr}$.

In view of Theorem 7.2.1 and (7.31) the following theorem is natural:

Theorem 7.5.1 (Langlands and Satake) *Assume that G is unramified over F . There is an isomorphism of algebras*

$$\mathcal{S} : C_c^\infty(G(F) // K) \longrightarrow \mathcal{O}(\widehat{T} \rtimes \text{Fr})^{F^N}.$$

The map $\mathcal{S}$ occurring in Theorem 7.5.1 is again referred to as the **Satake isomorphism**. This is reasonable, as in the special case where G is split Theorem 7.5.1 reduces to Theorem 7.2.1. We will first rephrase Theorem 7.5.1 in several manners as we did in the split case in §7.2. We will then define the $\mathcal{S}$ and prove that it is an isomorphism, although we will refer to the existing literature for the proof of the key step (Theorem 7.5.3) to simplify the exposition.

In analogy with the case where G is split, Theorem 7.5.1 and (7.31) induce bijections

$$\text{Hom}(C_c^\infty(G(F) // K), \mathbb{C}) \xrightarrow{\sim} (\widehat{T} \rtimes \text{Fr} / {}_FN)(\mathbb{C}) \xrightarrow{\sim} (\widehat{G} \rtimes \text{Fr} / \sim)(\mathbb{C}). \quad (7.32)$$

Here we take homomorphisms in the category of $\mathbb{C}$ -algebras. Proposition 7.1.2 implies that isomorphism classes of irreducible unramified representations of $G(F)$ are in natural bijection with

$$\mathrm{Hom}(C_c^\infty(G(F) // K), \mathbb{C}).$$

Thus, (7.32) implies the following analogue of Corollary 7.2.2:

Corollary 7.5.2 *Assume that G is unramified over F . The composite isomorphism (7.32) induces a bijection*

$$(\widehat{G} \rtimes \mathrm{Fr} / \sim)(\mathbb{C}) \longleftrightarrow \left\{ \begin{array}{l} \text{isomorphism classes of irreducible} \\ \text{unramified representations of } G(F) \end{array} \right\}.$$

□

By way of terminology, the Fr -semisimple conjugacy class attached to an isomorphism class of unramified representations is called its **Langlands class**. In the split case, the eigenvalues of a representative of the conjugacy class are called its **Satake parameters**.

There is a categorified version of the Satake correspondence due to Mirković and Vilonen that is known as the **geometric Satake correspondence** [MV07]. This result is an important tool in the geometric Langlands program. Moreover, it has now been extended to mixed characteristic by the work of Zhu [Zhu17].

To explicate the Satake isomorphism, recall that $T \leq G$ is a maximal torus contained in a Borel subgroup $B \leq G$ and that K is a hyperspecial subgroup of $G(F)$. We let N denote the unipotent radical of B . We say that K is in **good position with respect to** (B, T) if the Iwasawa decomposition

$$G(F) = B(F)K$$

holds and

$$B(F) \cap K = (T(F) \cap K)(N(F) \cap K).$$

We also assume that $K_T := T(F) \cap K$ is a maximal compact subgroup of $T(F)$. In fact, given B and T we can always choose K so that these assumptions hold (see Theorem B.1.1). Conversely, if we are given a hyperspecial subgroup, we can always choose B and T so that K is in good position with respect to (B, T) and $T(F) \cap K$ is a maximal compact subgroup of $T(F)$ (see Lemma B.5.2).

Using the Iwasawa decomposition $G(F) = B(F)K = T(F)N(F)K$, we have a decomposition of measures

$$dg = d_\ell b dk = dt dn dk \tag{7.33}$$

where $d_\ell b$ is a left Haar measure on $B(F)$ and dt, dn, dk are Haar measures on $T(F)$, $N(F)$, and K , respectively (see Proposition 3.2.2). Here we are using

the fact that $N(F)$ is unimodular (see Exercise 3.12). We assume for the rest of the chapter that

$$\text{meas}_{dk}(K) = \text{meas}_{dn}(N(F) \cap K) = \text{meas}_{dt}(K_T) = 1. \quad (7.34)$$

We define a $\mathbb{C}$ -linear map

$$\begin{aligned} C_c^\infty(G(F) // K) &\longrightarrow C_c^\infty(T(F)/K_T) \\ f &\longmapsto f^B \end{aligned} \quad (7.35)$$

where

$$f^B(t) := \delta_B^{1/2}(t) \int_{N(F)} f(tn) dn. \quad (7.36)$$

The function f^B is known as the **constant term** of f along B . There is only a single slash in $C_c^\infty(T(F)/K_T)$ in (7.35) above because $T(F)$ is commutative.

By Proposition 8.7.2 below, we have an algebra homomorphism

$$\begin{aligned} C_c^\infty(G(F) // K) &\longrightarrow C_c^\infty(T(F)/K_T)^{W(G,T)(F)} \\ f &\longmapsto f^B. \end{aligned} \quad (7.37)$$

The following is due to Satake [Sat63]. We refer to [Car79, §4.2] for an exposition.

Theorem 7.5.3 (Satake) *The map (7.37) is an algebra isomorphism. $\square$*

Proving Theorem 7.5.3 is the key step in the proof of Theorem 7.5.1. To complete the proof of Theorem 7.5.1 we require an algebraic interpretation of the image of (7.37).

Let $T_s \leq T$ be the maximal split torus in T .

Lemma 7.5.4 *We have that*

$$W(G, T)(F) = W(G, T_s)(F) = W(G, T_s)(F^{\text{sep}}). \quad (7.38)$$

Proof. The torus T_s is a maximal split torus of G [Bor91, Proposition 20.6(iii)]. Thus, the right equality is a special case of Lemma 1.7.9.

Now $C_G(T_s) = T = C_G(T)$ by [Bor91, Proposition 20.6(i)] and [Mil17, Proposition 17.61]. Thus, Lemma 1.7.9 implies that

$$W(G, T_s)(F) = N_G(T_s)(F)/T(F) = N_G(T_s)(F^{\text{sep}})/T(F^{\text{sep}})$$

and by Proposition 1.10.7, we have that $W(G, T)(F)$ is

$$\{g \in N_G(T)(F^{\text{sep}})/T(F^{\text{sep}}) : g\xi(g^{-1}) \in T(F^{\text{sep}}) \text{ for all } \xi \in \text{Gal}_F\}.$$

Any element $g \in W(G, T)(F)$ normalizes $T(F^{\text{sep}})$ and induces an isomorphism $T \rightarrow T$, hence normalizes T_s . Thus, we obtain an injection

$W(G, T)(F) \hookrightarrow W(G, T_s)(F)$. Suppose that $g \in G(F)$ normalizes T_s . Then $g^{-1}Tg$ is a torus containing T_s which is therefore contained in $C_G(T_s) = T$. In other words, any element of $G(F)$ normalizing T_s normalizes T . It follows that the map $W(G, T)(F) \rightarrow W(G, T_s)(F)$ is surjective. $\square$

In view of Lemma 7.5.4, we have an injection

$$W(G, T_s)(F) = W(G, T)(F) \hookrightarrow W(G, T)(F^{\text{sep}}) = W(\widehat{G}, \widehat{T})(\mathbb{C}) \quad (7.39)$$

with image ${}_FW(\mathbb{C})$. The closed immersion $T_s \rightarrow T$ induces a dual quotient map

$$\nu : \widehat{T} \longrightarrow \widehat{T}_s. \quad (7.40)$$

We leave the proof of the following lemma to the reader in Exercise 7.3:

Lemma 7.5.5 *One has that*

$$\ker(\nu)(\mathbb{C}) = \left\{ t^{-1}\text{Fr}(t) : t \in \widehat{T}(\mathbb{C}) \right\}.$$

$\square$

The fact that $W(G, T_s)(F)$ acts on T and preserves T_s implies that ${}_FW$ acts on $\widehat{T}$ and $\widehat{T}_s$ and the map (7.40) is ${}_FW$ -equivariant.

Lemma 7.5.6 *There is an isomorphism of $\mathbb{C}$ -algebras*

$$C_c^\infty(T(F)/K_T)^{W(G, T)(F)} \xrightarrow{\sim} \mathcal{O}(\widehat{T}_s)^{{}_FW}.$$

Proof. Let $K_s := T_s(F) \cap K$. It is a maximal compact subgroup of $T_s(F)$. Applying Exercise 7.4, restriction of functions induces an isomorphism of $\mathbb{C}$ -algebras

$$C_c^\infty(T(F)/K_T) \xrightarrow{\sim} C_c^\infty(T_s(F)/K_s). \quad (7.41)$$

The group $W(G, T)(F)$ preserves T_s as explained in the proof of Lemma 7.5.4 and it is clear that (7.41) is $W(G, T)(F)$ -equivariant. Hence, by Lemma 7.5.4, we obtain an isomorphism

$$C_c^\infty(T(F)/K_T)^{W(G, T)(F)} \xrightarrow{\sim} C_c^\infty(T_s(F)/K_s)^{W(G, T_s)(F)}. \quad (7.42)$$

Let $\mathbb{C}[M]$ denote the group algebra of a free abelian group M . Let $\varpi \in F$ be a uniformizer. There is a $W(G, T_s)(F)$ -equivariant isomorphism of $\mathbb{C}$ -algebras

$$\begin{aligned} \mathbb{C}[X_*(T_s)] &\xrightarrow{\sim} C_c^\infty(T_s(F)/K_s) \\ \sum_{\lambda \in X_*(T_s)} c_\lambda \lambda &\longmapsto \sum_{\lambda \in X_*(T_s)} c_\lambda \mathbb{1}_{\lambda(\varpi)K_s}. \end{aligned} \quad (7.43)$$

Here each c_λ is a complex number and $c_\lambda = 0$ for all but finitely many λ . The isomorphism (7.43) is independent of the choice of ϖ .

Thus, we have isomorphisms of $\mathbb{C}$ -algebras

$$C_c^\infty(T_s(F)/K_s) \xrightarrow{\sim} \mathbb{C}[X_*(T_s)] \xrightarrow{\sim} \mathbb{C}[X^*(\widehat{T}_s)] \quad (7.44)$$

where the left isomorphism is the inverse of (7.43) and the right isomorphism follows from the definition of the dual torus. The left isomorphism in (7.44) is $W(G, T_s)(F)$ -equivariant. Since T_s is split, we have $X_*(T_s) = X_*(T_{sF^{\text{sep}}})$. Thus, the isomorphism $\mathbb{C}[X_*(T_s)] \xrightarrow{\sim} \mathbb{C}[X^*(\widehat{T}_s)]$ in (7.44) is the restriction of the $W(G, T)(F^{\text{sep}}) = W(\widehat{G}, \widehat{T})(\mathbb{C})$ -equivariant isomorphism

$$\mathbb{C}[X_*(T_{F^{\text{sep}}})] \longrightarrow \mathbb{C}[X^*(\widehat{T})].$$

It follows that $\mathbb{C}[X^*(\widehat{T}_s)]$ is preserved by ${}_FW(\mathbb{C})$ and (7.44) is ${}_FW(\mathbb{C})$ -equivariant, where ${}_FW(\mathbb{C})$ acts on $\mathbb{C}[X_*(T_s)]$ via the isomorphism

$$W(G, T_s)(F) \xrightarrow{\sim} {}_FW(\mathbb{C})$$

induced by (7.39). In particular (7.44) induces an isomorphism

$$C_c^\infty(T_s(F)/K_s)^{W(G, T_s)(F)} \xrightarrow{\sim} \mathbb{C}[X^*(\widehat{T}_s)]^{{}_FW(\mathbb{C})}. \quad (7.45)$$

By Theorem 1.7.4, we have a canonical identification

$$\mathbb{C}[X^*(\widehat{T}_s)] \xrightarrow{\sim} \mathcal{O}(\widehat{T}_s) \quad (7.46)$$

and we deduce that there is an isomorphism

$$\mathbb{C}[X^*(\widehat{T}_s)]^{{}_FW(\mathbb{C})} \xrightarrow{\sim} \mathcal{O}(\widehat{T}_s)^{{}_FW}.$$

Combining this with (7.42) and (7.45), we deduce the lemma. $\square$

Let

$$\begin{aligned} \widehat{T}_s / {}_FW &:= \text{Spec} \left(\mathcal{O}(\widehat{T}_s)^{{}_FW} \right), \\ (\widehat{T} \rtimes \text{Fr}) / {}_FN &:= \text{Spec} \left(\mathcal{O}(T \rtimes \text{Fr})^{{}_FN} \right). \end{aligned} \quad (7.47)$$

Lemma 7.5.7 *The inclusion $T_s \rightarrow T$ induces an isomorphism*

$$(\widehat{T} \rtimes \text{Fr}) / {}_FN \xrightarrow{\sim} \widehat{T}_s / {}_FW.$$

Proof. Recall the map $\nu : \widehat{T} \rightarrow \widehat{T}_s$ of (7.40). We continue to denote by ν the map $\widehat{T} \rtimes \text{Fr} \rightarrow \widehat{T}_s$ given on $\mathbb{C}$ -algebras R by

$$\nu : (\widehat{T} \rtimes \text{Fr})(R) \longrightarrow \widehat{T}_s(R)$$

$$t \rtimes \text{Fr} \longmapsto \nu(t).$$

By [Bor79, Lemma 6.2], every element of ${}_FN(\mathbb{C})$ is of the form wu where $\text{Fr}(w) = w$ and $u \in \widehat{T}(\mathbb{C})$. Thus

$$(wu)^{-1}(t \rtimes \text{Fr})wu = u^{-1}\text{Fr}(u)w^{-1}tw \rtimes \text{Fr}$$

for all $t \rtimes \text{Fr} \in \widehat{T}(\mathbb{C}) \rtimes \text{Fr}$. This together with Lemma 7.5.5 implies $\nu : \widehat{T} \rtimes \text{Fr} \rightarrow \widehat{T}_s$ is equivariant with respect to the action of ${}_FN$, where ${}_FN$ acts via its quotient ${}_FW$ on $\widehat{T}_s$. In particular ν induces a morphism

$$(\widehat{T} \rtimes \text{Fr})/{}_FN \longrightarrow \widehat{T}_s/{}_FW. \quad (7.48)$$

It is surjective because ν is surjective. The $\mathbb{C}$ -points of (7.48) are

$$(\widehat{T}(\mathbb{C}) \rtimes \text{Fr})/{}_FN(\mathbb{C}) \longrightarrow \widehat{T}_s(\mathbb{C})/{}_FW(\mathbb{C}) \quad (7.49)$$

by Proposition 17.3.4.

To prove the injectivity of (7.49), assume that $t, t' \in \widehat{T}(\mathbb{C})$ and

$$\nu(t \rtimes \text{Fr}) = w^{-1}\nu(t' \rtimes \text{Fr})w$$

for some $w \in {}_FW(\mathbb{C})$. As mentioned above, we may take w to be fixed by Fr . Then $\nu(t) = \nu(w^{-1}t'w)$ so $t = xw^{-1}t'w$ for some $x \in (\ker \nu)(\mathbb{C})$. By Lemma 7.5.5, $x = u^{-1}\text{Fr}(u)$ for some $u \in \widehat{T}(\mathbb{C})$ and we deduce that $t \rtimes \text{Fr} = (wu)^{-1}(t' \rtimes \text{Fr})wu$.

Thus, (7.49) is bijective. By Proposition 1.2.9, to conclude that (7.48) is an isomorphism it suffices to check that $(\widehat{T} \rtimes \text{Fr})/{}_FN$ and $\widehat{T}_s/{}_FW$ are integral and $\widehat{T}_s/{}_FW$ is normal. The schemes $\widehat{T} \rtimes \text{Fr}$ and $\widehat{T}_s$ are smooth and irreducible, hence integral and normal. The quotients $(\widehat{T} \rtimes \text{Fr})/{}_FN$ and $\widehat{T}_s/{}_FW$ are therefore integral and normal by [MFK94, §0.2(2)] and the fact that GIT quotients are categorical quotients (see §17.2). $\square$

Combining Theorem 7.5.3, Lemma 7.5.6 and the map of coordinate rings induced by Lemma 7.5.7, we see that we have constructed a homomorphism

$$\mathcal{S} : C_c^\infty(G(F) // K) \longrightarrow C_c^\infty(T(F)/K_T)^{W(G,T)(F)} \longrightarrow \mathcal{O}(\widehat{T} \rtimes \text{Fr})^{F^N}. \quad (7.50)$$

This is the Satake isomorphism. Thus, we have completed the proof of Theorem 7.5.1. We will also write $\mathcal{S}$ for the isomorphism

$$\mathcal{S} : C_c^\infty(G(F) // K) \longrightarrow C_c^\infty(T(F)/K_T)^{W(G,T)(F)} \longrightarrow \mathcal{O}(\widehat{T}_s)^{F^N} \quad (7.51)$$

obtained by combining Theorem 7.5.3 and Lemma 7.5.6. This is harmless in view of Lemma 7.5.7.

The first map in this factorization of the Satake isomorphism is a special case of a parabolic descent map. In general, any construction that relates

objects on the group G to objects on Levi subgroups of its parabolic subgroups (in this case the Levi subgroup T of the Borel subgroup B) is known as **parabolic descent**. We will use this idea in the next section to give a more explicit parametrization of unramified representations.

7.6 The principal series

We now explain how to explicitly realize unramified representations. Let G be an unramified reductive group over a non-Archimedean local field F with Borel subgroup $B \leq G$ and maximal torus $T \leq B$. Let N be the unipotent radical of B and let $K \leq G(F)$ be a hyperspecial subgroup. We assume that K is in good position with respect to (B, T) and that $K_T := T(F) \cap K$ is a maximal compact open subgroup of $T(F)$. We assume measures are normalized as in (7.33) and (7.34).

We recall from Proposition 3.6.1 that the **modular quasi-character**

$$\delta_B := \delta_{B(F)} : B(F) \longrightarrow \mathbb{R}_{>0} \quad (7.52)$$

is

$$\delta_B(b) := |\det(\mathrm{Ad}(b) : \mathfrak{b} \longrightarrow \mathfrak{b})|,$$

where $\mathfrak{b} := \mathrm{Lie} B$. The last ingredient we need to define unramified principal series representations is unramified quasi-characters.

Definition 7.6.1. A quasi-character $\chi : T(F) \rightarrow \mathbb{C}^\times$ is **unramified** if it is trivial on K_T .

Let us explicitly describe unramified quasi-characters. As in the previous section, we let $T_s \leq T$ be the maximal split torus. We remind the reader that $X^*(T)$, the abelian group of homomorphisms $T \rightarrow \mathbb{G}_m$, is in general a proper subgroup of the group $X^*(T_{F^{\mathrm{sep}}})$ of homomorphisms $T_{F^{\mathrm{sep}}} \rightarrow \mathbb{G}_{mF^{\mathrm{sep}}}$.

In analogy with (4.29), we define a map

$$H_T : T(F) \longrightarrow \mathfrak{a}_T := \mathrm{Hom}(X^*(T), \mathbb{R})$$

via

$$e^{\langle H_T(t), \chi \rangle} = |\chi(t)|$$

for $\chi \in X^*(T)$. For each

$$\lambda \in \mathfrak{a}_{T\mathbb{C}}^* := X^*(T) \otimes_{\mathbb{Z}} \mathbb{C},$$

we then obtain a quasi-character

$$t \longmapsto e^{\langle H_T(t), \lambda \rangle}.$$

It is clearly unramified. We reinterpret this description of unramified quasi-characters as follows. First, observe that the identifications

$$X^*(T)_{\mathbb{C}} = X^*(T_s)_{\mathbb{C}} = X_*(\widehat{T}_s)_{\mathbb{C}} \quad (7.53)$$

allow us to identify $\mathfrak{a}_{T_{\mathbb{C}}}^*$ with the Lie algebra of $\widehat{T}_s$ (see Exercise 7.8). Thus

$$\widehat{T}_s(\mathbb{C}) = \{q^{-\lambda} : \lambda \in \mathfrak{a}_{T_{\mathbb{C}}}^*\}$$

where q , as usual, is the order of the residue field of $\mathcal{O}_F$ and $q^{-\lambda}$ is the matrix exponential $e^{-(\log q)\lambda}$. The group ${}_FW(\mathbb{C})$ of (7.29) acts on the set $\widehat{T}_s(\mathbb{C})$ as explained above Lemma 7.5.6. The map (7.28) restricts to an isomorphism

$$W(G, T)(F) \xrightarrow{\sim} {}_FW(\mathbb{C})$$

which we occasionally use to identify these two groups. With this convention in mind, the identifications (7.53) are ${}_FW(\mathbb{C})$ -equivariant.

Lemma 7.6.2 *One has a bijection*

$$\begin{aligned} \widehat{T}_s(\mathbb{C}) / {}_FW(\mathbb{C}) &\longrightarrow \text{Hom}(T(F)/K_T, \mathbb{C}^\times) / W(G, T)(F) \\ q^{-\lambda} &\longmapsto (t \mapsto e^{\langle H_T(t), \lambda \rangle}). \end{aligned}$$

In the lemma, homomorphisms are taken in the category of topological groups.

Proof. By Exercise 7.4, the map $T_s \rightarrow T$ induces an isomorphism of topological groups

$$T_s(F)/K_s \xrightarrow{\sim} T(F)/K_T, \quad (7.54)$$

where $K_s := T_s(F) \cap K$ is the maximal compact subgroup of $T_s(F)$. As explained in the proof of Lemma 7.5.4 the action of $W(G, T)(F)$ on $T(F)$ preserves $T_s(F)$. It follows that (7.54) is $W(G, T)(F)$ -equivariant. On the other hand, we have a ${}_FW(\mathbb{C})$ -equivariant map

$$\begin{aligned} \widehat{T}_s(\mathbb{C}) &\longrightarrow \text{Hom}(T_s(F)/K_s, \mathbb{C}^\times) \\ q^{-\lambda} &\longmapsto (t \mapsto e^{\langle H_T(t), \lambda \rangle}). \end{aligned} \quad (7.55)$$

Thus, we need only check that (7.55) is bijective. By choosing an isomorphism $T_s \cong \mathbb{G}_m^d$, we are reduced to observing that

$$\begin{aligned} \mathbb{C}^\times &\longrightarrow \text{Hom}(F^\times / \mathcal{O}_F^\times, \mathbb{C}^\times) \\ q^{-\lambda} &\longmapsto (t \mapsto e^{\lambda \log |t|}) \end{aligned}$$

is a bijection. Here we represent an element of $\mathbb{C}^\times$ as $q^{-\lambda}$ for some $\lambda \in \mathbb{C}$. $\square$

Let us relate Lemma 7.6.2 to the constructions in §7.5. We have a commutative diagram of $W(T, G)(F)$ -equivariant bijections

$$\begin{array}{ccc}
\mathrm{Hom}(C_c^\infty(T(F)/K_T), \mathbb{C}) & \xrightarrow{a} & \mathrm{Hom}(T(F)/K_T, \mathbb{C}^\times) \\
\downarrow & & \downarrow \\
\mathrm{Hom}(C_c^\infty(T_s(F)/K_s), \mathbb{C}) & \xrightarrow{b} & \mathrm{Hom}(T_s(F)/K_s, \mathbb{C}^\times).
\end{array}$$

Here homomorphisms on the left are taken in the category of $\mathbb{C}$ -algebras, and homomorphisms on the right are taken in the category of topological groups. The vertical arrows are given by restriction of functions and are isomorphisms by Exercise 7.4. The horizontal arrows are given by

$$a(\varphi)(t) := \varphi(\mathbb{1}_{tK_T}) \quad \text{and} \quad b(\varphi)(t) := \varphi(\mathbb{1}_{tK_s}).$$

One checks that b is a bijection, and hence so is a . The natural map

$$\mathrm{Hom}(C_c^\infty(T(F)/K), \mathbb{C})/W(G, T)(F) \longrightarrow \mathrm{Hom}(C_c^\infty(T(F)/K)^{W(G, T)(F)}, \mathbb{C})$$

is a bijection. Thus, we obtain a bijection

$$\mathrm{Hom}(T(F)/K_T, \mathbb{C}^\times)/W(G, T)(F) \xrightarrow{\sim} \mathrm{Hom}(C_c^\infty(T(F)/K_T)^{W(G, T)(F)}, \mathbb{C}) \quad (7.56)$$

where on the left homomorphisms are taken in the category of topological groups and on the right homomorphisms are taken in the category of $\mathbb{C}$ -algebras.

By unwinding the definition of the isomorphism in Lemma 7.5.6, one obtains the following lemma:

Lemma 7.6.3 *The bijection of Lemma 7.6.2 composed with (7.56) is the bijection*

$$\widehat{T}_s(\mathbb{C})/{}_FW(\mathbb{C}) \xrightarrow{\sim} \mathrm{Hom}(C_c^\infty(T(F)/K_T)^{W(G, T)(F)}, \mathbb{C})$$

induced by the isomorphism of Lemma 7.5.6 above. $\square$

For $\lambda \in \mathfrak{a}_{T\mathbb{C}}^*$, we define the representation

$$I(\lambda) := \mathrm{Ind}_B^G(t \mapsto e^{\langle H_T(t), \lambda \rangle}) \quad (7.57)$$

to be the (smooth) representation of $G(F)$ on the space of functions

$$\left\{ \varphi \in C^\infty(G(F)) : \varphi(tng) = \delta_B^{1/2}(t) e^{\langle H_T(t), \lambda \rangle} \varphi(g) \right\} \quad (7.58)$$

for all $(t, n, g) \in T(F) \times N(F) \times G(F)$. Here $G(F)$ acts via right translations

$$I(\lambda)(g)\varphi(x) := \varphi(xg). \quad (7.59)$$

This is an example of an induced representation (see §8.1 for the general construction). The factor of $\delta_B^{1/2}$ is present so that if

$$\lambda \in i\mathfrak{a}_T^* = i(X^*(T) \otimes_{\mathbb{Z}} \mathbb{R})$$

then $I(\lambda)$ is pre-unitarizable; this assertion is part of the following special case of propositions 8.2.2 and 8.2.3 below:

Proposition 7.6.4 *The representations $I(\lambda)$ are admissible and satisfy*

$$I(\lambda)^\vee \cong I(-\lambda).$$

If $\lambda \in i\mathfrak{a}_T^*$, then $I(\lambda)$ is pre-unitarizable. □

We note that the condition $\lambda \in i\mathfrak{a}_T^*$ is equivalent to the statement that the quasi-character $t \mapsto e^{\langle H_T(t), \lambda \rangle}$ is unitary (i.e. it is a character).

Definition 7.6.5. An **unramified principal series** representation of $G(F)$ is a representation isomorphic to $I(\lambda)$ for some $\lambda \in \mathfrak{a}_{T\mathbb{C}}^*$.

This definition is traditional, but it is a little inconsistent. We have assumed unramified representations are irreducible (see Definition 7.1.1), but unramified principal series representations need not be irreducible.

Lemma 7.6.6 *There is a unique line in $I(\lambda)$ fixed by K .*

Proof. Since $G(F) = B(F)K$, a function in $I(\lambda)$ is uniquely determined by its restriction to K . □

Our next aim is to explain how the Weyl group acts on unramified principal series representations. First, we define semisimplifications of smooth representations. A smooth representation (π, V) of $G(F)$ has **finite length** if there is a chain of $C_c^\infty(G(F))$ -submodules

$$0 = V_0 < V_1 < \cdots < V_{n-1} < V_n \tag{7.60}$$

(with $n \in \mathbb{Z}_{>0}$) such that V_i/V_{i-1} is irreducible for $0 < i \leq n$. The chain (7.60) is known as a composition series of V . If π has finite length, its **semisimplification** is the representation

$$\bigoplus_{i=1}^n V_i/V_{i-1}.$$

Up to isomorphism, the semisimplification depends only on the isomorphism type of V by the Jordan-Hölder theorem. An irreducible smooth representation of $G(F)$ is a subquotient of (π, V) if and only if it is isomorphic to V_i/V_{i-1} for some i .

The Weyl group $W(G, T)(F)$ acts on $X^*(T)$, hence on $\mathfrak{a}_{T\mathbb{C}}^*$. It therefore makes sense to consider $I(w(\lambda))$ for $\lambda \in \mathfrak{a}_{T\mathbb{C}}^*$ and $w \in W(G, T)(F)$. We say that $\lambda \in \mathfrak{a}_{T\mathbb{C}}^*$ is **regular** if $w(\lambda) = \lambda$ for $w \in W(G, T)(F)$ implies that w is the identity. Thus, λ is regular if and only if λ lies in an open Weyl chamber in $\mathfrak{a}_{T\mathbb{C}}^*$.

Theorem 7.6.7 *Let $\lambda \in \mathfrak{a}_{T\mathbb{C}}^*$ and $w \in W(G, T)(F)$.*

- (a) *The representation $I(\lambda)$ is of finite length.*
- (b) *The semisimplifications of $I(\lambda)$ and $I(w(\lambda))$ are isomorphic.*
- (c) *If $\lambda \in \mathfrak{ia}_T^*$ is regular then $I(\lambda)$ is irreducible.*

□

For the proof of the theorem, one can refer to [Car79, Theorem 3.3]. These results are due to Casselman.

Lemma 7.6.8 *The representation $I(\lambda)$ admits a unique (irreducible) unramified subquotient.*

Proof. By Theorem 7.6.7, there exists a composition series

$$0 = V_0 < \cdots < V_n = I(\lambda).$$

There is a unique line of K -fixed vectors in $I(\lambda)$ by Lemma 7.6.6. Let $1 \leq i \leq n$ be the smallest index such that V_i contains this line. Then V_i/V_{i-1} is unramified.

To prove that i is the unique index such that V_i/V_{i-1} is unramified, it suffices to observe that the map $V_j^K \rightarrow (V_j/V_{j-1})^K$ is surjective for all $1 \leq j \leq n$ by Exercise 7.9. □

We will also require the following result [Cas, Corollary 6.3.9]:

Theorem 7.6.9 *If π is an irreducible subquotient of $I(\lambda)$ then there exists a $w \in W(G, T)(F)$ such that π is a subrepresentation of $I(w(\lambda))$.* □

Let $J(\lambda)$ denote the unique (irreducible) unramified subquotient of $I(\lambda)$. By Theorem 7.6.7, $J(\lambda) \cong J(w(\lambda))$ for all $w \in W(G, T)(F)$.

The Satake isomorphism (7.51) identifies characters of $C_c^\infty(G(F) // K)$ with elements of $\widehat{T}_s(\mathbb{C})/{}_FW(\mathbb{C})$. These in turn may be identified with isomorphism classes of unramified representations by Proposition 7.1.2, yielding a bijection

$$\widehat{T}_s(\mathbb{C})/{}_FW(\mathbb{C}) \longleftrightarrow \left\{ \begin{array}{l} \text{isomorphism classes of irreducible} \\ \text{unramified representations of } G(F) \end{array} \right\}. \quad (7.61)$$

The following theorem makes this bijection explicit:

Theorem 7.6.10 *The bijection (7.61) is given by $q^{-\lambda} \mapsto J(\lambda)$. In particular, every unramified representation of $G(F)$ is isomorphic to a $J(\lambda)$ for some $\lambda \in \mathfrak{a}_{T\mathbb{C}}^*$. Moreover $J(\lambda) \cong J(\lambda')$ if and only if $\lambda = w(\lambda')$ for some $w \in W(G, T)(F)$.*

This theorem implies that the Langlands class of $J(\lambda)$ is the image of $q^{-\lambda}$ under the isomorphism

$$\widehat{T}_s(\mathbb{C})/{}_FW(\mathbb{C}) \xrightarrow{\sim} \widehat{T}(\mathbb{C})/{}_FN(\mathbb{C}) \xrightarrow{\sim} (\widehat{G} \rtimes \text{Fr} / \sim)(\mathbb{C})$$

given by Lemma 7.5.7 and (7.31).

Given the comments before the statement of Theorem 7.6.10, the theorem itself is an immediate consequence of the following proposition:

Proposition 7.6.11 *Let $f \in C_c^\infty(G(F) // K)$. For $\lambda \in \mathfrak{a}_{T\mathbb{C}}^*$ one has that*

$$\mathrm{tr} J(\lambda)(f) = \mathcal{S}(f)(q^{-\lambda}).$$

Here $\mathcal{S}$ is the Satake isomorphism of (7.51).

Proof. One has that

$$\mathrm{tr} I(\lambda)(f) = \mathrm{tr} e^{\langle H_T(\cdot), \lambda \rangle}(f^B)$$

by Proposition 8.6.2. On the other hand, by Lemma 7.6.3 one has

$$\mathrm{tr} e^{\langle H_T(\cdot), \lambda \rangle}(f^B) = \mathcal{S}(f)(q^{-\lambda}).$$

Thus, to complete the proof, it suffices to show that $\mathrm{tr} I(\lambda)(f) = \mathrm{tr} J(\lambda)(f)$. By Theorem 7.6.9 upon replacing λ by $w\lambda$ for some $w \in W(G, T)(F)$, we can assume that $J(\lambda)$ is a subrepresentation of $I(\lambda)$. The inclusion $J(\lambda) \rightarrow I(\lambda)$ induces a nonzero morphism of $C_c^\infty(G(F) // K)$ -modules

$$J(\lambda)^K \longrightarrow I(\lambda)^K.$$

This morphism is an isomorphism because both spaces are 1-dimensional by Lemma 7.6.6. $\square$

7.7 Weak global L -packets

Let G be a reductive group over a global field F . By convention, an admissible representation of $G(\mathbb{A}_F)$ is an admissible $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -module in the number field case and an admissible $G(\mathbb{A}_F)$ -module in the function field case.

Definition 7.7.1. Two irreducible admissible representations π_1, π_2 of $G(\mathbb{A}_F)$ are **weakly L -indistinguishable** if $\pi_1^S \cong \pi_2^S$ for some finite set of places S of F . An equivalence class of weakly L -indistinguishable admissible representations is called an **weak global L -packet** (of G). If a weak global L -packet contains an automorphic representation, we say that it is a **weak automorphic L -packet** (of G).

In §12.5, we will define a notion of local L -packets. The definition of local L -packets will be used to give a refinement of the notion of a global L -packet; this is why the adjective weak is used above.

For the remainder of this section, we will omit the word global to save ink. It may seem odd that we include merely admissible, and not automorphic,

representations in a weak automorphic L -packet, but this turns out to be convenient for technical reasons. In practice, one describes the weak L -packet of an automorphic representation and then asks which representations in the weak L -packet are automorphic.

To describe weak L -packets in more detail, we set some notational conventions. If S is a finite set of places of F including the infinite places such that G_{F_v} is unramified for $v \notin S$, we say that G is **unramified outside of S** . Now choose a maximal compact subgroup $K \leq G(\mathbb{A}_F)$. Let S be a finite set of places of F including the infinite places and assume that K_v (the projection of K to $G(F_v)$) is hyperspecial for $v \notin S$. We then say that π is **unramified outside of S** if π_v contains a nonzero vector fixed by K_v for $v \notin S$. This depends on the choice of K , but if K' is a different choice then π_v will be unramified with respect to K_v if and only if it is unramified with respect to K'_v for all but finitely many v by Corollary 2.4.11.

The Satake isomorphism provides a convenient way of thinking about weak L -packets. Assume that G is unramified outside of S . Notice that if π is an admissible representation of $G(\mathbb{A}_F)$ unramified outside of S then it defines a character

$$\mathrm{tr} \pi^S : C_c^\infty(G(\mathbb{A}_F^S) // K^S) \longrightarrow \mathbb{C}$$

where $K^S \leq G(\mathbb{A}_F^S)$ is a hyperspecial subgroup, i.e. $K_v \leq G(F_v)$ is hyperspecial for all $v \notin S$. By the Satake isomorphism, giving this character is equivalent to giving, for each $v \notin S$, a class $c_v \in (\widehat{G} \rtimes \mathrm{Fr}_v / \sim)(\mathbb{C})$. Here Fr_v is a choice of Frobenius element at v and the quotient is with respect to the action of $\widehat{G}$ given in (7.27). An element of $\widehat{G}(\mathbb{C}) \rtimes \mathrm{Fr}_v$ is **Fr-semisimple** if its $\widehat{G}$ -orbit is (Zariski) closed. As explained in a more general context in §17.3, $(\widehat{G} \rtimes \mathrm{Fr}_v / \sim)(\mathbb{C})$ is the set of $\widehat{G}(\mathbb{C})$ -orbits of Fr-semisimple elements of $\widehat{G}(\mathbb{C}) \rtimes \mathrm{Fr}_v$. When G is split, this set may be identified with semisimple conjugacy classes in $\widehat{G}(\mathbb{C})$. Let

$$\begin{aligned} \mathrm{LP}_w(G) &:= \{\text{weak } L\text{-packets of } G\}, \\ \mathrm{LP}_{w,\mathrm{aut}}(G) &:= \{\text{weak automorphic } L\text{-packets of } G\} \end{aligned} \tag{7.62}$$

and

$$c(G) := \left\{ (c_v) \in \prod_{v \notin S} (\widehat{G} \rtimes \mathrm{Fr}_v / \sim)(\mathbb{C}) \right\} / \sim$$

where we say that $\prod_{v \notin S} c_v \sim \prod_{v \notin S} c'_v$ if and only if $c_v = c'_v$ for almost every $v \notin S$ (i.e. all v outside of a finite set). There is no need to be specific about the set of places S in the notation here because we can always enlarge S by a finite set. Any weak L -packet has a representative that is unramified outside of some finite set of places S , so we have a bijection

$$c : \mathrm{LP}_w(G) \longrightarrow c(G) \tag{7.63}$$

defined in the obvious manner.

Let

$$r : {}^L H \longrightarrow {}^L G$$

be an L -map. This gives rise to a map

$$c(H) \longrightarrow c(G).$$

We can now state a weak form of the Langlands functoriality conjecture:

Conjecture 7.7.2 (Langlands) *Given a weak automorphic L -packet Π of H , there is a weak automorphic L -packet $r(\Pi)$ of G such that $c(r(\Pi)) = r(c(\Pi))$.*

Thus, Conjecture 7.7.2 asserts that there is a map making the following diagram commute:

$$\begin{array}{ccc} \mathrm{LP}_{\mathrm{w},\mathrm{aut}}(H) & \overset{?}{\dashrightarrow} & \mathrm{LP}_{\mathrm{w},\mathrm{aut}}(G) \\ \downarrow c & & \downarrow c \\ c(H) & \xrightarrow{r} & c(G). \end{array}$$

If the weak automorphic L -packet $r(\Pi)$ exists, it is called a **weak transfer** of Π with respect to r , or simply a weak transfer if r is understood.

The statement of Conjecture 7.7.2 is an oversimplification. In fact, it could be false as stated for completely general Π and r . However, it conveys the spirit of the precise version of the global Langlands functoriality conjecture which will be discussed in §12.6.

The fact that the class $c(\Pi)$ is only defined up to a finite set of places is an irritant, but it is not as horrible as it first appears. To explain, we will require some results described in more detail in §10.7. Let $n_1, \dots, n_d$ be a collection of positive integers with $\sum_{i=1}^d n_i = n$. Let $P = MN \leq \mathrm{GL}_n$ be the parabolic subgroup of type $(n_1, \dots, n_d)$ with its standard Levi decomposition (see Example 1.15). For each i let π_i be a cuspidal automorphic representation of $A_{\mathrm{GL}_{n_i}} \backslash \mathrm{GL}_{n_i}(\mathbb{A}_F)$ and let $\lambda \in \mathfrak{a}_{M\mathbb{C}}^*$. We can then form the induced representation

$$I(\otimes_{i=1}^d \pi_i, \lambda)$$

as in §10.3. In general, it may have several irreducible subquotients, all of which turn out to be automorphic. However it always has a canonical subquotient that will be described in more detail in §10.7. Automorphic representations equivalent to this canonical subquotient are known as **isobaric representations**. In particular, if $I(\otimes_{i=1}^d \pi_i, \lambda)$ is irreducible then it is isobaric; this holds in particular if $n_1 = n$ which is to say that the representation is cuspidal.

We now record the following foundational fact from [JS81b, JS81a]:

Theorem 7.7.3 (Jacquet and Shalika) *Any weak automorphic L -packet of GL_n contains a unique isobaric element.* $\square$

Thus, at least for GL_n , admissible representations that are automorphic and isobaric are very special; they are determined by their local factors at any cofinite set of places of F (see Exercise 7.11). In particular, forgetting about finite sets of places is not an issue for GL_n . For other reductive groups, the automorphic elements in a weak L -packet are supposed to be the fibers of any transfer of the packet to a general linear group.

Studying weak L -packets is no more or less difficult than studying combinatorics of Langlands classes. For general groups, describing which of the many admissible representations in a weak automorphic L -packet are automorphic representations is quite subtle. We refer to §12.6 for more details.

For a nonempty finite set of places S of F , we could ask which irreducible admissible representations of $G(F_S)$ are local components at S of automorphic representations. Assume that G is semisimple and simply connected over $\mathbb{Q}$ and that S is a finite set of places of $\mathbb{Q}$ such that $G(\mathbb{Q}_S)$ is noncompact. Then any irreducible admissible tempered representation of $G(\mathbb{Q}_S)$ is a limit, in a suitable sense, of local components at S of automorphic representations of $G(\mathbb{A}_{\mathbb{Q}})$. We refer to [Clo07, §3.3] for a precise statement. The method used in loc. cit. is due to Burger, Li, and Sarnak.

Exercises

7.1. Let S_2 act on $\mathbb{C}[t_1^{\pm}, t_2^{\pm}]$ by letting the nontrivial element act by $t_1 \mapsto t_2$. Define a linear map

$$\mathcal{S}' : C_c^{\infty}(\mathrm{GL}_2(F) // \mathrm{GL}_2(\mathcal{O}_F)) \longrightarrow \mathbb{C}[t_1^{\pm}, t_2^{\pm}]^{S_2}$$

by setting, for each $n \in \mathbb{Z}$, $k \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{S}' \left(\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \mathbb{1}_{\mathrm{GL}_2(\mathcal{O}_F)} \left(\varpi^{k-i+n} \varpi^{i+n} \right)_{\mathrm{GL}_2(\mathcal{O}_F)} \right) = q^{k/2} (t_1 t_2)^n \mathrm{Sym}^k(t_1, t_2).$$

Show that this is an algebra isomorphism and deduce that $\mathcal{S} = \mathcal{S}'$.

7.2. Let $\mathbb{C}[M]$ denote the group algebra of a free abelian group M . Let F be a field. For any torus T over F , construct Gal_F -equivariant isomorphisms

$$\mathbb{C}[X_*(T_{F^{\mathrm{sep}}})] \xrightarrow{\sim} \mathbb{C}[X^*(\widehat{T})] \xrightarrow{\sim} \mathcal{O}[\widehat{T}]$$

that induce natural isomorphisms between the functors

$$\mathbb{C}[X_*((\cdot)_{F^{\mathrm{sep}}})], \quad \mathbb{C}[X^*(\widehat{(\cdot)})], \quad \mathcal{O}[\widehat{(\cdot)}]$$

from the category of tori over F to the category of $\mathbb{C}$ -algebras.

7.3. Prove Lemma 7.5.5.

7.4. Let T be a torus over a non-Archimedean local field F that splits over an unramified extension. Let $T_s \leq T$ be the maximal split torus. Let $K_T \leq T(F)$ be the maximal compact subgroup of $T(F)$ and let $K_s := T_s(F) \cap K_T$ be the maximal compact subgroup of $T_s(F)$. Prove that the inclusion $T_s \rightarrow T$ induces an isomorphism of topological groups

$$T_s(F)/K_s \xrightarrow{\sim} T(F)/K_T.$$

7.5. Let E/F be a quadratic extension of number fields with nontrivial Galois automorphism σ . For any F -algebra R , let

$$U_n(R) := \{g \in \mathrm{GL}_2(E \otimes_F R) : \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \sigma(g)^{-t} \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} = g\}.$$

This is a unitary group in two variables over F . Prove that U_n is quasi-split over F , U_{nF_v} is split at every place v of F where E/F is split, and U_{nF_v} is nonsplit at every place v of F where E/F is nonsplit.

7.6. Prove the description of ${}^L\mathrm{Res}_{E/F}\mathrm{GL}_n$ given in (7.18) is true.

7.7. Prove that the description of ${}^L U_n$ given in (7.20) is true.

7.8. Let T be a split torus over an Archimedean local field F . Given $\lambda \in X_*(T)$, applying the Lie functor one obtains a map $d\lambda : F \rightarrow \mathrm{Lie} T$. Show that

$$\begin{aligned} X_*(T) &\longrightarrow \mathrm{Lie} T \\ \lambda &\longmapsto d\lambda(1) \end{aligned}$$

induces an isomorphism $X_*(T)_F := X_*(T) \otimes_{\mathbb{Z}} F \rightarrow \mathrm{Lie} T$.

7.9. Let G be a td-group and let $K \leq G$ be a compact open subgroup. Prove that the functor from smooth representations of G to $C_c^\infty(G//K)$ -modules given on objects by

$$V \longmapsto V^K$$

is exact. In other words, it sends exact sequences to exact sequences.

7.10. Let F be a local field and let T be a torus over F . Prove that if $T(F)$ is compact then T is anisotropic.

7.11. Let π be an isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ for some global field F . Show that if S is a nonempty finite set of places of F and $\pi'_S \not\cong \pi_S$ is an admissible representation of $\mathrm{GL}_n(F_S)$ then $\pi'_S \otimes \pi^S$ is never an isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$.

Chapter 8

Non-Archimedean Representation Theory



Let us assume we know nothing,
which is a reasonable
approximation.

D. Kazhdan

In this chapter, we explain how general admissible representations are built up out of supercuspidal representations via the process of parabolic induction.

8.1 Introduction

Let G be a reductive group over a non-Archimedean local field F , let $P \leq G$ be a parabolic subgroup, and let $P = MN$ be a Levi decomposition with $M \leq P$ a Levi subgroup and N the unipotent radical of P . This notation will be in force throughout the chapter.

We recall from Proposition 3.6.1 that the **modular quasi-character**

$$\delta_P := \delta_{P(F)} : P(F) \longrightarrow \mathbb{R}_{>0} \quad (8.1)$$

is

$$\delta_P(p) := |\det(\mathrm{Ad}(p) : \mathfrak{p} \longrightarrow \mathfrak{p})|,$$

where $\mathfrak{p} = \mathrm{Lie} P$. Let (σ, V) be a smooth irreducible representation of $M(F)$. Since $P/N \cong M$, σ extends to a representation of $P(F)$. Using the modular quasi-character, we define the **parabolically induced representation** (or simply **induced representation**) $I(\sigma) := \mathrm{Ind}_P^G(\sigma)$ to be the (smooth) representation of $G(F)$ on the space of functions

$$\mathrm{Ind}_P^G(V) = \left\{ \varphi \in C^\infty(G(F), V) : \varphi(mng) = \delta_P^{1/2}(m)\sigma(m)\varphi(g) \right\}$$

for all $(m, n, g) \in M(F) \times N(F) \times G(F)$. Here $C^\infty(G(F), V)$ denotes the space of locally constant functions from $G(F)$ to V as in Definition 5.2.1. The group $G(F)$ acts via right translation:

$$I(\sigma)(g)\varphi(x) = \varphi(xg). \quad (8.2)$$

The factor of $\delta_P^{1/2}$ is present so that, if σ is unitarizable, then $I(\sigma)$ is also unitarizable (see Proposition 8.2.2). This procedure yields a functor

$$\mathrm{Ind}_P^G : \mathrm{Rep}_{\mathrm{sm}} M(F) \longrightarrow \mathrm{Rep}_{\mathrm{sm}} G(F) \quad (8.3)$$

from smooth representations of $M(F)$ to smooth representations of $G(F)$ (in these categories, objects are smooth representations in the sense of §5.3 and morphisms are equivariant maps). This functor is called **parabolic induction**.

If V is a smooth representation of $G(F)$, then its contragredient $V^\vee$ is defined as in §5.4. Let

$$\langle , \rangle : V \times V^\vee \longrightarrow \mathbb{C}$$

denote the canonical pairing. A function of the form

$$m(g) := m_{\varphi, \varphi^\vee}(g) := \langle \pi(g)\varphi, \varphi^\vee \rangle \quad (8.4)$$

for $(\varphi, \varphi^\vee) \in V \times V^\vee$ is called a **matrix coefficient** of π (for the relationship between this notion and Definition 4.3.2, see Exercise 8.1).

Definition 8.1.1. A **supercuspidal** (resp. **quasi-cuspidal**) representation of $G(F)$ is an admissible (resp. smooth) representation all of whose matrix coefficients are compactly supported modulo the center of $G(F)$.

We have taken the term “quasi-cuspidal” from [Ber]. The term “supercuspidal” is encountered more often in the literature. We will show that an irreducible quasi-cuspidal representation is supercuspidal in Proposition 8.3.4 below.

The main result of this chapter is a theorem of Jacquet which states that every irreducible admissible representation is obtained as a subquotient of the parabolic induction of a supercuspidal representation (see Theorem 8.3.5 for the precise statement). One can profitably compare this result with the Langlands classification from §4.9 and the classification of automorphic representations explained in Chapter 10. These results can all be viewed as manifestations of what Harish-Chandra called the “philosophy of cusp forms” [HC70a]. Interpreted broadly, this is a slogan for the statement that all irreducible representations of reductive groups can be obtained via parabolic induction from cuspidal representations of Levi subgroups. Of course, the correct notion of cuspidal representation varies based on the context.

We will not give a full proof of Theorem 8.3.5, but we will develop much of the machinery used in the proof because it appears in many contexts. In particular, we define Jacquet modules in §8.3. In the case of GL_n , Bernstein and

Zelevinsky have refined Theorem 8.3.5 to an explicit classification of all irreducible admissible representations in terms of supercuspidal representations of Levi subgroups (see §8.4). We discuss the notion of a trace of an admissible representation in §8.5. In addition, we prove some of the basic properties of supercuspidal representations, including the fact that they admit coefficients (see Proposition 8.5.3). This fact can be applied to obtain simple trace formulae (see Lemma 16.4.2 and Theorem 18.2.5 below). Finally, we discuss the parallel notions of parabolic descent of representations in §8.6 and parabolic descent of orbital integrals in §8.7. The results in these two sections were already used in our discussion of the Satake isomorphism above, specifically in (7.37).

The canonical references for the topics covered in this chapter are the unpublished manuscripts of Casselman [Cas] and Bernstein [Ber]. The paper [Car79] contains sketches of proofs and refers to [Cas] for the details. The book [Ren10] is a useful resource as well. We have sometimes followed the expositions in [BH06] and [Mur], which fill in many details omitted in the previous sources.

8.2 Parabolic induction

Let us now prove some basic facts about parabolic induction. Let (σ, V) be a smooth representation of $M(F)$.

We start with the following observation:

Lemma 8.2.1 *If $K \leq G(F)$ is a compact open subgroup and $\varphi \in \text{Ind}_P^G(V)^K$, then*

$$\varphi(g) \in V^{M(F) \cap gKg^{-1}}.$$

Proof. Under the assumptions in the lemma if $k \in g^{-1}M(F)g \cap K$, we have

$$\varphi(g) = \varphi(gk) = \varphi(gkg^{-1}g) = \sigma(gkg^{-1})\varphi(g).$$

□

Proposition 8.2.2 *One has the following:*

- (a) *If σ is admissible, then $I(\sigma)$ is admissible.*
- (b) *If σ is unitary, then $I(\sigma)$ is unitarizable.*

Proof. Assume that (σ, V) is admissible. Let $K \leq G(F)$ be a compact open subgroup. We are to show that $\text{Ind}_P^G(V)^K$ is finite dimensional. An element $\varphi \in \text{Ind}_P^G(V)^K$ is determined by its value on any set of representatives X for the double cosets in

$$P(F) \backslash G(F) / K. \tag{8.5}$$

We can choose a maximal compact subgroup $K_{\max} \leq G(F)$ so that

$$P(F)K_{\max} = G(F) \quad (8.6)$$

and $M(F) \cap K_{\max}$ is a maximal compact subgroup of $M(F)$ by Theorem B.1.1. Hence,

$$P(F) \backslash G(F) / (K \cap K_{\max}) \cong (P(F) \cap K_{\max}) \backslash K_{\max} / (K \cap K_{\max})$$

is finite, and it follows that (8.5) is finite. On the other hand, for any $x \in X$, one has $\varphi(x) \in V^{M(F) \cap xKx^{-1}}$ by Lemma 8.2.1. For any compact open subgroup $K' \leq G(F)$, the intersection $M(F) \cap K'$ is a compact open subgroup of $M(F)$. Hence, $V^{M(F) \cap xKx^{-1}}$ is finite dimensional by admissibility. This proves (a).

For (b), let $(\cdot, \cdot)_V$ be an $M(F)$ -invariant inner product on V . As above, choose a maximal compact open subgroup $K_{\max} \leq G(F)$ so that the Iwasawa decomposition $G(F) = P(F)K_{\max}$ holds. Normalize the Haar measures dg on $G(F)$, dk on $K_{\max}$, and $d_\ell p$ on $P(F)$ so that $dg = d_\ell p dk$ (see Proposition 3.2.2).

For $\varphi_1, \varphi_2 \in \text{Ind}_P^G(V)$, we define

$$(\varphi_1, \varphi_2) := \int_{K_{\max}} (\varphi_1(k), \varphi_2(k))_V dk.$$

Since every element of $\text{Ind}_P^G(V)$ is determined by its restriction to $K_{\max}$, this is a positive definite inner product. We must check that it is $G(F)$ -invariant.

Choose a nonnegative $f \in C_c^\infty(G(F))$ such that

$$\int_{P(F)} f(pg) d_r p = 1$$

for all $g \in G(F)$ (see Exercise 8.3). Here we normalize the right Haar measure $d_r p$ so that $d_r p = \delta_P(p) d_\ell p$. For $x \in G(F)$ one has that

$$\begin{aligned} & \int_{K_{\max}} (\varphi_1(kx), \varphi_2(kx))_V dk \\ &= \int_{K_{\max}} (\varphi_1(kx), \varphi_2(kx))_V \int_{P(F)} f(pkx) d_r p dk \\ &= \int_{P(F) \times K_{\max}} (\varphi_1(kx), \varphi_2(kx))_V f(pkx) \delta_P(p) d_\ell p dk \\ &= \int_{P(F) \times K_{\max}} (\varphi_1(pkx), \varphi_2(pkx))_V f(pkx) d_\ell p dk \\ &= \int_{G(F)} (\varphi_1(gx), \varphi_2(gx))_V f(gx) dg \end{aligned}$$

$$= \int_{G(F)} (\varphi_1(g), \varphi_2(g))_V f(g) dg.$$

This expression is independent of x , so we deduce the invariance of our inner product. $\square$

Proposition 8.2.3 *One has that $I(\sigma^\vee) \cong I(\sigma)^\vee$.*

Proof. Fix a maximal compact open subgroup $K_{\max} \leq G(F)$ so that the Iwasawa decomposition $G(F) = P(F)K_{\max}$ holds.

Let $\varphi \in \text{Ind}_P^G(V)$ and $\varphi^\vee \in \text{Ind}_P^G(V^\vee)$. We then have a pairing

$$(\varphi, \varphi^\vee) := \int_{K_{\max}} \langle \varphi(k), \varphi^\vee(k) \rangle dk \quad (8.7)$$

where the pairing on the right is the pairing between V and $V^\vee$. Arguing as in the proof of part (b) of Proposition 8.2.2, we deduce that this pairing is $G(F)$ -invariant.

Since $\varphi^\vee$ is smooth, the linear form $(\cdot, \varphi^\vee)$ on V is smooth and we therefore obtain a $\mathbb{C}$ -linear map

$$\begin{aligned} \text{Ind}_P^G(V^\vee) &\longrightarrow \text{Ind}_P^G(V)^\vee \\ \varphi^\vee &\longmapsto (\cdot, \varphi^\vee). \end{aligned} \quad (8.8)$$

It is an intertwining map because (8.7) is $G(F)$ -equivariant. We must show that it is bijective. By smoothness, it suffices to show that it induces an isomorphism

$$\text{Ind}_P^G(V^\vee)^K \longrightarrow \text{Ind}_P^G(V)^\vee{}^K$$

for all compact open subgroups $K \leq K_{\max}$.

To do this, we first describe a basis of $\text{Ind}_P^G(V)^K$. As in the proof of Proposition 8.2.2, the set $P(F) \backslash G(F) / K$ is finite. Let $X \subset K_{\max}$ be a set of representatives for $P(F) \backslash G(F) / K$. For each $x \in X$, let $\mathcal{B}_x$ be a basis of $V^{M(F) \cap xKx^{-1}}$. By Lemma 8.2.1, $\varphi(x) \in V^{M(F) \cap xKx^{-1}}$. For each $w \in \mathcal{B}_x$ let $\varphi_{x,w}$ be the function supported on $P(F)xK$ such that

$$\varphi_{x,w}(mnxk) = \delta_P^{1/2}(m)\sigma(m)w$$

for $(m, n, k) \in M(F) \times N(F) \times K$. Then

$$\{\varphi_{x,w} : x \in X, w \in \mathcal{B}_x\} \quad (8.9)$$

is a $\mathbb{C}$ -basis of $\text{Ind}_P^G(V)^K$. Now for each x , let $\mathcal{B}_x^\vee$ be the basis of $(V^\vee)^{M(F) \cap xKx^{-1}}$ dual to $\mathcal{B}_x$. We define $\varphi_{x,w^\vee}$ by replacing V with $V^\vee$ in the construction above, and we obtain a basis

$$\{\varphi_{x,w^\vee} : x \in X, w^\vee \in \mathcal{B}_x^\vee\} \quad (8.10)$$

for $\text{Ind}_P^G(V^\vee)^K$. We are free to assume that the measure of K with respect to dk is 1, and if we take this convention, then (8.9) and (8.10) are a basis and its dual basis with respect to the pairing (8.7). $\square$

It is often useful to introduce a variant of the parabolic induction functor. We define a map

$$H_M : M(F) \longrightarrow \mathfrak{a}_M := \text{Hom}(X^*(M), \mathbb{R}) \quad (8.11)$$

via

$$\langle H_M(m), \lambda \rangle = \log |\lambda(m)|$$

for $\lambda \in X^*(M)$. This is the same map as (4.29) in §4.9, except now we are considering the case where F is non-Archimedean. As in §4.9 we let

$$H_P : P(M) \longrightarrow M(F) \longrightarrow \mathfrak{a}_M$$

be the map that is the composite of the canonical map $P(F) \rightarrow M(F)$ and H_M . For each $\lambda \in \mathfrak{a}_{M\mathbb{C}}^* := X^*(M) \otimes_{\mathbb{Z}} \mathbb{C}$, we obtain a quasi-character

$$m \longmapsto e^{\langle H_P(m), \lambda \rangle}.$$

Define

$$I(\sigma, \lambda) := I(\sigma \otimes e^{\langle H_P(\cdot), \lambda \rangle}). \quad (8.12)$$

The point of introducing this extra notation is that it makes clear that the induced representation $I(\sigma, \lambda)$ is part of a continuous family of representations indexed by $\mathfrak{a}_{M\mathbb{C}}^*$. In practice, this can be very useful. We employed this construction in the special case where P is a Borel subgroup and σ is trivial in our construction of the unramified principal series in §7.6.

8.3 Jacquet modules

We now define left adjoints to the functors Ind_P^G . Let (π, V) be a smooth representation of $G(F)$. Set

$$V(N) := \langle \pi(n)\varphi - \varphi : \varphi \in V, n \in N(F) \rangle \quad \text{and} \quad V_N := V/V(N). \quad (8.13)$$

The space V_N is referred to as the **space of coinvariants**. Since $M(F)$ normalizes $N(F)$, it preserves $V(N)$ and hence $M(F)$ acts on V_N . We define the **Jacquet functor**

$$\begin{aligned} (\cdot)_N : \text{Rep}_{\text{sm}} G(F) &\longrightarrow \text{Rep}_{\text{sm}} M(F) \\ (\pi, V) &\longmapsto (\pi_N, V_N), \end{aligned}$$

where

$$\pi_N := \pi|_{M(F)} \otimes \delta_P^{-1/2}.$$

The representation (π_N, V_N) is referred to as the **Jacquet module** of (π, V) (with respect to N).

A version of Frobenius reciprocity is valid for this functor:

Proposition 8.3.1 *The Jacquet functor is left adjoint to induction; in other words for smooth representations (π, V) of $G(F)$ and (σ, W) of $M(F)$, one has a bijection*

$$\mathrm{Hom}_{G(F)}(V, \mathrm{Ind}_P^G(W)) \xrightarrow{\sim} \mathrm{Hom}_{M(F)}(V_N, W)$$

that is functorial in (π, V) and (σ, W) . Here $M(F)$ acts on V_N via π_N .

Proof. There is an $M(F)$ -equivariant map

$$\begin{aligned} \mathrm{ev}_1 : \mathrm{Ind}_P^G(W) &\longrightarrow W \\ \varphi &\longmapsto \varphi(1) \end{aligned}$$

where $1 \in G(F)$ is the identity; it is clearly surjective. Thus, we have a $M(F)$ -equivariant map

$$\mathrm{ev}_1 \circ (\cdot)_N : \mathrm{Hom}_{G(F)}(V, \mathrm{Ind}_P^G(W)) \longrightarrow \mathrm{Hom}_{M(F)}(V_N, W)$$

given by composition with ev_1 . To construct the inverse, suppose we are given an $M(F)$ -intertwining map $\Phi : V_N \rightarrow W$. We define

$$\begin{aligned} \tilde{\Phi} : V &\longrightarrow \mathrm{Ind}_P^G(W) \\ \varphi &\longmapsto (g \mapsto \Phi(g \cdot \varphi)). \end{aligned}$$

The map $\Phi \mapsto \tilde{\Phi}$ is inverse to $\mathrm{ev}_1 \circ (\cdot)_N$. □

One would hope that the functor $(\cdot)_N$ preserves admissibility, and this is indeed the case:

Theorem 8.3.2 (Jacquet) *If (π, V) is a smooth admissible representation of $G(F)$, then (π_N, V_N) is a smooth admissible representation of $M(F)$. □*

Jacquet also proved the following elegant characterization of quasi-cuspidal representations using these functors:

Theorem 8.3.3 *A smooth irreducible representation (π, V) of $G(F)$ is quasi-cuspidal if and only if $V_N = 0$ for all parabolic subgroups $P \leq G$. □*

It is because of these theorems that V_N called the Jacquet module. Strictly speaking, Jacquet wrote up his results only for the case of GL_n [Jac71], but the proof carries over to the general case.

For the proof of Theorem 8.3.2, see [Cas, §3]. For the proof of Theorem 8.3.3, see [Ber, §I.5.1 and §II.2.2]. More details are given in [Ren10, §VI.2.1], but Renard assumes the field F is of characteristic zero. In the remainder of this section, we use theorems 8.3.2 and 8.3.3 to prove Theorem 8.3.5 below.

Before this, we prove a warm-up result. Recall that a supercuspidal representation is an admissible quasi-cuspidal representation.

Proposition 8.3.4 *An irreducible quasi-cuspidal representation is supercuspidal.*

Proof. Let (π, V) be an irreducible quasi-cuspidal representation of $G(F)$. Fix a nonzero $\varphi \in V$. For all compact open subgroups $K \leq G(F)$, one has that

$$V^K = \pi(e_K)V = \langle \pi(e_K)\pi(g)\varphi : g \in G(F) \rangle,$$

where

$$e_K := \frac{1}{\text{meas}_{dg}(K)} \mathbb{1}_K$$

is the idempotent attached to K . We must show V^K is finite dimensional. Since π admits a central quasi-character (see Exercise 5.8), if V^K is not finite dimensional, then there exists $\{g_n : n \in \mathbb{Z}_{>0}\} \subseteq G(F)$, all inequivalent modulo $Z_G(F)$, such that $\pi(e_K)\pi(g_n)\varphi$ are linearly independent. Let $W \subseteq V^K$ be a $\mathbb{C}$ -vector subspace such that

$$V^K = W \oplus \langle \pi(e_K)\pi(g_n)\varphi : g_n \in G(F), n \in \mathbb{Z}_{>0} \rangle.$$

As $V = V^K \oplus \ker \pi(e_K)$, we can define $\varphi^\vee \in \text{Hom}(V, \mathbb{C})$ such that

$$\langle \pi(e_K)\pi(g_n)\varphi, \varphi^\vee \rangle = n$$

for all n and $\varphi^\vee|_{W \oplus \ker \pi(e_K)} = 0$. Then $\varphi^\vee$ is fixed by K , hence is smooth, and hence is an element of $V^\vee$. On the other hand, by construction, the support of the matrix coefficient $\langle \pi(g)\varphi, \varphi^\vee \rangle$ is not compact modulo the center $Z_G(F)$. This contradiction implies the proposition. $\square$

Combined with Proposition 8.3.4, Theorem 8.3.2 and Theorem 8.3.3 allow us to deduce the following concrete manifestation of the philosophy of cusp forms:

Theorem 8.3.5 *If (π, V) is a smooth irreducible representation of $G(F)$, then the following are true:*

- (a) *There exists a parabolic subgroup $P = MN \leq G$, a supercuspidal representation (σ, W) of $M(F)$ and a nonzero intertwining map $V \rightarrow \text{Ind}_P^G(W)$.*
- (b) *The representation π is admissible.*

Proof. The first assertion implies the second, as induction preserves admissibility by Proposition 8.2.2. For the first, we proceed by induction on the

dimension of G (as an F -algebraic group). If G has dimension 1, then it is a torus (this follows from the proof of [Mil17, Corollary 17.27], for example), and so the result is trivial.

Assume that, for all proper parabolic subgroups P , there is no nonzero intertwining map $V \rightarrow \text{Ind}_P^G(W)$ where (σ, W) is a smooth irreducible representation of $M(F)$. Applying Frobenius reciprocity (Proposition 8.3.1), we see that

$$\text{Hom}_{M(F)}(V_N, W) = 0$$

for all parabolic subgroups $P = MN$ and all smooth representations (σ, W) of $M(F)$, which implies π is supercuspidal by Theorem 8.3.3 and Proposition 8.3.4.

Now assume that there is a proper parabolic subgroup $P < G$, a Levi subgroup $M \leq P$, a smooth representation (σ, W) of $M(F)$, and a nonzero (hence injective) intertwining map $V \rightarrow \text{Ind}_P^G(W)$. By Frobenius reciprocity, there is a nonzero intertwining map

$$V_N \longrightarrow W$$

of representations of $M(F)$ so we can apply our inductive hypothesis to deduce that there is a parabolic subgroup $Q \leq M$ with Levi subgroup M_Q , a supercuspidal representation (ρ, U) of $M_Q(F)$ and a nonzero intertwining map

$$V_N \longrightarrow W \longrightarrow \text{Ind}_Q^M(U).$$

Applying Frobenius reciprocity again, we obtain a nonzero intertwining map

$$V \longrightarrow \text{Ind}_P^G \circ \text{Ind}_Q^M(U) \cong \text{Ind}_{QN}^G(U)$$

(see Exercise 8.4 for the last isomorphism). Here we are implicitly using the fact that QN is a parabolic subgroup of G (see Proposition 1.9.8). $\square$

A **cuspidal datum** is a tuple

$$(M, (\sigma, W)) \tag{8.14}$$

consisting of a Levi subgroup $M \leq G$ and a supercuspidal representation (σ, W) of $M(F)$. We say that two cuspidal data $(M, (\sigma, W))$ and $(M', (\sigma', W'))$ are **equivalent** if they are $G(F)$ -conjugate, that is, there is a $g \in G(F)$ such that $gM(F)g^{-1} = M'(F)$ and $m' \mapsto \sigma(g^{-1}m'g)$ is isomorphic to σ' .

Here is an amplification of the preceding theorem:

Theorem 8.3.6 *Assume that F has characteristic zero. Suppose that π is a smooth irreducible representation of $G(F)$ and $(M, (\sigma, W)), (M', (\sigma', W'))$ are cuspidal data. Assume that P and P' are parabolic subgroups of G with Levi subgroups M and M' , respectively, and that there are nonzero intertwining maps*

$$V \longrightarrow \mathrm{Ind}_P^G(W) \quad \text{and} \quad V' \longrightarrow \mathrm{Ind}_{P'}^G(W').$$

Then $(M, (\sigma, W))$ and $(M', (\sigma', W'))$ are equivalent. $\square$

The only published reference for this theorem we know is [Ren10, Theorem VI.5.4] which assumes that F is of characteristic zero.

The **cuspidal support** of an irreducible admissible representation π is the equivalence class of a cuspidal datum $(M, (\sigma, W))$ such that there is a nonzero intertwining map $V \rightarrow \mathrm{Ind}_P^G(W)$ for some parabolic subgroup P containing M . This is well-defined by theorems 8.3.5 and 8.3.6.

8.4 The Bernstein-Zelevinsky classification

Theorem 8.3.5 provides the first step towards a classification of all irreducible admissible representations of $G(F)$ in terms of supercuspidal representations. The remaining task is to understand isomorphisms between subquotients of induced representations.

It is useful to start by stepping backwards. In §4.8 we defined, for any local field, the notions of square integrable (or discrete) representations, tempered representations, essentially square integrable representations, and essentially tempered representations. These notions were employed to state the Archimedean Langlands classification (Theorem 4.9.2). We will now state the Langlands classification in the non-Archimedean setting. It classifies irreducible admissible representations in terms of tempered representations.

Fix a minimal parabolic subgroup $P_0 \leq G$ and call a parabolic subgroup standard if it contains P_0 . Let (σ, V) be an irreducible admissible representation of $M(F)$. Given $\lambda \in \mathfrak{a}_{MC}^*$ we can form the induced representation $I(\sigma, \lambda)$ as in (8.12).

The following is the analogue of Theorem 4.9.1 in this setting:

Theorem 8.4.1 *If σ is unitary and tempered and $\mathrm{Re}(\lambda)$ is in the positive Weyl chamber, then $I(\sigma, \lambda)$ admits a unique irreducible quotient $J(\sigma, \lambda)$. $\square$*

The representation $J(\sigma, \lambda)$ is known as the **Langlands quotient** of $I(\sigma, \lambda)$.

We now state the non-Archimedean analogue of Theorem 4.9.2:

Theorem 8.4.2 *Every irreducible admissible representation of $G(F)$ is isomorphic to some $J(\sigma, \lambda)$ with σ unitary and tempered and $\mathrm{Re}(\lambda)$ in the positive Weyl chamber. Moreover if we insist that the parabolic subgroup defining $J(\sigma, \lambda)$ is standard, fix a Levi decomposition $P = MN$ of each standard parabolic subgroup, insist that σ is trivial on A_M , and stipulate that $\mathrm{Re}(\lambda)$ is in the positive Weyl chamber, then every irreducible admissible representation of $G(F)$ is isomorphic to a $J(\sigma, \lambda)$ that is unique up to replacing σ by another representation of $M(F)$ equivalent to σ . $\square$*

In the degenerate case $P = G$, the assumption that $\mathrm{Re}(\lambda)$ is in the positive Weyl chamber is automatically satisfied. For the proofs of these theorems, we

refer to [BW00, §XI.2]. See also [Kon03]. Theorem 8.4.2 allows us to attach, to every irreducible admissible representation π of $G(F)$, parameters (M, σ, λ) as in the theorem. These are called the **Langlands data** of π . They are unique in the sense explained in the theorem.

In view of these theorems, to classify the irreducible admissible representations of $G(F)$ it suffices to classify the irreducible tempered representations of $G(F)$. This work is largely complete in the case when G is a classical group, see [Mœg02, MT02, Jan14] for example. The situation for general linear groups is arguably the simplest and we will discuss it in this section. The results are due to Bernstein and Zelevinsky [BZ77, Zel80] and the theory is known as the **Bernstein-Zelevinsky classification**.

We start with the classification of representations that are essentially square integrable. Let $n > 1$, let $a|n$, and let σ be an irreducible supercuspidal representation of $\mathrm{GL}_{n/a}(F)$. The external tensor product $\sigma^a := \sigma^{\otimes a}$ is a (supercuspidal) representation of $\mathrm{GL}_{n/a}^a$, which we view as a Levi subgroup of the standard parabolic subgroup of GL_n of type $(n/a, \dots, n/a)$. We refer to the reader Example 1.15 for our conventions regarding standard parabolic subgroups of general linear groups.

There is a unique irreducible quotient $Q(\sigma^a, \lambda_a)$ of $I(\sigma^a, \lambda_a)$ where

$$\lambda_a := \left(-\frac{a-1}{2}, -\frac{a-3}{2}, \dots, \frac{a-3}{2}, \frac{a-1}{2} \right) \in X^*(M) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Here we have identified $X^*(M) \otimes_{\mathbb{Z}} \mathbb{C}$ with $\mathbb{C}^a$ in the unique manner such that

$$e \left\langle {}^{H_M} \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_a \end{pmatrix}, \lambda_a \right\rangle = \prod_{i=1}^a |\det m_i|^{i-1-\frac{a-1}{2}}. \quad (8.15)$$

We use analogous conventions below. The quotient $Q(\sigma^a, \lambda_a)$ is essentially square integrable. Conversely, all irreducible essentially square integrable representations arise in this manner:

Theorem 8.4.3 (Bernstein) *Every irreducible admissible essentially square integrable representation π of $\mathrm{GL}_n(F)$ is isomorphic to $Q(\sigma^a, \lambda_a)$ for a pair (σ, a) where $a|n$ and σ is an irreducible supercuspidal representation of $\mathrm{GL}_{n/a}(F)$. $\square$*

The theorem is stated in [Zel80, §9.3]. There is a sketch of the proof in [JS83, §1.2]. We observe that unlike in the setting of Theorem 8.4.2, the parameter λ_a does not lie in the positive Weyl chamber. If σ is a unitary supercuspidal representation, then the Langlands data of $Q(\sigma^a, \lambda_a)$ are $(\mathrm{GL}_n, Q(\sigma^a, \lambda_a), 0)$ and a similar statement holds if we merely assume σ is supercuspidal.

To proceed we require the notion of linked representations. Following [BZ77], one calls any set of supercuspidal representations of $\mathrm{GL}_r(F)$ for some r of the form

$$\left\{ \sigma, \sigma \otimes e^{\langle H_{\mathrm{GL}_r}(\cdot), 1 \rangle}, \dots, \sigma \otimes e^{\langle H_{\mathrm{GL}_r}(\cdot), d \rangle} \right\} \quad (8.16)$$

for some integer d a **segment**. Two segments are **linked** if neither is included in the other and their union is a segment.

To each $Q(\sigma^a, \lambda_a)$, we associate the segment

$$\left\{ \sigma \otimes e^{\langle H_{\mathrm{GL}_{n/a}}(\cdot), -\frac{a-1}{2} \rangle}, \dots, \sigma \otimes e^{\langle H_{\mathrm{GL}_{n/a}}(\cdot), \frac{a-1}{2} \rangle} \right\}.$$

We say that

$$Q(\sigma^a, \lambda_a) \quad \text{and} \quad Q(\sigma'^{a'}, \lambda_{a'}) \quad (8.17)$$

are **linked** if their associated segments are linked.

Now assume $\sum_{i=1}^k n_i = n$ and $a_i | n_i$ for all i . If P is the standard parabolic of type $(n_1, \dots, n_k)$, then, for any collection of square integrable representations $Q(\sigma_i^{a_i}, \lambda_{a_i})$ of $\mathrm{GL}_{n_i}(F)$ as above, we can form the induced representation

$$I(Q(\sigma_1^{a_1}, \lambda_{a_1}) \otimes \dots \otimes Q(\sigma_k^{a_k}, \lambda_{a_k}), 0). \quad (8.18)$$

In the case where $Q(\sigma_i^{a_i}, \lambda_{a_i})$ is linked with $Q(\sigma_j^{a_j}, \lambda_{a_j})$ if and only if $i = j$ we say that none of the $Q(\sigma_i^{a_i}, \lambda_{a_i})$ are linked.

We require the notion of a **generic representation** to state the next theorem. Generic representations are defined in §11.3. The reader can treat the definition as a black box for the time being. To aid the reader, we point out that irreducible admissible representations are nondegenerate in the terminology of [BZ77, Zel80] if and only if they are generic (see [BZ76]).

The following theorem is [Zel80, Theorem 9.7]:

Theorem 8.4.4 *If the representation (8.18) is irreducible, then it is generic. The representation (8.18) is irreducible if and only if none of the $Q(\sigma_i^{a_i}, \lambda_{a_i})$ are linked. Any irreducible admissible generic representation π is isomorphic to (8.18) for some $Q(\sigma_i^{a_i}, \lambda_{a_i})$, $1 \leq i \leq k$, uniquely determined up to reordering indices.* $\square$

Theorem 8.4.5 *An admissible representation (π, V) of $\mathrm{GL}_n(F)$ is tempered and irreducible if and only if it is of the form (8.18) where all of the $Q(\sigma_i^{a_i}, \lambda_{a_i})$ are square integrable.*

To clarify the assumptions in the theorem, we note that the $Q(\sigma_i^{a_i}, \lambda_{a_i})$ are always essentially square integrable, so the assumption of square integrability in the theorem amounts to assuming that the central character of $Q(\sigma_i^{a_i}, \lambda_{a_i})$ is unitary for each i .

Proof. Assume (π, V) is tempered and irreducible. Then by [Wal03, Proposition III.4.1] and Theorem 8.4.3, π is a subquotient of a representation π' of the form (8.18) for some square integrable $Q(\sigma_i^{a_i}, \lambda_{a_i})$. Since the $Q(\sigma_i^{a_i}, \lambda_{a_i})$

are all square integrable, none of them are linked by Exercise 8.11. Thus, π' is irreducible by Theorem 8.4.4 and we deduce that $\pi' \cong \pi$.

Conversely, suppose that an admissible representation π is of the form (8.18) where all of the $Q(\sigma_i^{a_i}, \lambda_{a_i})$ are square integrable. Then none of the $Q(\sigma_i^{a_i}, \lambda_{a_i})$ are linked (see Exercise 8.11), so π is irreducible by Theorem 8.4.4. Since the (normalized) induction of a tempered representation is tempered [Wal03, Lemme III.2.3], we conclude that π is tempered. $\square$

Combining the Langlands classification (Theorem 8.4.2) and the subsequent description of tempered representations given by Theorem 8.4.5, we can give an analogue of the Langlands classification in terms of square integrable representations. We defer a discussion of this to §10.6.

8.5 Traces, characters, and coefficients

Let (π, V) be an admissible representation of $G(F)$. Then, for all $f \in C_c^\infty(G(F))$, one has an operator

$$\pi(f) : V \longrightarrow V.$$

There is a compact open subgroup $K \leq G(F)$ such that $f \in C_c^\infty(G(F) // K)$ and hence $\pi(f)$ induces an operator

$$\pi(f) : W \longrightarrow W$$

for any finite dimensional subspace $V^K \leq W \leq V$. We define the **trace of** $\pi(f)$ by

$$\mathrm{tr} \pi(f) := \mathrm{tr} \pi(f)|_W \tag{8.19}$$

for any such W . The notion of the trace of a representation will play a crucial role in the trace formula in later chapters.

In this non-Archimedean setting, a **distribution** on $G(F)$ is simply a complex linear functional on $C_c^\infty(G(F))$. Thus the trace map

$$\mathrm{tr} \pi : C_c^\infty(G(F)) \longrightarrow \mathbb{C}$$

is a distribution on $G(F)$, called the **character** of π . Of course, this distribution depends on a choice of Haar measure. For $\gamma \in G(F)$, the centralizer $C_\gamma \leq G$ is defined by setting

$$C_\gamma(R) := \{g \in G(R) : g^{-1}\gamma g = \gamma\} \tag{8.20}$$

for F -algebras R . Let $G^{\mathrm{reg}, \mathrm{ss}} \leq G$ denote the (open) subscheme consisting of regular semisimple elements. The notion of a regular semisimple element

will be discussed in more detail in §17.2 later. For the moment, it suffices to know that

$$G^{\text{reg,ss}}(F) = \left\{ \gamma \in G(F) : C_\gamma^\circ \leq G \text{ is a maximal torus} \right\}. \quad (8.21)$$

The following is a fundamental result [HC99]:

Theorem 8.5.1 (Harish-Chandra) *Assume that F has characteristic zero. The distribution $\text{tr } \pi$ is represented by a locally constant function Θ_π with support in $G^{\text{reg,ss}}(F)$. $\square$*

In other words, if F has characteristic zero there is a locally constant function Θ_π on $G^{\text{reg,ss}}(F)$ such that

$$\text{tr } \pi(f) = \int_{G(F)} \Theta_\pi(g) f(g) dg$$

for all $f \in C_c^\infty(G(F))$. This result tells us that we can almost regard $\text{tr } \pi$ as a function.

The following is a version of linear independence of characters adapted to this setting:

Proposition 8.5.2 *Let $\{(\pi_i, V_i) : 1 \leq i \leq n\}$ be a finite set of irreducible admissible representations. If $\pi_i \cong \pi_j$ implies $i = j$, then the distributions $\text{tr } \pi_i$ are linearly independent.*

Proof. We use admissibility to reduce the assertion to a finite dimensional setting. Fix a compact open subgroup $K \leq G(F)$ such that $V_i^K \neq 0$ for all i . This implies that $\{V_i^K\}$ is a finite set of finite dimensional $\mathbb{C}$ -vector spaces with an action of $C_c^\infty(G(F) // K)$. They are all simple (that is, irreducible) for this action by Proposition 5.3.4. Moreover, they are pairwise nonisomorphic as $C_c^\infty(G(F) // K)$ -modules by Proposition 7.1.2. Let

$$A := \text{Im} \left(C_c^\infty(G(F) // K) \longrightarrow \prod_i \text{End}_{\mathbb{C}}(V_i^K) \right).$$

Then A is a finite dimensional $\mathbb{C}$ -algebra and the $\{V_i^K\}$ are a finite family of finite dimensional nonisomorphic simple A -modules. Hence, the traces $\text{tr } \pi_i|_{C_c^\infty(G(F) // K)}$ are linearly independent [GW09, Lemma 4.1.18]. $\square$

Thus, traces can be used to distinguish between a finite set of representations. In particular, if $\{\pi_1, \dots, \pi_n\}$ is a finite set of pairwise nonisomorphic irreducible representations, then linear independence of characters implies that we can find an $f \in C_c^\infty(G(F))$ such that

$$\text{tr } \pi_i(f) = 0 \text{ if and only if } i \neq 1.$$

For a refinement of this result at the level of operators, see Exercise 8.9.

One can ask for more. Let π be an irreducible admissible representation. A **coefficient** of π is a smooth function $f_\pi \in C_c^\infty(G(F))$ such that $\text{tr } \pi(f_\pi) \neq 0$ and $\text{tr } \pi'(f_\pi) = 0$ for $\pi' \not\cong \pi$. Thus if a coefficient for π exists, we can use it to isolate π among any set of irreducible admissible representations, finite or not. We observe that if a coefficient exists it is not unique. For example, any $G(F)$ -conjugate of a coefficient is another coefficient. For general π , functions f_π as above need not exist (see Exercise 8.13). However, in certain circumstances, one can construct coefficients:

Proposition 8.5.3 *Assume that $Z_G(F)$ is compact and let (π, V) be an irreducible supercuspidal representation of $G(F)$. Then for all $f \in C_c^\infty(G(F))$ there exists a unique $f_\pi \in C_c^\infty(G(F))$ such that $\pi(f_\pi) = \pi(f)$ and if π' is an irreducible admissible representations of $G(F)$ with $\pi' \not\cong \pi$, then $\pi'(f_\pi) = 0$.*

As an immediate consequence, we see that coefficients of supercuspidal representations exist.

To prove Proposition 8.5.3, we collect some observations on the space of endomorphisms of a representation. For any smooth representation (π, V) of $G(F)$, consider the $\mathbb{C}$ -vector space

$$\text{End}(V) := \text{Hom}_{\mathbb{C}}(V, V).$$

Here V is given the discrete topology, so $\text{Hom}_{\mathbb{C}}(V, V)$ consists of arbitrary $\mathbb{C}$ -linear maps. There is an action of $G(F) \times G(F)$ on $\text{End}(V)$ given by

$$\begin{aligned} G(F) \times G(F) \times \text{End}(V) &\longrightarrow \text{End}(V) \\ ((g_1, g_2), A) &\longmapsto \pi(g_1) \circ A \circ \pi(g_2^{-1}). \end{aligned}$$

We let

$$\text{End}_{\text{sm}}(V) \leq \text{End}(V)$$

denote the subspace consisting of **smooth endomorphisms**, that is, endomorphisms that are left and right invariant by a compact open subgroup $K \leq G(F)$. This is a smooth representation of $G(F) \times G(F)$.

The usual morphism

$$V \otimes \text{Hom}_{\mathbb{C}}(V, \mathbb{C}) \longrightarrow \text{End}(V)$$

given on pure tensors by

$$\varphi \otimes \varphi^\vee \longmapsto (\varphi_0 \mapsto \langle \varphi_0, \varphi^\vee \rangle \varphi)$$

is $G(F) \times G(F)$ -equivariant and upon restriction induces an isomorphism $V \otimes V^\vee \rightarrow \text{End}_{\text{sm}}(V)$. Here all of the tensor products are over $\mathbb{C}$. Thus, the action of $G(F) \times G(F)$ on $\text{End}_{\text{sm}}(V)$ can be reasonably denoted by $\pi \otimes \pi^\vee$ (for more details on product representations see Theorem 5.7.2 above). If (π, V) is admissible, then $\text{End}_{\text{sm}}(V)$ is also admissible (see Exercise 8.14).

One also has an intertwining map

$$\beta : (\pi \otimes \pi^\vee, \text{End}_{\text{sm}}(V)) \longrightarrow (\rho, C^\infty(G(F)))$$

given by

$$\beta(A)(g) = \text{tr}(\pi(g^{-1}) \circ A).$$

Here ρ acts via $\rho(g_1, g_2)(f)(h) = f(g_1^{-1}hg_2)$. We note that for each $A \in \text{End}_{\text{sm}}(V)$, the function $\beta(A)$ is a sum of matrix coefficients of $\pi^\vee$. Indeed, if we choose a compact open subgroup $K \leq G(F)$ such that A is fixed on the left and right under K , choose a basis $\varphi_1, \dots, \varphi_n$ of V^K and a dual basis $\varphi_1^\vee, \dots, \varphi_n^\vee$ of $V^{\vee K}$, then

$$\beta(A)(g) = \sum_{i=1}^n \langle \pi(g^{-1}) \circ A\varphi_i, \varphi_i^\vee \rangle = \sum_{i=1}^n \langle A\varphi_i, \pi^\vee(g)\varphi_i^\vee \rangle. \quad (8.22)$$

Proof of Proposition 8.5.3: Since $Z_G(F)$ is compact by assumption we deduce that $\beta(\text{End}_{\text{sm}}(V)) \leq C_c^\infty(G(F))$. Since there are $\varphi \in V$ and $\varphi^\vee \in V^\vee$ such that

$$\langle \varphi, \varphi^\vee \rangle = \langle \pi(1)\varphi, \varphi^\vee \rangle \neq 0,$$

we have that β is not identically zero. Thus, since the exterior tensor product $\pi \otimes \pi^\vee$ is irreducible, β is an embedding.

Consider

$$\begin{aligned} \beta' : (\rho, C_c^\infty(G(F))) &\longrightarrow (\pi \otimes \pi^\vee, \text{End}_{\text{sm}}(V)) \\ f &\longmapsto \pi(f). \end{aligned} \quad (8.23)$$

Then $\beta' \circ \beta$ is an endomorphism of the irreducible representation $\text{End}_{\text{sm}}(V)$ of $G(F) \times G(F)$. Hence, $\beta' \circ \beta$ is multiplication by a scalar by Schur's lemma (see Exercise 5.6), say $\beta' \circ \beta = \lambda I$ for some $\lambda \in \mathbb{C}$ where $I : V \rightarrow V$ is the identity. We will now show that $\lambda \neq 0$ and that we can take

$$f_\pi := \lambda^{-1} \beta \circ \beta'(f).$$

To show that $\lambda \neq 0$, note that we can find $f \in C_c^\infty(G(F))$ such that $\pi(f) \neq 0$ (take, for example, f to be the characteristic function of a sufficiently small compact open subgroup). Thus, $\beta'(f) = \pi(f) \neq 0$ and since β is an embedding, we deduce that $\beta \circ \beta'(f) \neq 0$.

To show that we can take f_π as claimed, we first observe that

$$\pi(\beta \circ \beta'(f)) = \beta' \circ \beta \circ \beta'(f) = \lambda \beta'(f) = \lambda \pi(f).$$

Second, let (π', V') be a smooth irreducible representation of $G(F)$, let $\varphi' \in V'$ be a nonzero vector, and let

$$\begin{aligned} \gamma : (\rho|_{G(F) \times 1}, C_c^\infty(G(F))) &\longrightarrow (\pi', V') \\ f &\longmapsto \pi'(f)\varphi'. \end{aligned} \quad (8.24)$$

As a representation of $G(F)$, one has that $\text{End}_{\text{sm}}(V)|_{G(F) \times 1}$ is isomorphic to a direct sum of copies of π , and thus the same is true of $\gamma(\beta(\text{End}_{\text{sm}}(V)))$. Thus, $\gamma(\beta(\text{End}_{\text{sm}}(V))) = 0$ unless $\pi' \cong \pi$ (since whenever the former is nonzero we obtain an intertwining operator between π and π'). It follows that $\pi'(f_\pi) = 0$ if $\pi' \not\cong \pi$. This completes the proof of the proposition.

8.6 Parabolic descent of representations

Parabolic descent is a term for the process by which one passes from objects (say, representations) on a group G to objects on a Levi subgroup of a parabolic subgroup of G . We end this chapter with two examples of this phenomenon. First we show that the character of a parabolically induced representation is related in a straightforward manner to the character of the inducing representation (see Proposition 8.6.2). Then in §8.7 we prove an analogous result (Proposition 8.7.1) for orbital integrals.

Let $P \leq G$ be a parabolic subgroup with Levi decomposition $P(F) = M(F)N(F)$ and let $K \leq G(F)$ be a maximal compact subgroup.

Definition 8.6.1. A maximal compact subgroup $K \leq G(F)$ is said to be **in good position with respect to** (P, M) if $G(F) = P(F)K$ and $P(F) \cap K = (M(F) \cap K)(N(F) \cap K)$.

Given a parabolic subgroup P with Levi decomposition $P = MN$, we can always find a maximal compact subgroup in good position with respect to (P, M) . We can even arrange that

$$K_M := M(F) \cap K$$

is a maximal compact subgroup of $M(F)$ (see Theorem B.4.3); we will also assume this in what follows.

We assume throughout this section that Haar measures are chosen so that with respect to the decomposition $G(F) = KM(F)N(F)$ one has

$$dg = \delta_P(m) dk dm dn \quad (8.25)$$

where

$$\text{meas}_{dk}(K) = \text{meas}_{dn}(N(F) \cap K) = \text{meas}_{dm}(K_M) = 1.$$

This is equivalent to stipulating that with respect to the decomposition $G(F) = M(F)N(F)K$ one has

$$dg = dmdndk. \quad (8.26)$$

The **constant term of $f \in C_c^\infty(G(F))$ along P** is the function

$$f^P(m) := \delta_P^{1/2}(m) \int_{N(F)} f(mn)dn. \quad (8.27)$$

This is a generalization of (7.36). Let

$$C_c^\infty(G(F); K) := \{f \in C_c^\infty(G(F)) : f(k^{-1}xk) = f(x)\}$$

for all $k \in K$. We note that $C_c^\infty(G(F); K)$ is the image of the map

$$\begin{aligned} C_c^\infty(G(F)) &\longrightarrow C_c^\infty(G(F); K) \\ f &\longmapsto f_K \end{aligned} \quad (8.28)$$

where $f_K(x) := \int_K f(k^{-1}xk)dk$. The space $C_c^\infty(G(F); K) \leq C_c^\infty(G(F))$ is a subalgebra under convolution and one trivially has that

$$\mathrm{tr} \pi(f) = \mathrm{tr} \pi(f_K). \quad (8.29)$$

Proposition 8.6.2 *The constant term defines a map*

$$\begin{aligned} C_c^\infty(G(F); K) &\longrightarrow C_c^\infty(M(F); K_M) \\ f &\longmapsto f^P. \end{aligned}$$

If (σ, V) is an admissible representation of $M(F)$, then

$$\mathrm{tr} I(\sigma)(f) = \mathrm{tr} \sigma(f^P).$$

The Archimedean analogue of this proposition is valid as well [Kna86, (10.23)].

Proof. We follow the proof of [Lau96, Lemma 7.5.7]. Since K is compact, $\delta_P(k) = 1$ for all $k \in K_M$. It follows that $f^P \in C_c^\infty(M(F); K_M)$ as claimed.

We now prove the equality of traces. Consider the space $\mathrm{Ind}_P^G(V)^\circ$ of locally constant functions

$$\varphi : K \longrightarrow V$$

such that $\varphi(mnk) = \sigma(m)\varphi(k)$ for $(m, n, k) \in K_M \times (N(F) \cap K) \times K$. There is a representation $I(\sigma)^\circ$ of K on this space given by

$$I(\sigma)^\circ(k)\varphi(g) = \varphi(gk). \quad (8.30)$$

By the Iwasawa decomposition, restriction of functions to K induces an isomorphism

$$\mathrm{Ind}_P^G(V) \longrightarrow \mathrm{Ind}_P^G(V)^\circ. \quad (8.31)$$

Thus, for all $f \in C_c^\infty(G(F))$, $I(\sigma)(f)$ induces an endomorphism of $\text{Ind}_P^G(V)^\circ$. Let us compute this endomorphism. Define

$$\Phi_{k,k'}(m) := \delta_P^{1/2}(m) \int_{N(F)} f(k^{-1}mnk')dn$$

for $k, k' \in K$ and $m \in M(F)$. Then if $\varphi \in \text{Ind}_P^G(V)$ one has that

$$\begin{aligned} I(\sigma)(f)\varphi(k) &= \int_{G(F)} f(g)\varphi(kg)dg \\ &= \int_{G(F)} f(k^{-1}g)\varphi(g)dg \\ &= \int_{M(F)} \int_{N(F)} \int_K f(k^{-1}mnk')\varphi(mnk')dk'dndm \\ &= \int_{M(F)} \int_{N(F)} \int_K f(k^{-1}mnk')\delta_P^{1/2}(m)\sigma(m)\varphi(k')dk'dndm \\ &= \int_K \sigma(\Phi_{k,k'})\varphi(k')dk'. \end{aligned}$$

Thus, the endomorphism on $\text{Ind}_P^G(V)^\circ$ induced by $I(\sigma)(f)$ is

$$\varphi \longmapsto \left(k \mapsto \int_K \sigma(\Phi_{k,k'})\varphi(k')dk' \right). \quad (8.32)$$

In particular, $\text{tr } I(\sigma)(f)$ is the trace of the endomorphism (8.32). By Exercise (8.16), the trace of (8.32) is

$$\int_K \text{tr } \sigma(\Phi_{k,k})dk. \quad (8.33)$$

On the other hand

$$f^P = \int_K \Phi_{k,k}dk.$$

The lemma follows. □

As an addendum, we prove the following lemma:

Lemma 8.6.3 *The constant term provides an algebra homomorphism*

$$C_c^\infty(G(F) // K) \longrightarrow C_c^\infty(M(F) // K_M).$$

Proof. Let $f_1, f_2 \in C_c^\infty(G(F) // K)$. One has that

$$(f_1 * f_2)^P(m) = \delta_P^{1/2}(m) \int_{N(F)} (f_1 * f_2)(mn)dn$$

$$= \delta_P^{1/2}(m) \int_{N(F)} \left(\int_{G(F)} f_1(mng^{-1}) f_2(g) dg \right) dn.$$

We rewrite this as

$$\begin{aligned} & \delta_P^{1/2}(m) \int_{N(F) \times K \times M(F) \times N(F)} \delta_P(m') f_1(mn(k'm'n')^{-1}) f_2(k'm'n') dndk' dm' dn' \\ &= \delta_P^{1/2}(m) \int_{N(F) \times M(F) \times N(F)} \delta_P(m') f_1(mnn'^{-1}m'^{-1}) f_2(m'n') dndm' dn'. \end{aligned}$$

Changing variables $n \mapsto nn'$ this is

$$\delta_P^{1/2}(m) \int_{N(F) \times M(F) \times N(F)} \delta_P(m') f_1(mnm'^{-1}) f_2(m'n') dndm' dn'$$

and then changing variables $n \mapsto m'^{-1}nm'$ this is

$$\begin{aligned} & \delta_P^{1/2}(m) \int_{N(F) \times M(F) \times N(F)} f_1(mm'^{-1}n) f_2(m'n') dm' dn' dn \\ &= \int_{M(F)} \left(\delta_P^{1/2}(mm'^{-1}) \int_{N(F)} f_1(mm'^{-1}n) dn \delta_P^{1/2}(m') \int_{N(F)} f_2(m'n') dn' \right) dm' \\ &= (f_1^P * f_2^P)(m). \end{aligned}$$

This proves our assertion that the constant term is an algebra morphism. $\square$

8.7 Parabolic descent of orbital integrals

In our discussion of the Satake isomorphism in §7.5, we claimed that the constant term map has image in Weyl invariant functions on the maximal torus. We prove this in the current section and use it as an opportunity to discuss descent of orbital integrals.

We keep the notation of the previous section; thus, $P = MN$ is a parabolic subgroup of G , $K \leq G(F)$ is a maximal compact subgroup in good position with respect to (P, M) , and $K_M := M(F) \cap K$ is maximal compact subgroup of $M(F)$.

Let $\gamma \in G(F)$ and let C_γ be the centralizer of γ (see (8.20)). Earlier in Definition 1.5.4, we only defined semisimple elements for perfect fields. For general fields F we say that $\gamma \in G(F)$ is **absolutely semisimple** if the image of γ in $G(\overline{F})$ is semisimple. If $\gamma \in G(F)$ is absolutely semisimple, then the orbit $O(\gamma) \subset G$ of γ is closed [Ste65, Corollary 6.13, Proposition 6.14]. For an absolutely semisimple element γ , the neutral component of the centralizer C_γ is reductive (see Theorem 17.2.9). For more details on orbits and stabilizers, we refer to §17.2.

Assume that $\gamma \in G(F)$ is absolutely semisimple. Then $C_\gamma(F)$ is unimodular, and upon choosing a Haar measure dg_γ on $C_\gamma^\circ(F)$ we can form a right $G(F)$ -invariant Radon measure $\frac{dg}{dg_\gamma}$ on the quotient $C_\gamma^\circ(F) \backslash G(F)$ (see Theorem 3.2.3). We then define the **orbital integral**

$$O_\gamma^\circ(f) := \int_{C_\gamma^\circ(F) \backslash G(F)} f(g^{-1}\gamma g) \frac{dg}{dg_\gamma}. \quad (8.34)$$

Note that this is not the same as the local version of the orbital integral appearing in (18.26). It differs in that in (18.26) we use C_γ instead of C_γ° . This is why there is a superscript $\circ$ in the notation. The quotient map

$$C_\gamma^\circ(F) \backslash G(F) \longrightarrow C_\gamma(F) \backslash G(F)$$

is proper as is the map

$$\begin{aligned} C_\gamma(F) \backslash G(F) &\longrightarrow G(F) \\ C_\gamma(F)g &\longmapsto g^{-1}\gamma g \end{aligned}$$

by the argument in the proof of Theorem 17.5.3. It follows that the integral (8.34) is absolutely convergent.

Let $H \leq G$ be a connected subgroup. For F -algebras R and $h \in H(R)$, it is convenient to define

$$D_{H \backslash G}(h) := \det(I - \text{Ad}(h^{-1}) : \mathfrak{h} \backslash \mathfrak{g} \longrightarrow \mathfrak{h} \backslash \mathfrak{g}) \quad (8.35)$$

where $\mathfrak{h} := \text{Lie } H$, and $\mathfrak{g} := \text{Lie } G$, and I is the identity map.

There is a dual relationship between orbital integrals and characters that can be made precise in many ways; one of the most profound is the trace formula (see §18.4). In view of Proposition 8.6.2 the following proposition is consonant with this duality:

Proposition 8.7.1 *Assume that $\gamma \in M(F)$ is absolutely semisimple and $C_\gamma^\circ \leq M$. If $D_{M \backslash G}(\gamma) \neq 0$, then one has that*

$$O_\gamma^\circ(f) = |D_{M \backslash G}(\gamma)|^{-1/2} O_\gamma^\circ((f_K)^P).$$

Here $(f_K)^P$ is defined using (8.27) and (8.28), we normalize the Haar measure dg on $G(F)$ as in (8.25) and we take the Haar measures on $C_\gamma^\circ(F)$ to be equal on either side of the equation. If we replace the assumption that $C_\gamma^\circ \leq M$ with the stronger assumption that $C_\gamma \leq M$ and define $O_\gamma^\circ(f)$ with $C_\gamma^\circ(F)$ replaced by $C_\gamma(F)$, then the proposition remains valid (the proof is even slightly easier).

The following proof is taken from [Lau96, Proposition 4.3.11]:

Proof. One has that

$$O_\gamma^\circ(f) = \int_{C_\gamma^\circ(F) \backslash M(F)} \int_{N(F)} \int_K f(k^{-1}n^{-1}m^{-1}\gamma mnk) dk dn \frac{dm}{dm_\gamma}.$$

For each $m \in M(F)$ there is a morphism of affine F -schemes $N \rightarrow N$ given on points in an F -algebra R by

$$\begin{aligned} N(R) &\longrightarrow N(R) \\ n &\longmapsto (m^{-1}\gamma m)^{-1}n^{-1}(m^{-1}\gamma m)n. \end{aligned} \quad (8.36)$$

We claim that this morphism is an isomorphism. We have an action of N on itself given by

$$\begin{aligned} N(R) \times N(R) &\longrightarrow N(R) \\ (n_0, n) &\longmapsto (m^{-1}\gamma m)^{-1}n^{-1}(m^{-1}\gamma m)n_0n. \end{aligned}$$

The image of (8.36) is the orbit $O(I)$ of the identity $I \in N(F)$ under this action and is therefore closed in N by Theorem 17.2.10 below. The stabilizer N_I of I in N is

$$C_{m^{-1}\gamma m} \cap N.$$

The group $C_{m^{-1}\gamma m}(\overline{F})/C_{m^{-1}\gamma m}^\circ(\overline{F})$ consists of semisimple elements by [Ste68, Corollary 9.4]. Since N is unipotent, the image of the natural map

$$C_{m^{-1}\gamma m}(\overline{F}) \cap N(\overline{F}) \longrightarrow C_{m^{-1}\gamma m}(\overline{F})/C_{m^{-1}\gamma m}^\circ(\overline{F})$$

is trivial, and we deduce that

$$N_I = C_{m^{-1}\gamma m} \cap N \leq C_{m^{-1}\gamma m}^\circ \leq M.$$

Hence, N_I is trivial. We conclude from Proposition 17.2.2 that (8.36) is a isomorphism onto its image, a closed subscheme of N . By considering dimensions we deduce that (8.36) is an isomorphism.

The Jacobian of (8.36) depends only on γ and is equal to

$$J(\gamma) = \left| \det (I - \text{Ad}(\gamma^{-1}) : \mathfrak{n} \longrightarrow \mathfrak{n}) \right|.$$

Thus, we can change variables and deduce that the integral above is

$$\int_{C_\gamma^\circ(F) \backslash M(F)} \int_{N(F)} \int_K f(k^{-1}m^{-1}\gamma mnk) dk \frac{dn}{J(\gamma)} \frac{dm}{dm_\gamma}.$$

Observing that

$$|D_{M \backslash G}(\gamma)| = \delta_P(\gamma) J(\gamma)^2$$

we deduce the proposition. $\square$

This is the first time we have used the usual change of variables formula in the non-Archimedean setting. A general discussion of integration on

manifolds over non-Archimedean local fields, including the change of variables formula, is in [Igu00, Chapter 7]. We refer especially to [Igu00, Proposition 7.4.1].

The following proposition was applied to construct the map (7.37) above:

Proposition 8.7.2 *Assume that G is quasi-split with Borel subgroup B and that $M \leq B$ is a maximal torus. The constant term induces a homomorphism*

$$C_c^\infty(G(F) // K) \longrightarrow C_c^\infty(M(F)/K_M)^{W(G,M)(F)}.$$

To prove Proposition 8.7.2 we require the following proposition:

Proposition 8.7.3 *With assumptions as in Proposition 8.7.2, for any $\gamma \in M(F)$*

$$|D_{M \setminus G}(\gamma)| \neq 0$$

if and only if $C_\gamma^\circ = M$. The set

$$\{\gamma \in M(F) : C_\gamma^\circ = M\} \quad (8.37)$$

is dense in $M(F)$ in the natural topology.

Proof. Clearly $M \leq C_\gamma^\circ$ because M is connected and commutative. Let $\mathfrak{g}_\gamma$ be the Lie algebra of C_γ° and $\mathfrak{m}$ be the Lie algebra of M . Since $I - \text{Ad}(\gamma^{-1})$ acts by zero on the subspace

$$\mathfrak{m} \setminus \mathfrak{g}_\gamma \leq \mathfrak{m} \setminus \mathfrak{g}$$

we deduce that $|D_{M \setminus G}(\gamma)| \neq 0$ if and only if $\mathfrak{m} = \mathfrak{g}_\gamma$. Since M is smooth and C_γ° is connected, $\mathfrak{m} = \mathfrak{g}_\gamma$ if and only if $C_\gamma^\circ = M$ [Mil17, Proposition 10.15].

To complete the proof we observe that the complement of the vanishing locus of a nonzero polynomial on a (non-Archimedean) analytic manifold is dense in the manifold. $\square$

There are surprisingly few textbook treatments of analytic manifolds in the non-Archimedean setting; two are [Ser06, Sch11] and some additional topics are in [Igu00]. Perhaps the reason is that more sophisticated treatments involving rigid analytic spaces, Berkovich spaces, etc. are often needed and have drawn more attention.

Proof of Proposition 8.7.2: In view of Lemma 8.6.3 it suffices to check that the image of the constant term map lies in the subalgebra of $C_c^\infty(M(F)/K_M)$ fixed by $W(G, M)(F)$.

Let $T_s \leq M$ be a maximal split torus. By Lemma 7.5.4 the natural map

$$W(G, M)(F) \longrightarrow W(G, T_s)(F)$$

is an isomorphism. Moreover, by Lemma 1.7.9, every element of $W(G, T_s)(F)$ has a representative $w \in N_G(T_s)(F)$, the normalizer of $T_s(F)$ in $G(F)$. Thus, it suffices to show that for $f \in C_c^\infty(G(F) // K)$ one has that

$$f^B(w^{-1}\gamma w) = f^B(\gamma) \quad (8.38)$$

for all $(w, \gamma) \in N_G(T_s)(F) \times M(F)$.

Assume that $\gamma \in M(F)$ satisfies $C_\gamma^\circ = M$. By Proposition 8.7.3 this implies $|D_{M \setminus G}(\gamma)| \neq 0$. We then have that

$$\begin{aligned} O_\gamma^\circ(f) &= |D_{M \setminus G}(\gamma)|^{-1/2} O_\gamma^\circ(f^B) \\ &= |D_{M \setminus G}(\gamma)|^{-1/2} f^B(\gamma) \end{aligned}$$

by Proposition 8.7.1. Here the last equality follows since $M(F)$ is commutative. It is clear that $O_{w^{-1}\gamma w}^\circ(f) = O_\gamma^\circ(f)$, so at least for $\gamma \in M(F)$ such that $C_\gamma^\circ = M$, we deduce (8.38). This set is dense in $M(F)$ by Proposition 8.7.3. It is clear that f^B is continuous, so we deduce (8.38) for all $\gamma \in M(F)$. $\square$

Exercises

In all of these exercises G is a reductive group over a non-Archimedean local field F and $P \leq G$ is a parabolic subgroup with a fixed Levi subgroup $M \leq P$ and unipotent radical N .

8.1. Let (π, V) be a unitary representation of $G(F)$ on a Hilbert space V . Let $V_{\text{sm}} \leq V$ be the subspace of smooth vectors (see Exercise 5.4). Construct an isomorphism $V_{\text{sm}} \cong (V_{\text{sm}})^\vee$. Using this isomorphism, show that a matrix coefficient in the sense of (8.4) is a matrix coefficient in the sense of Definition 4.3.2.

8.2. Assume that $K \leq G(F)$ is a maximal compact subgroup in good position with respect to (P, M) . Show that we can normalize the Haar measures dg, dk, dm, dn on $G(F), K, M(F)$ and $N(F)$, respectively, so that with respect to the decomposition $G(F) = KM(F)N(F)$ one has that

$$dg = \delta_P(m) dk dm dn$$

where

$$\text{meas}_{dk}(K) = \text{meas}_{dn}(N(F) \cap K) = \text{meas}_{dm}(M(F) \cap K) = 1.$$

8.3. Show that we can choose a function $f \in C_c^\infty(G(F))$ such that

$$\int_{P(F)} f(px) d_r p = 1$$

for all $x \in G(F)$.

8.4. Let $Q \leq M$ be a parabolic subgroup. Prove there is a natural transformation of functors

$$\mathrm{Ind}_P^G \circ \mathrm{Ind}_Q^M \cong \mathrm{Ind}_{QN}^G.$$

State and prove the analogous statement for Jacquet modules.

8.5. Prove that the functor Ind_P^G is exact (that is, it preserves exact sequences) and additive (that is, takes direct sums to direct sums).

8.6. Prove that the functor $\pi \mapsto \pi_N$ is exact (sends exact sequences to exact sequences) and additive (that is, takes direct sums to direct sums).

8.7. Let (π, V) be a smooth representation of $G(F)$. Prove that an element $\varphi \in V$ is in $V(N)$ if and only if $\int_{K_N} \pi(n)\varphi dn = 0$ for some compact subgroup $K_N \leq N(F)$.

8.8. Prove that $G^{\mathrm{reg}, \mathrm{ss}}$, defined as in (8.21), is an open subscheme of G .

8.9. Let $\{(\pi_i, V_i) : 1 \leq i \leq n\}$ be a finite set of irreducible admissible representations of $G(F)$ such that $\pi_i \cong \pi_j$ if and only if $i = j$. Let $K \leq G(F)$ be a compact open subgroup such that $V_1^K \neq 0$. Show that there exists an $f \in C_c^\infty(G(F) // K)$ such that $\pi_1(f)|_{V_1^K}$ is the identity and $\pi_i(f) = 0$ if $1 < i \leq n$.

8.10. Prove that an unramified irreducible admissible representation of $\mathrm{GL}_n(F)$ is tempered if and only if its Satake parameters have complex norm 1.

8.11. In the notation of §8.4, show that if $Q(\sigma^a, \lambda_a)$ and $Q(\sigma'^a, \lambda_{a'})$ are both square integrable (not just essentially square integrable), then they are not linked.

8.12. Suppose that π is an irreducible supercuspidal representation of $G(F)$. Show that there exists a function $f_\pi \in C_c^\infty(G(F))$ such that $\mathrm{tr} \pi(f_\pi) = 1$ and if $\mathrm{tr} \pi'(f_\pi) \neq 0$ for some irreducible admissible representation π' of $G(F)$, then $\pi \cong \pi' \otimes \chi$ for some quasi-character $\chi : Z_G(F) \rightarrow \mathbb{C}^\times$.

8.13. Let π be an irreducible principal series representation of $G(F)$. Prove that there are no coefficients of π in the sense of §8.5.

8.14. Let (π, V) be an admissible representation of a td-type group H . Prove that the space of smooth endomorphisms $\mathrm{End}_{\mathrm{sm}}(V)$ is an admissible representation of $H \times H$.

8.15. Prove that (8.29) holds.

8.16. Using the basis for $\mathrm{Ind}_P^G(V)$ constructed in the proof of Proposition 8.2.3, prove that (8.32) has trace (8.33).

Chapter 9

The Cuspidal Spectrum



God exists since mathematics is consistent, and the Devil exists since we cannot prove it.

André Weil

The cuspidal spectrum of $L^2([G])$ decomposes discretely into a Hilbert space direct sum with finite multiplicities. We give a proof of this fact in this chapter. We also prove that cuspidal automorphic forms are rapidly decreasing in the number field case and are compactly supported in the function field case.

9.1 Introduction

Let G be a reductive group over a global field F . As in (2.21), we set

$$[G] := A_G G(F) \backslash G(\mathbb{A}_F).$$

For $x \in A_G \backslash G(\mathbb{A}_F)$, one has the operator

$$\begin{aligned} R(x) : L^2([G]) &\longrightarrow L^2([G]) \\ \varphi &\longmapsto (g \mapsto \varphi(gx)). \end{aligned} \tag{9.1}$$

This defines the regular representation of $A_G \backslash G(\mathbb{A}_F)$ on $L^2([G])$. As we have seen in various circumstances, it is often convenient to work instead with the operators

$$\begin{aligned} R(f) : L^2([G]) &\longrightarrow L^2([G]) \\ \varphi &\longmapsto \left(g \mapsto \int_{A_G \backslash G(\mathbb{A}_F)} f(x) \varphi(gx) dx \right) \end{aligned} \tag{9.2}$$

for $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$. Here, as in (6.1),

$$C_c^\infty(A_G \backslash G(\mathbb{A}_F)) = C_c^\infty(A_G \backslash G(F_\infty)) \otimes C_c^\infty(G(\mathbb{A}_F^\infty))$$

where the tensor product is algebraic. Later in this book we will be interested in various notions of the trace for operators of this type. There is a problem, however, in that $R(f)$ does not in general have a well-defined trace. Instead, one restricts the operator to the cuspidal subspace $L_{\text{cusp}}^2([G])$ defined in (6.20). The main goal of the current chapter is to prove that this restriction has a well-defined trace (see Theorem 9.1.2 below). One is then left with understanding the complement of the cuspidal subspace; this is possible thanks to the theory of Eisenstein series which will be discussed in Chapter 10.

We now explain what is meant by a trace in this infinite dimensional setting. Let V be a Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|_2$. We recall from §3.1 that we always assume that our Hilbert spaces are separable. When speaking of an orthonormal basis we mean an orthonormal basis in the Hilbert space sense.

For all bounded operators $A : V \rightarrow V$, there exists a unique bounded operator $|A|$ characterized by the property that $|A| \circ |A| = A \circ A^*$ (where the $*$ denotes the adjoint) [DE09, Theorem 5.1.3]. As the notation suggests, the operator $|A|$ is **positive** in the sense that $(|A|\varphi, \varphi) \geq 0$ for all $\varphi \in V$.

Definition 9.1.1. A bounded linear operator $A : V \rightarrow V$ is **Hilbert-Schmidt** if there is an orthonormal basis $\{\varphi_i\}_{i=1}^\infty$ of V such that

$$\sum_{i=1}^\infty \|A\varphi_i\|_2^2 < \infty.$$

A bounded linear operator $A : V \rightarrow V$ is of **trace class** if there is an orthonormal basis $\{\varphi_i\}_{i=1}^\infty$ of V such that

$$\sum_{i=1}^\infty (|A|\varphi_i, \varphi_i) < \infty.$$

If A is Hilbert-Schmidt (resp. of trace class), we let

$$\|A\|_{\text{HS}} = \left(\sum_{i=1}^\infty \|A\varphi_i\|_2^2 \right)^{1/2} \quad \text{and} \quad \|A\|_{\text{tr}} = \sum_{i=1}^\infty (|A|\varphi_i, \varphi_i). \quad (9.3)$$

This is known as the **Hilbert-Schmidt norm** (resp. **trace norm**) of A . If A is of trace class we define its **trace** to be

$$\text{tr } A := \sum_{i=1}^\infty (A\varphi_i, \varphi_i).$$

The Hilbert-Schmidt norm, trace norm, and trace are independent of the choice of orthonormal basis $\{\varphi_i\}_{i=1}^\infty$ (see Exercise 9.1).

We recall from (6.20) that $\varphi \in L^2([G])$ is **cuspidal** if for every proper parabolic subgroup $P < G$ with unipotent radical N , one has that

$$\int_{[N]} \varphi(n g) dn = 0$$

for almost every $g \in A_G \backslash G(\mathbb{A}_F)$. Here and below, we mean almost every g with respect to a Haar measure on $A_G \backslash G(\mathbb{A}_F)$. The subspace of cuspidal $\varphi \in L^2([G])$ is denoted $L^2_{\text{cusp}}([G])$. We will check in Lemma 9.2.1 below that $L^2_{\text{cusp}}([G])$ is closed. Assuming this, $R(f)$ restricts to define a bounded linear operator

$$R_{\text{cusp}}(f) : L^2_{\text{cusp}}([G]) \longrightarrow L^2_{\text{cusp}}([G]) \quad (9.4)$$

(see Exercise 3.19).

As mentioned above, our aim in this chapter is to prove the following fundamental theorem:

Theorem 9.1.2 (Gelfand and Piatetski-Shapiro) *The subspace*

$$L^2_{\text{cusp}}([G]) \leq L^2([G])$$

is closed and $R_{\text{cusp}}(f)$ is of trace class for all $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$.

Here $R_{\text{cusp}}(f)$ is the restriction of $R(f)$ to the cuspidal subspace. Most of this chapter is devoted to the proof of Theorem 9.1.2. We complete the proof in §9.5. We will follow [God66], filling in some details using results from [Bor97]. These two references proceed classically whereas we will proceed adelically. This leads to some (mostly notational) simplifications.

We will show Theorem 9.1.2 implies the following corollary in §9.3 (see Corollary 9.3.2):

Corollary 9.1.3 *The subspace $L^2_{\text{cusp}}([G])$ decomposes into a Hilbert space direct sum of irreducible representations of $A_G \backslash G(\mathbb{A}_F)$, each occurring with finite multiplicity.*

In other words, we can write

$$L^2_{\text{cusp}}([G]) = \widehat{\bigoplus_{\pi} L^2_{\text{cusp}}(\pi)}, \quad (9.5)$$

where the Hilbert space direct sum is over isomorphism classes of cuspidal automorphic representations π of $A_G \backslash G(\mathbb{A}_F)$ and the multiplicity of π in the π -isotypic subspace $L^2_{\text{cusp}}(\pi)$ is finite.

The techniques used to prove Theorem 9.1.2 will also allow us to prove that cuspforms are rapidly decreasing in the sense of Definition 9.6.1 below in the

number field case (see Corollary 9.6.5). They are even compactly supported in the function field case (see Theorem 9.5.1). These facts are used constantly when working with cuspidal automorphic forms. The techniques also allow us to prove Theorem 6.5.3 above in §9.7.

We close this section by discussing a generalization of Theorem 9.1.2. The **discrete spectrum**

$$L^2_{\text{disc}}([G]) \leq L^2([G]) \quad (9.6)$$

is the largest closed subspace of $L^2([G])$ that decomposes as a Hilbert space direct sum of irreducible subrepresentations of $L^2([G])$. As in Definition 3.7.1, a discrete automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ is a unitary representation of $A_G \backslash G(\mathbb{A}_F)$ equivalent to a subrepresentation of $L^2_{\text{disc}}([G])$. Thus, cuspidal representations are discrete by Corollary 9.1.3.

Example 9.1. The constant functions on $[G]$ lie in $L^2([G])$ by Theorem 2.6.3. The space of all constant functions on $[G]$ is a closed 1-dimensional subspace of $L^2([G])$ isomorphic to the trivial representation of $A_G \backslash G(\mathbb{A}_F)$. If G^{der} is isotropic, then this representation is not cuspidal.

Hence, discrete automorphic representations need not be cuspidal. More complicated examples are given in Theorem 10.8.1 below.

A far more difficult theorem than Corollary 9.1.3 is the following:

Theorem 9.1.4 *Assume F is a number field. The restriction of $R(f)$ to the full discrete spectrum of $L^2([G])$ is of trace class.* $\square$

Müller [Mül89] first proved the theorem in the special case where f is finite under a maximal compact subgroup $K_\infty \leq G(F_\infty)$. Ji [Ji98] and Müller [Mül98] then independently removed this assumption.

9.2 The cuspidal subspace

We begin with the easiest part of Theorem 9.1.2:

Lemma 9.2.1 *The space $L^2_{\text{cusp}}([G])$ is closed in $L^2([G])$.*

We will use the proof of this lemma as an opportunity to introduce Poincaré series. To keep straight the techniques used in the proof, we do not assume that G is reductive or even connected; any affine algebraic group over a global field will do. Let $H \leq G$ be a subgroup. Assume that

$$A_G H(F) \backslash A_G H(\mathbb{A}_F)$$

is compact and let $\chi : A_G \backslash A_G H(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ be a quasi-character trivial on $H(F)$. It defines a function on $A_G H(F) \backslash A_G H(\mathbb{A}_F)$ that we again denote by χ .

The fact that $A_G H(F) \backslash A_G H(\mathbb{A}_F)$ is compact implies that $A_G \backslash A_G H(\mathbb{A}_F)$ is unimodular (see Exercise 9.6); we let dh be a choice of Haar measure on it. When $N \leq G$ is a unipotent group then the natural map $N(\mathbb{A}_F) \rightarrow A_G \backslash A_G N(\mathbb{A}_F)$ is an isomorphism. This is the situation of interest in Lemma 9.2.1.

For continuous functions $\varphi : [G] \rightarrow \mathbb{C}$, we let

$$\mathcal{P}_\chi(\varphi) := \int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} \varphi(h) \chi^{-1}(h) dh. \quad (9.7)$$

This is an example of a **period integral**. It is absolutely convergent since $A_G H(F) \backslash A_G H(\mathbb{A}_F)$ is compact.

We consider the subspace

$$V := \{ \varphi \in L^2([G]) : \mathcal{P}_\chi(R(g)\varphi) = 0 \text{ for almost every } g \in A_G \backslash G(\mathbb{A}_F) \},$$

where R is defined as in (9.1). We will show that V is closed:

Proposition 9.2.2 *The subspace $V \leq L^2([G])$ is closed.*

The proof of Proposition 9.2.2 will be given at the end of this section. Lemma 9.2.1 is a direct consequence of Proposition 9.2.2:

Proof of Lemma 9.2.1: For parabolic subgroups $P \leq G$ with unipotent radical N_P , let V_P be the space V of Proposition 9.2.2 in the special case where $H = N_P$ and χ is the trivial character. Then

$$L^2_{\text{cusp}}([G]) = \bigcap_P V_P.$$

Thus, $L^2_{\text{cusp}}([G])$, being the intersection of closed subspaces, is closed. $\square$

We now prepare to prove Proposition 9.2.2. The definition of V can be phrased in a more convenient manner as follows:

Lemma 9.2.3 *Let $\varphi \in L^2([G])$. One has $\mathcal{P}_\chi(R(g)\varphi) = 0$ for almost every $g \in A_G \backslash G(\mathbb{A}_F)$ if and only if $\mathcal{P}_\chi(R(f)\varphi) = 0$ for all $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$.*

Proof. Since $A_G H(F) \backslash A_G H(\mathbb{A}_F)$ is compact, the Fubini-Tonelli theorem implies that

$$\int_{A_G \backslash G(\mathbb{A}_F)} f(g) \mathcal{P}_\chi(R(g)\varphi) d_r g = \mathcal{P}_\chi(R(f)\varphi)$$

for all $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$. If $\mathcal{P}_\chi(R(g)\varphi) = 0$ for almost every g then the left-hand side is zero for all $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ and we deduce that $\mathcal{P}_\chi(R(f)\varphi) = 0$ for all $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$. Suppose $\mathcal{P}_\chi(R(g)\varphi) \neq 0$ for some set of $g \in A_G \backslash G(\mathbb{A}_F)$ of positive measure. Then there is a function $f \in L^1(A_G \backslash G(\mathbb{A}_F))$ such that $\int_{A_G \backslash G(\mathbb{A}_F)} f(g) \mathcal{P}_\chi(R(g)\varphi) d_r g \neq 0$. Since $C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ is dense in $L^1(A_G \backslash G(\mathbb{A}_F))$, we may assume $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$, and we deduce $\mathcal{P}_\chi(R(f)\varphi) \neq 0$. $\square$

For $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$, we consider the **kernel function**

$$K_f(x, y) = \sum_{\gamma \in G(F)} f(x^{-1}\gamma y). \quad (9.8)$$

It is easy to see that the sum is absolutely convergent (in fact finite for x and y in a fixed compact set) and that $K_f(x, y)$ defines a smooth function on $A_G \backslash G(\mathbb{A}_F) \times A_G \backslash G(\mathbb{A}_F)$ (see §16.1). We define the **Poincaré series**

$$\text{Pé}_{f,\chi}(g) := \int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} K_f(h, g) \chi^{-1}(h) dh. \quad (9.9)$$

Since $A_G H(F) \backslash A_G H(\mathbb{A}_F)$ is compact, the integral converges absolutely. If G is reductive, $H = N$ is the unipotent radical of a Borel subgroup and χ is a generic character of $N(\mathbb{A}_F)$ in the sense of §11.2 below, then this coincides with the Poincaré series studied in many places, including [Fri87] and [Ste87].

Lemma 9.2.4 *One has that $\text{Pé}_{f,\chi} \in L^2([G])$.*

Proof. One has that

$$\begin{aligned} \text{Pé}_{f,\chi}(g) \overline{\text{Pé}_{f,\chi}(g)} &= \int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} \sum_{\gamma_1 \in G(F)} f(h_1^{-1} \gamma_1 g) \chi^{-1}(h_1) dh_1 \\ &\quad \times \int_{A_G H(F) \backslash H(\mathbb{A}_F)} \sum_{\gamma_2 \in G(F)} \overline{f(h_2^{-1} \gamma_2 g)} \chi(h_2) dh_2. \end{aligned} \quad (9.10)$$

Choose a relatively compact measurable fundamental domain $\mathcal{F} \subset A_G H(\mathbb{A}_F)$ for the action of $A_G H(F)$. Then

$$\tilde{f}(g) := \int_{\mathcal{F}} f(h^{-1}g) \chi^{-1}(h) dh \quad (9.11)$$

is in $C_c^\infty(A_G \backslash G(\mathbb{A}_F))$. Thus, from (9.10) and (9.11), we obtain that

$$\int_{[G]} \text{Pé}_{f,\chi}(g) \overline{\text{Pé}_{f,\chi}(g)} d_r g = \int_{[G]} \left(\sum_{\gamma_1 \in G(F)} \tilde{f}(\gamma_1 g) \sum_{\gamma_2 \in G(F)} \overline{\tilde{f}(\gamma_2 g)} \right) d_r g.$$

Since $\sum_{\gamma \in G(F)} \tilde{f}(\gamma g)$ is a compactly supported continuous function on $[G]$, this integral converges. $\square$

In the proof of Lemma 9.2.6 below, we will encounter for the first time the trivial, but useful, technique known as **unfolding**. The following lemma is a special case of Theorem 3.2.3:

Lemma 9.2.5 *Suppose that G is a locally compact group with right Haar measure $d_r g$. If $f \in L^1(G)$ and $\Gamma \leq G$ is a discrete subgroup such that the modular character of G is trivial on Γ , then*

$$\int_{\Gamma \backslash G} \left(\sum_{\gamma \in \Gamma} f(\gamma g) \right) d_r g = \int_G f(g) d_r g.$$

□

Here in the lemma, we have continued to denote by $d_r g$ the G -invariant measure on $\Gamma \backslash G$ induced by $d_r g$. Unfolding is using the identity of Lemma 9.2.5 to simplify the left hand side of the equality. Using this technique, we prove the following lemma.

Lemma 9.2.6 *If $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ and $\varphi \in L^2([G])$, then*

$$\int_{[G]} \text{Pé}_{f, \chi \delta_G}(g) \varphi(g) d_r g = \mathcal{P}_\chi(R(f) \varphi).$$

Proof. Let $\varphi \in L^2([G])$. Then by Lemma 9.2.4, the integral

$$\int_{[G]} \text{Pé}_{f, \chi \delta_G}(g) \varphi(g) d_r g$$

is absolutely convergent. It is equal to

$$\int_{[G]} \int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} K_f(h, g) \chi^{-1}(h) dh \varphi(g) d_r g.$$

Since $A_G H(F) \backslash A_G H(\mathbb{A}_F)$ is compact, there is a compact subset $\Omega \subset [G]$ such that for $h \in A_G H(F) \backslash H(\mathbb{A}_F)$ the support of $K_f(h, g)$ as a function of g lies in Ω . Applying the Fubini-Tonelli theorem, this implies

$$\begin{aligned} & \int_{[G]} \int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} K_f(h, g) (\chi \delta_G)^{-1}(h) dh \varphi(g) d_r g \\ &= \int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} \left(\int_{[G]} K_f(h, g) \varphi(g) d_r g \right) (\chi \delta_G)^{-1}(h) dh. \end{aligned}$$

Unfolding as in Lemma 9.2.5, this is

$$\begin{aligned} & \int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} \left(\int_{A_G \backslash G(\mathbb{A}_F)} f(g) \varphi(hg) \delta_G(h) d_r g \right) (\chi \delta_G)^{-1}(h) dh \\ &= \int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} \left(\int_{A_G \backslash G(\mathbb{A}_F)} f(g) \varphi(hg) d_r g \right) \chi^{-1}(h) dh. \end{aligned}$$

□

Given our preparation, the proof of Proposition 9.2.2 is now straightforward:

Proof of Proposition 9.2.2: Each $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ gives rise to a linear form $(\cdot, \text{Pé}_{f, \chi \delta_G})$ on $L^2([G])$. It is continuous by Lemma 9.2.4. In view of Lemma 9.2.6 and Lemma 9.2.3, the space V is the intersection over all $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ of the kernels of these linear forms. $\square$

9.3 Deduction of the discreteness of the spectrum

We now explain how to deduce the discreteness of the spectrum in Corollary 9.1.3 from the assertion that certain operators are of trace class in Theorem 9.1.2. Our treatment is adapted from [Bor97] with little modification. As in the previous section, we do not yet assume that G is reductive.

We recall that a compact operator T on a Hilbert space V with an inner product $(\cdot, \cdot)$ can be written as

$$T = \sum_{n=1}^{\infty} \lambda_n (\varphi_n, \cdot) \varphi'_n, \quad (9.12)$$

where $\{\varphi_n\}_{n=1}^{\infty}$ and $\{\varphi'_n\}_{n=1}^{\infty}$ are orthonormal bases of vectors in V and $\{\lambda_n\}_{n=1}^{\infty} \subset \mathbb{C}$ is a sequence of numbers such that $\lim_{n \rightarrow \infty} \lambda_n = 0$. If T is self-adjoint, then we can take $\varphi_n = \varphi'_n$, and in particular in this case T has an orthonormal basis of eigenvectors (see [Zhu07, §1.3], for example). We observe that if T is a compact operator on V with nonzero eigenvalue λ then the λ -eigenspace is finite dimensional. This follows from the fact that $\lim_{n \rightarrow \infty} \lambda_n = 0$ in the decomposition (9.12).

A **Dirac sequence** on $A_G \backslash G(\mathbb{A}_F)$ is a sequence of nonnegative real valued functions $f_n \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ indexed by $n \in \mathbb{Z}_{>0}$ such that

- (a) for any open neighborhood $U \ni 1$ in $A_G \backslash G(\mathbb{A}_F)$ with compact closure, the support of f_n is contained in U for n large enough,
- (b) one has that $f_n(x) = f_n(x^{-1})\delta_G^{-1}(x)$, and
- (c) one has that

$$\int_{A_G \backslash G(\mathbb{A}_F)} f_n(x) dx = 1$$

for all n .

It is not difficult to see that a Dirac sequence on $A_G \backslash G(\mathbb{A}_F)$ exists (see Exercise 9.7). They are known as Dirac sequences because for any representation (R, V) of $A_G \backslash G(\mathbb{A}_F)$ on a locally convex and quasi-complete vector space V and any $\varphi \in V$ one has $R(f_n)\varphi \rightarrow \varphi$ as $n \rightarrow \infty$ by Corollary 3.4.2. We also note that if $\{f_n\}$ is a Dirac sequence and (R, V) is a unitary representation of $A_G \backslash G(\mathbb{A}_F)$ then $R(f_n)$ is self-adjoint for every n by condition (b).

Lemma 9.3.1 *Let (R, V) be a unitary representation of $A_G \backslash G(\mathbb{A}_F)$. Assume that there is a Dirac sequence $\{f_n\}$ on $A_G \backslash G(\mathbb{A}_F)$ such that $R(f_n)$ is compact for all n . Then (R, V) is the Hilbert space direct sum of its irreducible subrepresentations. The multiplicity of each subrepresentation is finite.*

Proof. We remind the reader that a subspace $W \leq V$ is a subrepresentation if and only if it is closed and $A_G \backslash G(\mathbb{A}_F)$ -invariant. We first show that a nonzero subrepresentation $W \leq V$ contains a minimal nonzero subrepresentation. Let W be a subrepresentation of V . Clearly there exists a j such that $R(f_j)|_W \neq 0$. Let λ be a nonzero eigenvalue of $R(f_j)$ in W and let $E_\lambda \leq W$ be the corresponding finite dimensional eigenspace (it is nonzero). Let M be a minimal space among the nonzero intersections of E_λ with the subrepresentations of W ; such a space exists because E_λ is finite dimensional. Let φ be a nonzero element in M and let $P \leq W$ be the smallest subrepresentation containing φ . By construction, $P \cap E_\lambda = M$. We claim that P is a minimal nonzero subrepresentation of W . Let Q be a nonzero subrepresentation of P and $Q^\perp$ be its orthogonal complement in P . Then $P = Q \oplus Q^\perp$. The spaces P , Q , and $Q^\perp$ are invariant under $R(f_j)$, hence $M = P \cap E_\lambda = (Q \cap E_\lambda) \oplus (Q^\perp \cap E_\lambda)$. Therefore $M = Q \cap E_\lambda$, $P = Q$ and $Q^\perp = 0$.

Next, we prove that (R, V) has a discrete decomposition into closed irreducible subrepresentations. Let S be the set consisting of the nonzero subrepresentations of V admitting a discrete decomposition. If we take a minimal subrepresentation M of V (which exists by the previous paragraph) then M is irreducible, hence S is not empty. The set S is partially ordered by the relation $X \leq Y$ if the space of X is contained in that of Y . Let W be the space given by a maximal element. If $W \neq V$, then we could add to W a minimal subrepresentation in the orthogonal complement of W , which exists by the previous claim. This contradiction implies that $W = V$.

Finally, we will show that the multiplicities are finite. Let π be an irreducible unitary representation occurring in V and let m_π be its multiplicity. Let j be such that $\pi(f_j)$ is not zero on the space of π , let λ be a nonzero eigenvalue of $\pi(f_j)$ and m_λ be its multiplicity. Then the dimension of the eigenspace of $R(f_j)$ in V with eigenvalue λ is finite and is at least $m_\lambda m_\pi$. Therefore, m_π is finite. $\square$

Corollary 9.3.2 *Theorem 9.1.2 implies Corollary 9.1.3.*

Proof. Trace class operators are Hilbert-Schmidt and Hilbert-Schmidt operators are compact (see Exercise 9.3). $\square$

One has the following sufficient condition for an operator to be of trace class:

Lemma 9.3.3 *Assume $\text{meas}_{d_r, g}([G]) < \infty$. Let $V \leq L^2([G])$ be a closed $A_G \backslash G(\mathbb{A}_F)$ -invariant subspace. For $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$, if*

$$\|R(f)\varphi\|_\infty \ll_f \|\varphi\|_2 \quad (9.13)$$

for all $\varphi \in V$ then the operator $R(f)|_V$ is of trace class.

In the lemma and below $\|\cdot\|_\infty$ denotes the uniform norm. We point out that if G is reductive then $\text{meas}_{d_r g}([G]) < \infty$ by Theorem 2.6.3.

Proof. For given $x \in A_G \backslash G(\mathbb{A}_F)$, the assumption implies that the map $\varphi \mapsto R(f)\varphi(x)$ is a continuous linear form on V . Hence by the Riesz representation theorem, there exists an element $K_{R(f),x} \in V$ such that

$$R(f)\varphi(x) = \int_{[G]} K_{R(f),x}(y)\varphi(y)d_r y$$

for all $\varphi \in V$. Taking $\varphi = \overline{K_{R(f),x}}$ we obtain

$$\|K_{R(f),x}\|_2^2 \leq \|R(f)\overline{K_{R(f),x}}\|_\infty \ll_f \|K_{R(f),x}\|_2.$$

Thus $\|K_{R(f),x}\|_2 \leq C$ for some constant $C > 0$ independent of x . Write

$$K_{R(f)}(x, y) := K_{R(f),x}(y)$$

for $x, y \in [G]$. Then

$$\begin{aligned} \int_{[G]} \int_{[G]} |K_{R(f)}(x, y)|^2 d_r x d_r y &= \int_{[G]} \left(\int_{[G]} \overline{K_{R(f),x}(y)} K_{R(f),x}(y) d_r y \right) d_r x \\ &\leq C^2 \int_{[G]} d_r x \end{aligned}$$

is finite and

$$R(f)\varphi(x) = \int_{[G]} K_{R(f)}(x, y)\varphi(y)d_r y$$

for $\varphi \in V$. Therefore the operator $R(f)$ on V is represented by an L^2 -kernel. This implies that it is Hilbert-Schmidt (see Exercise 9.4).

It follows from the Dixmier Malliavin lemma (Theorem 4.2.9) that f is a finite linear combination of convolutions $f_1 * f_2$ with $f_1, f_2 \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ (see Exercise 9.8). Therefore $R(f)$ is a finite linear combination of convolutions of two Hilbert-Schmidt operators and hence is of trace class (see Exercise 9.2). $\square$

In view of lemmas 9.2.1 and 9.3.3, to prove Theorem 9.1.2, it suffices to prove that for $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ one has an estimate (9.13) for $\varphi \in L_{\text{cusp}}^2([G])$. This estimate is really the heart of the proof. In the number field case we derive (9.13) using the argument of [God66]. In the function field case we will be able to deduce something even stronger that will allow us to argue directly without invoking Lemma 9.3.3.

In the next sections we will begin to employ our assumption that G is reductive. We have been careful in isolating when this assumption is used for

two reasons. First, it is helpful to see at what point the argument passes from abstract representation theory to something that uses the specific structure of the adelic quotient in the reductive case. Second, there ought to be a notion of a cuspidal automorphic representation on certain groups that are not necessarily reductive, but we do not know the natural level of generality. This has been studied in the case where G is the Jacobi group, a semidirect product of SL_2 and the 3-dimensional Heisenberg group [BS98, §4.2]. The book [Sli84] contains relevant information on the representation theory of the adelic points of affine algebraic groups that are not necessarily reductive.

9.4 The basic estimate

We now assume that G is reductive and that F is a number field. Our aim is to prove (9.13). As explained in the previous section, this will complete the proof of Theorem 9.1.2 in the number field case.

By applying Weil restriction of scalars we can assume $F = \mathbb{Q}$. Let P be a fixed minimal parabolic subgroup of G with unipotent radical N . We assume $P \neq G$; otherwise there is nothing to prove. Let $\mathfrak{n}$ be the Lie algebra of N . The exponential map induces an isomorphism of affine schemes over $\mathbb{Q}$:

$$\exp : \mathfrak{n} \longrightarrow N$$

[Mil17, Proposition 14.32]. Here we are slightly abusing notation in that we are denoting by $\mathfrak{n}$ the commutative algebraic group over $\mathbb{Q}$ whose points in a $\mathbb{Q}$ -algebra R are

$$\mathfrak{n}(R) := \mathfrak{n} \otimes_{\mathbb{Q}} R.$$

This group scheme is isomorphic to $\mathbb{G}_a^{\dim_{\mathbb{Q}} \mathfrak{n}}$. Note that we are not claiming that the exponential map is an isomorphism with respect to any sort of group structure.

The method of proof we will employ is to analyze $R(f)\varphi$ by applying Poisson summation on $\mathfrak{n}(\mathbb{Q})$. This allows us to write $R(f)\varphi$ in terms of a sum over $\eta \in \mathfrak{n}(\mathbb{Q})$ of Fourier transforms of compactly supported smooth functions on $\mathfrak{n}(\mathbb{A}_{\mathbb{Q}})$ depending on f . These Fourier transforms can be bounded using reduction theory except when η is degenerate in a certain sense. Happily, the assumption that φ is cuspidal causes the contribution of degenerate η to vanish.

Let us begin this process. Any element $\varphi \in L^2([G])$ is left-invariant under $G(\mathbb{Q})$. Thus for $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_{\mathbb{Q}}))$ we can write

$$\begin{aligned}
R(f)\varphi(x) &= \int_{A_G \backslash G(\mathbb{A}_{\mathbb{Q}})} f(x^{-1}y)\varphi(y)dy \\
&= \int_{A_G N(\mathbb{Q}) \backslash G(\mathbb{A}_{\mathbb{Q}})} \sum_{\gamma \in N(\mathbb{Q})} f(x^{-1}\gamma y)\varphi(y)dy \\
&= \int_{A_G N(\mathbb{Q}) \backslash G(\mathbb{A}_{\mathbb{Q}})} K_{f,P}(x,y)\varphi(y)dy,
\end{aligned} \tag{9.14}$$

where for $x, y \in G(\mathbb{A}_{\mathbb{Q}})$ we set

$$K_{f,P}(x,y) = \sum_{\gamma \in N(\mathbb{Q})} f(x^{-1}\gamma y) = \sum_{\eta \in \mathfrak{n}(\mathbb{Q})} f(x^{-1}\exp(\eta)y). \tag{9.15}$$

We view $\mathfrak{n}$ as an affine abelian algebraic group over $\mathbb{Q}$ with addition as the law of composition. In particular,

$$[\mathfrak{n}] := \mathfrak{n}(\mathbb{Q}) \backslash \mathfrak{n}(\mathbb{A}_{\mathbb{Q}})$$

is an abelian group. We note that for fixed $x, y \in G(\mathbb{A}_{\mathbb{Q}})$ the function

$$n \mapsto f(x^{-1}\exp(n)y)$$

is smooth on $\mathfrak{n}(\mathbb{A}_{\mathbb{Q}})$. Let $\psi : \mathbb{Q} \backslash \mathbb{A}_{\mathbb{Q}} \rightarrow \mathbb{C}^{\times}$ be a nontrivial character and let

$$\langle \cdot, \cdot \rangle : \mathfrak{n} \times \mathfrak{n} \longrightarrow \mathbb{G}_a$$

be a perfect pairing (over $\mathbb{Q}$). Then by Poisson summation one has

$$K_{f,P}(x,y) = \sum_{\eta \in \mathfrak{n}(\mathbb{Q})} \int_{\mathfrak{n}(\mathbb{A}_{\mathbb{Q}})} f(x^{-1}\exp(n)y)\psi(\langle \eta, n \rangle)dn \tag{9.16}$$

for an appropriate choice of (additive) Haar measure dn on $\mathfrak{n}(\mathbb{A}_{\mathbb{Q}})$. For a primer on Poisson summation over global fields, see Appendix C. Using (9.14) and (9.16) we obtain

$$\begin{aligned}
R(f)\varphi(x) &= \int_{A_G N(\mathbb{Q}) \backslash G(\mathbb{A}_{\mathbb{Q}})} K_{f,P}(x,y)\varphi(y)dy \\
&= \int_{A_G N(\mathbb{Q}) \backslash G(\mathbb{A}_{\mathbb{Q}})} \left(\sum_{\eta \in \mathfrak{n}(\mathbb{Q})} \int_{\mathfrak{n}(\mathbb{A}_{\mathbb{Q}})} f(x^{-1}\exp(n)y)\psi(\langle \eta, n \rangle)dn \right) \varphi(y)dy.
\end{aligned} \tag{9.17}$$

We estimate $R(f)\varphi(x)$ in Proposition 9.4.7 below using this expansion.

Let $T \leq P$ be a maximal split torus. Then $M := C_G(T)$ is a Levi subgroup of P . Let $Z_0 \leq Z_G$ be a maximal split torus and let $T_0 \leq G^{\text{der}}$ be a maximal split torus such that $T_0 Z_0 = T$. Thus $A_{Z_0} = A_G$. The torus T is a maximal split central torus of M and hence

$$A_M = A_{Z_0} A_{T_0}$$

where the product is direct. It follows that

$$A_{T_0} M(\mathbb{A}_{\mathbb{Q}})^1 = M(\mathbb{A}_{\mathbb{Q}}) \cap G(\mathbb{A}_{\mathbb{Q}})^1. \quad (9.18)$$

On the other hand every element of A_{T_0} lies in $A_{T_0}(t)$, defined as in (2.26), for $t \in \mathbb{R}_{>0}$ small enough. Thus by the Iwasawa decomposition (2.24) we can choose a maximal compact subgroup $K \leq G(\mathbb{A}_{\mathbb{Q}})$ such that any element $y \in G(\mathbb{A}_{\mathbb{Q}})^1$ can be written as

$$y = u_y m_y s_y k_y \quad (9.19)$$

with $(u_y, m_y, s_y, k_y) \in N(\mathbb{A}_{\mathbb{Q}}) \times M(\mathbb{A}_{\mathbb{Q}})^1 \times A_{T_0}(t) \times K$ for some $t > 0$ (depending on y). To analyze what part of this decomposition of y is uniquely determined by y , it is convenient to introduce a certain map. Let $\mathfrak{a}_M := \text{Hom}(X^*(M), \mathbb{R})$. Define the homomorphism

$$H_M : M(\mathbb{A}_{\mathbb{Q}}) \longrightarrow \mathfrak{a}_M \quad (9.20)$$

by

$$\langle H_M(m), \lambda \rangle = \log |\lambda(m)|$$

for $\lambda \in X^*(M)$. Thus $\ker H_M = M(\mathbb{A}_{\mathbb{Q}})^1$ and in particular H_M is trivial on $K \cap M(\mathbb{A}_{\mathbb{Q}})$. The map mentioned above is

$$\begin{aligned} H_P : G(\mathbb{A}_{\mathbb{Q}}) &\longrightarrow \mathfrak{a}_M \\ nmk &\longmapsto H_M(m). \end{aligned} \quad (9.21)$$

Here $(n, m, k) \in N(\mathbb{A}_{\mathbb{Q}}) \times M(\mathbb{A}_{\mathbb{Q}}) \times K$. The restriction of H_P to A_M induces an isomorphism

$$H_P : A_M \xrightarrow{\sim} \mathfrak{a}_M. \quad (9.22)$$

This implies the following lemma:

Lemma 9.4.1 *If $y = u_{iy} m_{iy} s_{iy} k_{iy}$ with*

$$(u_{iy}, m_{iy}, s_{iy}, k_{iy}) \in N(\mathbb{A}_{\mathbb{Q}}) \times M(\mathbb{A}_{\mathbb{Q}})^1 \times A_{T_0}(t) \times K$$

for $1 \leq i \leq 2$ then $s_{1y} = s_{2y}$. □

We shall denote by

$$\mathfrak{S}(t) = \Omega_N \Omega_M A_{T_0}(t) K \quad (9.23)$$

a fixed Siegel set in $G(\mathbb{A}_{\mathbb{Q}})^1$. Here Ω_N and Ω_M are fixed compact subsets of $N(\mathbb{A}_{\mathbb{Q}})$ and $M(\mathbb{A}_{\mathbb{Q}})^1$, respectively.

Let Φ be the set of roots of T in G . The subset $\Phi^+ \subset \Phi$ of roots appearing in $\mathfrak{n}$ is a set of positive roots. Let $\mathfrak{n}_\lambda \leq \mathfrak{n}$ be the root space of a root $\lambda \in \Phi^+$.

Lemma 9.4.2 *There is a compact subset $\Omega'_N \subset N(\mathbb{A}_{\mathbb{Q}})$ depending on Ω_N and t such that $s^{-1}\Omega_N s \subset \Omega'_N$ for all $s \in A_{T_0}(t)$.*

Proof. It suffices to check that for any compact set $\Omega \subset N(\mathbb{R})$ there is a compact subset $\Omega' \subset N(\mathbb{R})$ depending on Ω and t such that $s^{-1}\Omega s \subset \Omega'$ for all $s \in A_{T_0}(t)$. For this, since the exponential map $\exp : \mathfrak{n}(\mathbb{R}) \rightarrow N(\mathbb{R})$ is a homeomorphism it suffices to check that for any compact set $\omega \subset \mathfrak{n}(\mathbb{R})$ there is a compact subset $\omega' \subset \mathfrak{n}(\mathbb{R})$ depending on t and ω such that

$$\text{Ad}(s^{-1})(\omega) \subset \omega'$$

for $s \in A_{T_0}(t)$. We have

$$\mathfrak{n} = \bigoplus_{\lambda \in \Phi^+} \mathfrak{n}_\lambda$$

and $\text{Ad}(s^{-1})$ preserves each $\mathfrak{n}_\lambda(\mathbb{R})$. Thus it suffices to treat the special case in which $\omega \subset \mathfrak{n}_\lambda(\mathbb{R})$. Now if λ is a simple root, then $\lambda(s) \geq t$ for $s \in A_{T_0}(t)$ by definition of $A_{T_0}(t)$. In general if we write $\lambda = \sum_{i=1}^n \alpha_i$ for some set of simple roots α_i (not necessarily distinct) then $\lambda(s) \geq t^n$ for $s \in A_{T_0}(t)$. Since the restriction of $\text{Ad}(s^{-1})$ to $\mathfrak{n}_\lambda$ is multiplication by $\lambda(s)^{-1}$ we deduce the special case isolated above. $\square$

Example 9.2. We illustrate the lemma in a special case. Let $G = \text{GL}_3$, let $P = B$ be the Borel subgroup of upper triangular matrices and let $T < B$ be the subgroup of diagonal matrices; it is a maximal torus. For $t_0 \in \mathbb{R}_{>0}$ and $\ell \in \mathbb{Z}_{>0}$ let

$$\Omega(\ell, t_0) := \left\{ \begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix} : x_{ij}^\infty \in \ell^{-1}\widehat{\mathbb{Z}}, |x_{ij}|_\infty \leq t_0 \right\}.$$

This is a compact subset of $N(\mathbb{A}_{\mathbb{Q}})$, and every compact subset of $N(\mathbb{A}_{\mathbb{Q}})$ is contained in $\Omega(\ell, t_0)$ for suitable ℓ and t_0 . For

$$s = \begin{pmatrix} s_1 & & \\ & s_2 & \\ & & s_3 \end{pmatrix} \in A_{T_0}$$

we have

$$\begin{pmatrix} s_1 & & \\ & s_2 & \\ & & s_3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & x_{12} & x_{13} \\ & 1 & x_{23} \\ & & 1 \end{pmatrix} \begin{pmatrix} s_1 & & \\ & s_2 & \\ & & s_3 \end{pmatrix} = \begin{pmatrix} 1 & s_1^{-1}s_2x_{12} & s_1^{-1}s_3x_{13} \\ & 1 & s_2^{-1}s_3x_{23} \\ & & 1 \end{pmatrix}. \quad (9.24)$$

One has $s \in A_{T_0}(t)$ if and only if $s_1s_2^{-1} \geq t$ and $s_2s_3^{-1} \geq t$. If $s_1s_2^{-1} \geq t$ and $s_2s_3^{-1} \geq t$ then $s_1s_3^{-1} \geq t^2$. Thus

$$\text{Ad}(s^{-1})(\Omega(\ell, t_0)) \subseteq \Omega(\ell, \max(t_0t^{-1}, t_0t^{-2}))$$

for $s \in A_{T_0}(t)$.

Lemma 9.4.3 *There is a compact subset $\Omega_G \subset G(\mathbb{A}_\mathbb{Q})^1$ depending only on $\mathfrak{S}(t)$ such that if $x \in \mathfrak{S}(t)$ then $s_x^{-1}x \in \Omega_G$.*

Proof. If $x \in \mathfrak{S}(t)$ we have

$$x \in \Omega_N \Omega_M s_x K = \Omega_N s_x \Omega_M K.$$

Here we are using the fact that s_x commutes with $M(\mathbb{A}_\mathbb{Q}) = C_G(T)(\mathbb{A}_\mathbb{Q})$.

There is a compact subset $\Omega'_N \subset N(\mathbb{A}_\mathbb{Q})$ such that $s_x^{-1}\Omega_N s_x \subset \Omega'_N$ for all $x \in \mathfrak{S}(t)$ by Lemma 9.4.2. Thus $x \in s_x \Omega_G$ for some compact subset Ω_G of $G(\mathbb{A}_\mathbb{Q})^1$. $\square$

Lemma 9.4.4 *Fix a compact subset $\Omega'_N \subset N(\mathbb{A}_\mathbb{Q})$. Assume that $x \in \mathfrak{S}(t)$ and $y \in \Omega'_N M(\mathbb{A}_\mathbb{Q})K \cap G(\mathbb{A}_\mathbb{Q})^1$. Let $\Omega_G \subseteq G(\mathbb{A}_\mathbb{Q})^1$ be a compact set. If*

$$x^{-1}ny \in \Omega_G$$

for some $n \in N(\mathbb{A}_\mathbb{Q})$, then $s_x^{-1}y$ lies in a compact subset of $G(\mathbb{A}_\mathbb{Q})^1$ depending only on Ω_G , $\mathfrak{S}(t)$ and Ω'_N .

Proof. To avoid distracting repetition, for this proof by “a compact subset” we mean “a compact subset depending only on Ω_G , $\mathfrak{S}(t)$, and Ω'_N .” By Lemma 9.4.3 $s_x^{-1}x \in \Omega'_G$ for some compact subset $\Omega'_G \subset G(\mathbb{A}_\mathbb{Q})^1$. Thus

$$\Omega'_G \Omega_G \ni (s_x^{-1}x)x^{-1}ny = s_x^{-1}ny = s_x^{-1}nu_y s_x m_y s_x^{-1}s_y k_y. \quad (9.25)$$

Here we assume $u_y \in \Omega'_N$. Now $s_x^{-1}nu_y s_x \in N(\mathbb{A}_\mathbb{Q})$, $m_y \in M(\mathbb{A}_\mathbb{Q})^1$, and $s_x^{-1}s_y \in A_{T_0}$. Thus

$$H_P((s_x^{-1}x)x^{-1}ny) = H_P(s_x^{-1}s_y) \in H_P(\Omega'_G \Omega_G K).$$

In view of the isomorphism (9.22) we see that $s_x^{-1}s_y$ lies in a compact subset of A_{T_0} . Since $s_x \in A_{T_0}(t)$ this implies $s_y \in A_{T_0}(t')$ for some $t' > 0$ depending only on Ω_G , $\mathfrak{S}(t)$, and Ω'_N .

Using the fact that $s_x^{-1}s_y$ lies in a compact subset of A_{T_0} we deduce from (9.25) that there is a compact subset Ω''_G of $G(\mathbb{A}_\mathbb{Q})^1$ such that $s_x^{-1}nu_y s_x m_y \in \Omega''_G$. Since $N(\mathbb{A}_\mathbb{Q}) \cap M(\mathbb{A}_\mathbb{Q})^1$ is trivial this implies that m_y lies in a compact subset Ω'_M of $M(\mathbb{A}_\mathbb{Q})^1$. By assumption, $u_y \in \Omega'_N$. Thus y lies in the Siegel set $\Omega'_N \Omega'_M A_{T_0}(t')K$. Applying Lemma 9.4.3 we deduce that $s_y^{-1}y$ lies in a compact subset of $G(\mathbb{A}_\mathbb{Q})^1$. But we have already shown that $s_x^{-1}s_y$ lies in a compact subset of A_{T_0} , so we deduce that $s_x^{-1}y$ lies in a compact subset of $G(\mathbb{A}_\mathbb{Q})^1$. $\square$

In the previous two lemmas we worked with subsets of $G(\mathbb{A}_\mathbb{Q})^1$ instead of $A_G \backslash G(\mathbb{A}_\mathbb{Q})$. This was merely for notational convenience. Indeed, Lemma 2.6.2 implies the natural map

$$G(\mathbb{A}_\mathbb{Q})^1 \longrightarrow A_G \backslash G(\mathbb{A}_\mathbb{Q}) \quad (9.26)$$

is an isomorphism of topological groups. Moreover for any subsets $\Omega_1, \Omega_2 \subset G(\mathbb{A}_{\mathbb{Q}})^1$ the map (9.26) induces a homeomorphism

$$\Omega_1 M(\mathbb{A}_{\mathbb{Q}}) \Omega_2 \cap G(\mathbb{A}_{\mathbb{Q}})^1 \xrightarrow{\sim} A_G \backslash \Omega_1 M(\mathbb{A}_{\mathbb{Q}}) \Omega_2.$$

Using (9.26) we will identify elements of $G(\mathbb{A}_{\mathbb{Q}})^1$ with their images in $A_G \backslash G(\mathbb{A}_{\mathbb{Q}})$ when this is convenient.

Choose a norm

$$\|\cdot\|_{\mathfrak{n}}$$

on the real vector space $\mathfrak{n}(\mathbb{R})$. The next proposition, Proposition 9.4.5, gives us good analytic control over many of the terms in the sum over η in (9.17). It will turn out that the terms that cannot be estimated using Proposition 9.4.5 vanish since φ is cuspidal (see the proof of Proposition 9.4.7 below).

Proposition 9.4.5 *Let $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_{\mathbb{Q}}))$. Assume that*

$$y \in A_G \backslash \Omega'_N M(\mathbb{A}_{\mathbb{Q}}) K$$

for some fixed compact subset $\Omega'_N \subset N(\mathbb{A}_{\mathbb{Q}})$. Let $B \in \mathbb{R}_{\geq 0}$ and assume $\eta \in \mathfrak{n}(\mathbb{Q})$ is not zero. There is a compact subset $\Omega_{\mathfrak{n}}^\infty \subset \mathfrak{n}(\mathbb{A}_{\mathbb{Q}}^\infty)$ depending on f , Ω'_N and $\mathfrak{S}(t)$ with the following property: for any $x \in \mathfrak{S}(t)$ one has

$$\begin{aligned} & \int_{\mathfrak{n}(\mathbb{A}_{\mathbb{Q}})} f(x^{-1} \exp(n)y) \psi(\langle \eta, n \rangle) dn \\ & \ll_{f, B, \Omega'_N, \mathfrak{S}(t)} \|\eta\|_{\mathfrak{n}}^{-B} \mathbb{1}_{\Omega_{\mathfrak{n}}^\infty}(\eta) \delta_P(s_x) \lambda(s_x)^{-B} \end{aligned}$$

for any $\lambda \in \Phi^+$ such that $\langle \eta, \cdot \rangle|_{\mathfrak{n}_\lambda} \neq 0$.

Proof. Since $C_c^\infty(A_G \backslash G(\mathbb{A}_{\mathbb{Q}})) = C_c^\infty(A_G \backslash G(\mathbb{R})) \otimes C_c^\infty(G(\mathbb{A}_{\mathbb{Q}}^\infty))$ we may assume that $f = f_\infty f^\infty$ for some $f_\infty \in C_c^\infty(A_G \backslash G(\mathbb{R}))$ and $f^\infty \in C_c^\infty(G(\mathbb{A}_{\mathbb{Q}}^\infty))$. One has that

$$\begin{aligned} & \int_{\mathfrak{n}(\mathbb{A}_{\mathbb{Q}})} f(x^{-1} \exp(n)y) \psi(\langle \eta, n \rangle) dn \\ &= \int_{\mathfrak{n}(\mathbb{A}_{\mathbb{Q}})} f(x^{-1} s_x \exp(\text{Ad}(s_x^{-1})n) s_x^{-1} y) \psi(\langle \eta, n \rangle) dn \\ &= \delta_P(s_x) \int_{\mathfrak{n}(\mathbb{A}_{\mathbb{Q}})} f(x^{-1} s_x \exp(n) s_x^{-1} y) \psi(\langle \eta, \text{Ad}(s_x)n \rangle) dn \quad (9.27) \\ &= \delta_P(s_x) \int_{\mathfrak{n}(\mathbb{R})} f_\infty(x_\infty^{-1} s_x \exp(n) s_x^{-1} y_\infty) \psi(\langle \text{Ad}^\vee(s_x^{-1})\eta, n \rangle) dn \\ & \quad \times \int_{\mathfrak{n}(\mathbb{A}_{\mathbb{Q}}^\infty)} f^\infty((x^\infty)^{-1} \exp(n) y^\infty) \psi(\langle \eta, n \rangle) dn. \end{aligned}$$

Consider the function

$$G(\mathbb{A}_{\mathbb{Q}}^{\infty})^2 \times \mathfrak{n}(\mathbb{A}_{\mathbb{Q}}^{\infty}) \longrightarrow \mathbb{C}$$

$$(x^{\infty}, y^{\infty}, n') \longmapsto \int_{\mathfrak{n}(\mathbb{A}_{\mathbb{Q}}^{\infty})} f^{\infty}((x^{\infty})^{-1} \exp(n) y^{\infty}) \psi(\langle n', n \rangle) dn. \quad (9.28)$$

As a function of n' (9.28) is the Fourier transform of a compactly supported smooth function on $\mathfrak{n}(\mathbb{A}_{\mathbb{Q}}^{\infty})$. Hence it is compactly supported and smooth as a function of $n' \in \mathfrak{n}(\mathbb{A}_{\mathbb{Q}}^{\infty})$ for each x^{∞}, y^{∞} (see Lemma C.3.1). We now take Ω_G to be the support of f and apply Lemma 9.4.3 and Lemma 9.4.4 to see that if (9.28) is nonzero then x^{∞} and y^{∞} lie in a compact set Ω'_G depending only on f , Ω'_N and $\mathfrak{S}(t)$. There is a compact open subgroup $K^{\infty} \leq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ such that (9.28) is invariant under $(x^{\infty}, y^{\infty}) \mapsto (x^{\infty} k_1, y^{\infty} k_2)$ for $k_1, k_2 \in K^{\infty}$. Thus as x^{∞} and y^{∞} vary over Ω'_G , the expression (9.28) varies over a finite set of functions of n' . We deduce that

$$\int_{\mathfrak{n}(\mathbb{A}_{\mathbb{Q}}^{\infty})} f^{\infty}((x^{\infty})^{-1} \exp(n) y^{\infty}) \psi(\langle n', n \rangle) dn \ll_{f, \Omega'_N, \mathfrak{S}(t)} \mathbb{1}_{\Omega_{\mathfrak{n}}^{\infty}}(n') \quad (9.29)$$

for some compact set $\Omega_{\mathfrak{n}}^{\infty} \subset \mathfrak{n}(\mathbb{A}_{\mathbb{Q}}^{\infty})$ depending on f , Ω'_N , and $\mathfrak{S}(t)$.

We now turn to the contribution of the infinite places to (9.27). For $n' \in \mathfrak{n}(\mathbb{R})$ consider

$$\delta_P(s_x) \int_{\mathfrak{n}(\mathbb{R})} f(x_{\infty}^{-1} s_x \exp(n) s_x^{-1} y_{\infty}) \psi(\langle \text{Ad}^{\vee}(s_x^{-1}) n', n \rangle) dn. \quad (9.30)$$

One can estimate this directly using an integration by parts argument. The key point is to control the derivatives of $f(x_{\infty}^{-1} s_x \exp(n) s_x^{-1} y_{\infty})$ as a function of n . It is slightly more elegant (but equivalent) to phrase this in terms of functional analysis. We have continuous maps

$$\begin{aligned} \text{res} : C_c^{\infty}(G(\mathbb{R})) &\longrightarrow C_c^{\infty}(\mathfrak{n}(\mathbb{R})) \\ f &\longmapsto (n \mapsto f(\exp(n))) \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}_{\mathfrak{n}} : C_c^{\infty}(\mathfrak{n}(\mathbb{R})) &\longrightarrow \mathcal{S}(\mathfrak{n}(\mathbb{R})) \\ f &\longmapsto \left(n' \mapsto \int_{\mathfrak{n}(\mathbb{R})} f(n) \psi(\langle n', n \rangle) dn \right). \end{aligned}$$

Here $\mathcal{S}(\mathfrak{n}(\mathbb{R}))$ is the usual Schwartz space on the vector space $\mathfrak{n}(\mathbb{R})$, and all spaces are given their usual topologies. The map $\mathcal{F}_{\mathfrak{n}}$ is just a particular normalization of the Fourier transform. We also have a continuous action

$$\begin{aligned} R : G(\mathbb{R}) \times G(\mathbb{R}) \times C_c^{\infty}(G(\mathbb{R})) &\longrightarrow C_c^{\infty}(G(\mathbb{R})) \\ (g_1, g_2, f) &\longmapsto (g \mapsto f(g_1^{-1} g g_2)). \end{aligned}$$

Now $f \mapsto \sup_{n' \in \mathfrak{n}(\mathbb{R})} \|n'\|^B |f|(n')$ is a continuous seminorm on $\mathcal{S}(\mathfrak{n}(\mathbb{R}))$ for any $B \in \mathbb{R}_{\geq 0}$. Therefore, if $\Omega \subset G(\mathbb{R}) \times G(\mathbb{R})$ is any compact set one has

$$\sup_{(g_1, g_2) \in \Omega} \left(\max_{n' \in \mathfrak{n}(\mathbb{R})} \|n'\|^B \mathcal{F}_{\mathfrak{n}} \circ \text{res}(R(g_1, g_2)f)(n') \right) < \infty. \quad (9.31)$$

Applying Lemma 9.4.3 and Lemma 9.4.4 we deduce from (9.31) that (9.30) is bounded by a constant depending on f, B, Ω'_N , and $\mathfrak{S}(t)$ times

$$\delta_P(s_x) \max(1, \|\text{Ad}^\vee(s_x^{-1})n'\|_{\mathfrak{n}})^{-B}.$$

Combining this estimate with the estimate (9.29) at the finite places and the identity (9.27), we see that for $\eta \in \mathfrak{n}(\mathbb{Q}) - \{0\}$

$$\int_{\mathfrak{n}(\mathbb{A}_{\mathbb{Q}})} f(x^{-1} \exp(n)y) \psi(\langle \eta, n \rangle) dn \ll \delta_P(s_x) \|\text{Ad}^\vee(s_x^{-1})\eta\|_{\mathfrak{n}}^{-B} \mathbb{1}_{\Omega_{\mathfrak{n}}^\infty}(\eta), \quad (9.32)$$

where the implied constant depends on f, B, Ω'_N and $\mathfrak{S}(t)$.

For each $\lambda \in \Phi^+$ let $\mathfrak{n}_\lambda^\vee(\mathbb{R})$ be the $\mathbb{R}$ -linear dual of $\mathfrak{n}_\lambda(\mathbb{R})$. Then $\text{Ad}^\vee(s_x^{-1})$ is multiplication by $\lambda(s_x)$ on $\mathfrak{n}_\lambda^\vee(\mathbb{R})$. For all $\lambda \in \Phi^+$ we have that $\lambda(s_x)$ is bounded below by a nonzero constant depending on t . It follows that

$$\|\text{Ad}^\vee(s_x^{-1})\eta\|_{\mathfrak{n}} \gg_t \|\eta\|_{\mathfrak{n}}. \quad (9.33)$$

for $x \in \mathfrak{S}(t)$.

If $\eta \in \Omega_{\mathfrak{n}}^\infty \cap \mathfrak{n}(\mathbb{Q})$ then $\langle \eta, \cdot \rangle|_{\mathfrak{n}_\lambda}$ lies in a lattice in $\mathfrak{n}_\lambda^\vee(\mathbb{R})$ depending only on $\Omega_{\mathfrak{n}}^\infty$. In particular, regarded as an element of the real vector space $\mathfrak{n}_\lambda^\vee(\mathbb{R})$, the linear form $\langle \eta, \cdot \rangle|_{\mathfrak{n}_\lambda}$ is either zero or lies in the complement of an open neighborhood of zero depending only on $\Omega_{\mathfrak{n}}^\infty$. Thus if $\langle \eta, \cdot \rangle|_{\mathfrak{n}_\lambda} \neq 0$, then

$$\mathbb{1}_{\Omega_{\mathfrak{n}}^\infty}(\eta) \|\text{Ad}^\vee(s_x^{-1})\eta\|_{\mathfrak{n}} \gg \mathbb{1}_{\Omega_{\mathfrak{n}}^\infty}(\eta) \lambda(s_x) \quad (9.34)$$

Here the implied constant depends on t and $\Omega_{\mathfrak{n}}^\infty$. Combining (9.32), (9.33) and (9.34) we deduce the proposition. $\square$

Let Δ be the set of simple roots attached to the set of positive roots Φ^+ . For each simple root $\alpha \in \Delta$, let

$$\Phi_\alpha^+ \subset \Phi^+$$

be the set of roots of T in G of the form $\lambda = \sum_{\beta \in \Delta} m_\beta \beta$ with $m_\beta \geq 0$ for all $\beta \neq \alpha$ and $m_\alpha > 0$. Let

$$\mathfrak{n}(\Phi_\alpha^+) := \bigoplus_{\lambda \in \Phi_\alpha^+} \mathfrak{n}_\lambda. \quad (9.35)$$

It is clear that this is an ideal in $\mathfrak{n}$. Let $N(\Phi_\alpha^+) \trianglelefteq N$ be the connected normal subgroup of N with Lie algebra $\mathfrak{n}(\Phi_\alpha^+)$. It is the unipotent radical of a maximal parabolic subgroup of G containing P . In fact, if we take $P_0 = P$

in the notation of §1.9, then in the notation of Theorem 1.9.4 the maximal parabolic with unipotent radical $N(\Phi_\alpha^+)$ is $P_{\Delta-\{\alpha\}}$.

Lemma 9.4.6 *Let $\alpha \in \Delta$ and assume that $\langle \eta, \cdot \rangle$ is trivial on $\mathfrak{n}(\Phi_\alpha^+)(\mathbb{A}_\mathbb{Q})$. Then*

$$y \longmapsto \int_{\mathfrak{n}(\mathbb{A}_\mathbb{Q})} f(x^{-1} \exp(n)y) \psi(\langle \eta, n \rangle) dn$$

is invariant under multiplication by $N(\Phi_\alpha^+)(\mathbb{A}_\mathbb{Q})$ on the left.

Proof. Let $n \in \mathfrak{n}(\mathbb{A}_\mathbb{Q})$ and $u \in N(\Phi_\alpha^+)(\mathbb{A}_\mathbb{Q})$. Choose $n' \in \mathfrak{n}(\mathbb{A}_\mathbb{Q})$ such that $\exp(n') = u$. One has

$$\exp(n)u = \exp(n + w(n, n'))$$

for some $w(n, n') \in \mathfrak{n}(\mathbb{A}_\mathbb{Q})$ by the Baker-Campbell-Hausdorff formula. As mentioned above, $\mathfrak{n}(\Phi_\alpha^+)$ is an ideal in $\mathfrak{n}$. Combining this with the fact that $w(n, n')$ is given in terms of commutators of n and n' we deduce that $w(n, n') \in \mathfrak{n}(\Phi_\alpha^+)(\mathbb{A}_\mathbb{Q})$. Thus

$$\begin{aligned} & \int_{\mathfrak{n}(\mathbb{A}_\mathbb{Q})} f(x^{-1} \exp(n)uy) \psi(\langle \eta, n \rangle) dn \\ &= \int_{\mathfrak{n}(\mathbb{A}_\mathbb{Q})} f(x^{-1} \exp(n + w(n, n'))y) \psi(\langle \eta, n \rangle) dn \\ &= \int_{\mathfrak{n}(\mathbb{A}_\mathbb{Q})} f(x^{-1} \exp(n)y) \psi(\langle \eta, n - w(n, n') \rangle) d(n - w(n, n')) \\ &= \int_{\mathfrak{n}(\mathbb{A}_\mathbb{Q})} f(x^{-1} \exp(n)y) \psi(\langle \eta, n \rangle) dn. \end{aligned}$$

□

The main result of this section is the following proposition:

Proposition 9.4.7 *Let $\varphi \in L^2_{\text{cusp}}([G])$. Then for any $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_\mathbb{Q}))$, any $B \in \mathbb{R}_{>0}$, and any $\alpha \in \Delta$ one has that*

$$|R(f)\varphi(x)| \ll_{f,B,\mathfrak{S}(t)} \alpha(s_x)^{-B} \|\varphi\|_2$$

for $x \in \mathfrak{S}(t)$.

Combined with Lemma 9.2.1 and Lemma 9.3.3 this completes the proof of Theorem 9.1.2 in the number field case. We will complete the proof of Theorem 9.1.2 in the function field case in §9.5 below.

Proof. In view of (9.17) to prove the proposition it suffices to bound

$$\int_{A_G N(\mathbb{Q}) \backslash G(\mathbb{A}_\mathbb{Q})} \left(\sum_{\eta \in \mathfrak{n}(\mathbb{Q})} \int_{\mathfrak{n}(\mathbb{A}_\mathbb{Q})} f(x^{-1} \exp(n)y) \psi(\langle \eta, n \rangle) dn \right) \varphi(y) dy \quad (9.36)$$

by a constant times

$$\delta_P(s_x)\alpha(s_x)^{-B}\|\varphi\|_2. \quad (9.37)$$

Here and throughout the proof all constants are allowed to depend on f , B , and $\mathfrak{S}(t)$. Indeed, given (9.37) we deduce the bound in the proposition by increasing B and varying α to remove the factor of $\delta_P(s_x)$.

If $\eta \in \mathfrak{n}(\mathbb{Q})$ has the property that $\langle \eta, \cdot \rangle$ is identically zero on $\mathfrak{n}(\Phi_\alpha^+)(\mathbb{A}_\mathbb{Q})$ then

$$y \longmapsto \int_{\mathfrak{n}(\mathbb{A}_\mathbb{Q})} f(x^{-1} \exp(n)y) \psi(\langle \eta, n \rangle) dn$$

is left-invariant under $N(\Phi_\alpha^+)(\mathbb{A}_\mathbb{Q})$ by Lemma 9.4.6. Thus

$$\begin{aligned} & \int_{A_G N(\mathbb{Q}) \backslash G(\mathbb{A}_\mathbb{Q})} \left(\int_{\mathfrak{n}(\mathbb{A}_\mathbb{Q})} f(x^{-1} \exp(n)y) \psi(\langle \eta, n \rangle) dn \right) \varphi(y) dy \\ &= \int \left(\int_{N(\mathbb{Q}) \backslash N(\mathbb{Q})N(\Phi_\alpha^+)(\mathbb{A}_\mathbb{Q})} \left(\int_{\mathfrak{n}(\mathbb{A}_\mathbb{Q})} f(x^{-1} \exp(n)uy) \psi(\langle \eta, n \rangle) dn \right) \varphi(uy) \right) \frac{dy}{du} \\ &= \int \left(\int_{\mathfrak{n}(\mathbb{A}_\mathbb{Q})} f(x^{-1} \exp(n)y) \psi(\langle \eta, n \rangle) dn \int_{[N(\Phi_\alpha^+)]} \varphi(uy) du \right) \frac{dy}{du} \\ &= 0 \end{aligned}$$

since φ is cuspidal. Here the unmarked integrals are over

$$A_G N(\mathbb{Q}) N(\Phi_\alpha^+)(\mathbb{A}_\mathbb{Q}) \backslash G(\mathbb{A}_\mathbb{Q}).$$

We point out that we have used fact that $N(\Phi_\alpha^+)$ is normal in N . To check that our manipulations with the integrals above are justified we observe that

$$\int_{A_G N(\mathbb{Q}) \backslash G(\mathbb{A}_\mathbb{Q})} \left(\int_{\mathfrak{n}(\mathbb{A}_\mathbb{Q})} |f(x^{-1} \exp(n)y)| dn \right) |\varphi(y)| dy$$

is finite since f is compactly supported. Thus in (9.36) one can omit the summands indexed by those η for which $\langle \eta, \cdot \rangle$ is identically zero on $\mathfrak{n}(\Phi_\alpha^+)(\mathbb{A}_\mathbb{Q})$.

By choosing an appropriate measurable fundamental domain for $N(\mathbb{Q})$ acting on $A_G \backslash G(\mathbb{A}_\mathbb{Q})$ in (9.36) we may assume that the integral over y is supported in $A_G \backslash \Omega'_N M(\mathbb{A}_\mathbb{Q}) K$ for some compact subset $\Omega'_N \subset N(\mathbb{A}_\mathbb{Q})$ and some maximal compact subgroup $K \leq A_G \backslash G(\mathbb{A}_\mathbb{Q})$. Suppose that $\eta \in \mathfrak{n}(\mathbb{Q})$ has the property that $\langle \eta, \cdot \rangle$ is not identically zero on $\mathfrak{n}(\Phi_\alpha^+)(\mathbb{A}_\mathbb{Q})$. Then $\langle \eta, \cdot \rangle_{\mathfrak{n}_\lambda} \neq 0$ for some $\lambda \in \Phi_\alpha^+$. Since we have assumed $x \in \mathfrak{S}(t)$, $\alpha'(s_x)$ is bounded below by a constant depending on t for all $\alpha' \in \Delta$. Given this and the definition of Φ_α^+ , Proposition 9.4.5 implies there is a compact subset $\Omega_\mathfrak{n}^\infty \subset \mathfrak{n}(\mathbb{A}_\mathbb{Q}^\infty)$ such that for any $B > 0$ one has that

$$\int_{\mathfrak{n}(\mathbb{A}_{\mathbb{Q}})} f(x^{-1} \exp(n)y) \psi(\langle \eta, n \rangle) dn \ll \|\eta\|_{\mathfrak{n}}^{-B} \mathbb{1}_{\Omega_{\mathfrak{n}}^{\infty}}(\eta) \delta_P(s_x) \alpha(s_x)^{-B}. \quad (9.38)$$

Note that $\sum_{\eta \in \mathfrak{n}(\mathbb{Q})} \|\eta\|_{\mathfrak{n}}^{-B} \mathbb{1}_{\Omega_{\mathfrak{n}}^{\infty}}(\eta)$ is bounded for B sufficiently large. Thus for B sufficiently large, Lemma 9.4.4, (9.36), and (9.38) imply that

$$|R(f)\varphi(x)| \ll \delta_P(s_x) \alpha(s_x)^{-B} \int_{s_x \Omega_G} |\varphi(y)| dy$$

where $\Omega_G \subset A_G \backslash G(\mathbb{A}_{\mathbb{Q}})$ is a compact set. By the Hölder inequality this is bounded by

$$\begin{aligned} & \delta_P(s_x) \alpha(s_x)^{-B} \text{meas}_{dy}(s_x \Omega_G)^{1/2} \left(\int_{s_x \Omega_G} |\varphi(y)|^2 dy \right)^{1/2} \\ & \leq \delta_P(s_x) \alpha(s_x)^{-B} \text{meas}_{dy}(\Omega_G)^{1/2} \|\varphi\|_2. \end{aligned}$$

□

9.5 The function field case

We still owe the reader a proof of Theorem 9.1.2 when F is a function field. Thus we continue to assume that G is reductive, but now assume that F is a function field. Rather than prove (9.13) we will argue more directly. What makes this possible is that the asymptotic behavior of cuspidal automorphic forms is much simpler in the function field case than in the number field case:

Theorem 9.5.1 (Harder) *Let $\varphi \in L_{\text{cusp}}^2([G])$ be a function that is invariant under a compact open subgroup $K' \leq A_G \backslash G(\mathbb{A}_F)$. Then $\varphi(x) = 0$ for x outside a compact subset of $[G]$ that depends only on K' .*

We follow Harder's original proof [Har74] but generalize the argument from split to arbitrary reductive groups. Let P be a fixed minimal parabolic subgroup of G . If $P = G$ then the theorem is trivial so we assume $P \neq G$. Let N be the unipotent radical of P and let $T \leq P$ be a maximal split torus. Then $M := C_G(T)$ is a Levi subgroup of P . Let $Z_0 \leq Z_G$ be a maximal split torus and let $T_0 \leq G^{\text{der}}$ be a maximal split torus such that $T_0 Z_0 = T$. Choose a maximal compact subgroup $K \leq G(\mathbb{A}_F)$ such that the Iwasawa decomposition $P(\mathbb{A}_F)K = G(\mathbb{A}_F)$ holds. By replacing the group K' in the theorem with $K' \cap K$ we can assume $K' \leq K$. Fix compact subsets $\Omega_N \subset N(\mathbb{A}_F)$, $\Omega_M \subset M(\mathbb{A}_F)$, and $t \in \mathbb{R}_{>0}$ and let $\mathfrak{S}(t) = \Omega_N \Omega_M A_{T_0}(t)K$ be the corresponding Siegel set as in (2.25). We assume that Ω_N, Ω_M and K are chosen so that, for t sufficiently small, one has that

$$G(F)\mathfrak{S}(t) = G(\mathbb{A}_F)^1;$$

this is possible by Theorem 2.7.2.

Let Φ^+ be the set of positive roots of the split maximal torus $T \leq M$ acting on N and Δ the corresponding base. For each simple root $\alpha \in \Delta$, let

$$\Phi_\alpha^+ \subset \Phi^+$$

be the set of positive roots of T in G of the form $\lambda = \sum_{\beta \in \Delta} m_\beta \beta$ with $m_\beta \geq 0$ for all $\beta \neq \alpha$ and $m_\alpha > 0$. Define the Lie algebra $\mathfrak{n}(\Phi_\alpha^+)$ as in (9.35) and let $N(\Phi_\alpha^+) \trianglelefteq N$ be the unique smooth connected subgroup of N with Lie algebra $\mathfrak{n}(\Phi_\alpha^+)$ that is fixed under conjugation by T [CGP10, Proposition 3.3.6].

Lemma 9.5.2 *If $t > 0$ is chosen sufficiently large, then for any $h \in \mathfrak{S}(t)$ one has that*

$$N(\Phi_\alpha^+)(F)(N(\Phi_\alpha^+)(\mathbb{A}_F) \cap hK'h^{-1}) = N(\Phi_\alpha^+)(\mathbb{A}_F).$$

□

Proof. Write $h = nmsk = nsmk$ with

$$(n, m, s, k) \in \Omega_N \times \Omega_M \times A_{T_0}(t) \times K$$

for compact sets $\Omega_N \times \Omega_M \subset N(\mathbb{A}_F) \times M(\mathbb{A}_F)^1$. As (m, k) varies over $\Omega_M \times K$ the groups $mkK'(mk)^{-1}$ vary over a finite set of subgroups of $G(\mathbb{A}_F)^1$. Therefore we may assume $h = ns$ at the expense of allowing K' to be an arbitrary compact open subgroup of $G(\mathbb{A}_F)^1$.

We apply an inductive argument. Let $\Phi_{\alpha, \text{nd}}^+ \subset \Phi_\alpha^+$ denote the subset of nondivisible roots, that is, roots λ such that $\lambda/2$ is not a root. It is helpful to know that if α and $r\alpha$ are both roots for some $r \in \mathbb{Q}_{>0}$ then $r = 2$ or $r = \frac{1}{2}$ [Bor91, §21.7]. For $\lambda \in \Phi^+$ let N_λ denote the root group of (1.45). If $\lambda \in \Phi^+$ and $2\lambda \notin \Phi^+$ let $N_{(\lambda)} = N_\lambda$. If $\lambda \in \Phi^+$ and $2\lambda \in \Phi^+$, then by [CGP10, Proposition 3.3.6] there is a unique smooth connected subgroup $N_{(\lambda)}$ of N that is stable under the action of T with Lie algebra $\mathfrak{g}_\lambda + \mathfrak{g}_{2\lambda}$. There is a natural partial order on Φ^+ and hence on Φ_α^+ . Extend it to a total order $\lambda_1 \leq \dots \leq \lambda_j$ on Φ_α^+ .

Consider the product map

$$\prod N_{(\lambda)} \longrightarrow N \tag{9.39}$$

where the product is over $\lambda \in \Phi_\alpha^+$ with $\lambda \geq \lambda_i$ that are either nondivisible, or are divisible and satisfy $\lambda/2 < \lambda_i$. The map (9.39) is an isomorphism onto its image N_i , a unipotent subgroup of N , and $N(\Phi_\alpha^+) = N_1$ by [Bor91, Proposition 21.9]. By loc. cit. the image is independent of the ordering of the subgroups $N_{(\lambda)}$ in the product. Each of the groups N_i is isomorphic to $\mathbb{G}_a^{d_i}$ for some integer d_i by [Bor91, §21.10] and they are all normal in N (this follows from [CGP10, Proposition 3.3.5]).

For $1 \leq i \leq j$ suppose

$$N_{i+1}(F)(N_{i+1}(\mathbb{A}_F) \cap nsKs^{-1}n^{-1}) = N_{i+1}(\mathbb{A}_F)$$

for all $n \in \Omega_N$ and $s \in A_{T_0}(t)$ for t sufficiently large. Here when $i = j$ we take N_{j+1} to be the trivial group, so the statement is vacuous.

It follows from [CGP10, Proposition 3.3.5] that the adjoint action of N on $N_i/N_{i+1} \cong N_{\lambda_i}$ is trivial. Thus

$$\begin{aligned} N_i(F)(N_i(\mathbb{A}_F) \cap nsKs^{-1}n^{-1})/N_{i+1}(\mathbb{A}_F) \\ = N_i(F)(N_i(\mathbb{A}_F) \cap sKs^{-1})/N_{i+1}(\mathbb{A}_F). \end{aligned}$$

Now if $\Omega \subset \mathbb{A}_F$ is a compact open set and $a \in A_{\mathbb{G}_m}(t)$ with t sufficiently large then

$$F + a\Omega = \mathbb{A}_F.$$

On the other hand $N_i/N_{i+1} \cong N_{\lambda_i}$ is isomorphic to $\mathbb{G}_a^d$ for some d [CGP10, Lemma 3.3.8]. Thus if t is large enough then

$$N_i(F)(N_i(\mathbb{A}_F) \cap sKs^{-1})/N_{i+1}(\mathbb{A}_F) = N_i(\mathbb{A}_F)/N_{i+1}(\mathbb{A}_F)$$

completing the induction. $\square$

We point out that the number field analogue of the assertion of Lemma 9.5.2 is false. See Exercise 9.9 below.

Proof of Theorem 9.5.1: Let $\varphi \in L_{\text{cusp}}^2([G])^{K'}$. By Lemma 2.6.2, $A_G G(\mathbb{A}_F)^1$ is a finite index subgroup of $G(\mathbb{A}_F)$. Choose a set of representatives $g_1, \dots, g_j$ for $A_G G(\mathbb{A}_F)^1 \backslash G(\mathbb{A}_F)$. Then upon replacing K' by a finite index subgroup if necessary, we can assume that the functions

$$x \mapsto \varphi(xg_i)$$

all lie in $L_{\text{cusp}}^2([G])^{K'}$.

Choose t so that the conclusion of Lemma 9.5.2 holds. For $h \in \mathfrak{S}(t)$ the functions

$$\begin{aligned} N(\Phi_\alpha^+)(\mathbb{A}_F) &\longrightarrow \mathbb{C} \\ n &\longmapsto \varphi(nhg_i) \end{aligned}$$

are left invariant under $N(\Phi_\alpha^+)(F)$ and right invariant under $N(\Phi_\alpha^+)(\mathbb{A}_F) \cap hK'h^{-1}$. Thus by Lemma 9.5.2 they are constant. On the other hand, $N(\Phi_\alpha^+)$ is the unipotent radical of a proper parabolic subgroup of G and we obtain

$$0 = \int_{[N(\Phi_\alpha^+)]} \varphi(nhg_i) dn = \text{meas}_{dn}([N(\Phi_\alpha^+)]) \varphi(hg_i).$$

Thus φ vanishes on $\mathfrak{S}(t)g_i$ for sufficiently large t for each i . Hence the support of φ lies in the relatively compact subset

$$\bigcup_{i=1}^j A_G G(F)(\mathfrak{S}(t_0) - \mathfrak{S}(t))g_i$$

of $[G]$ for any t_0 small enough that $G(F)\mathfrak{S}(t_0) = G(\mathbb{A}_F)^1$. $\square$

Using Theorem 9.5.1 we prove Theorem 9.1.2 in the function field case:

Proof of Theorem 9.1.2 (for function fields): By Lemma 9.2.1 the cuspidal subspace $L^2_{\text{cusp}}([G])$ is closed in $L^2([G])$. Let $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$. We must show $R_{\text{cusp}}(f)$ is of trace class. As explained in more detail in §16.1, the operator

$$R(f) : L^2([G]) \longrightarrow L^2([G])$$

is represented by the kernel

$$K_f(x, y) := \sum_{\gamma \in G(F)} f(x^{-1}\gamma y).$$

It is not hard to see that this function is continuous as a function of $(x, y) \in [G] \times [G]$ (for more details see §16.1). Thus for any compact subset $\Omega \subseteq [G]$ the restriction of $K_f(x, y)$ to $\Omega \times \Omega$ is in $L^2(\Omega \times \Omega)$, and hence is a Hilbert-Schmidt operator.

Now there is a compact open subgroup $K \leq A_G \backslash G(\mathbb{A}_F)$ such that $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F) // K)$. Thus for any $\varphi \in L^2_{\text{cusp}}([G])$ such that $R(f)\varphi \neq 0$ one has $\varphi \in L^2_{\text{cusp}}([G])^K$. Using Theorem 9.5.1 choose a compact subset $\Omega \subset [G]$ such that any element of $L^2_{\text{cusp}}([G])^K$ is supported in Ω . In other words,

$$R_{\text{cusp}}(f) = R(f)|_{L^2(\Omega) \cap L^2_{\text{cusp}}([G])}.$$

Now $R(f)|_{L^2(\Omega)}$ is Hilbert-Schmidt, so its restriction to the closed subspace $L^2(\Omega) \cap L^2_{\text{cusp}}([G])$ of $L^2(\Omega)$ is also Hilbert-Schmidt (see Exercise 9.5).

Thus we have proven that

$$R_{\text{cusp}}(f) : L^2_{\text{cusp}}([G]) \longrightarrow L^2_{\text{cusp}}([G])$$

is Hilbert-Schmidt. On the other hand, every $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ is a finite linear combination of convolutions: $f = \sum_{i=1}^k h_{i1} * h_{i2}$ where $h_{ij} \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ (see Exercise 9.8). We deduce that

$$R_{\text{cusp}}(f) = \sum_{i=1}^k R(h_{i1}) * R(h_{i2}).$$

This says that $R_{\text{cusp}}(f)$ is a finite sum of convolutions of two Hilbert-Schmidt operators, and hence is of trace class (see Exercise 9.2). $\square$

9.6 Rapidly decreasing functions

We now assume that F is a number field. We fix a Siegel set $\mathfrak{S}(t)$ as in (9.23). For $x \in \mathfrak{S}(t)$ we define $s_x \in A_{T_0}(t)$ as in (9.19). The estimate of Proposition 9.4.7 for cuspidal functions motivates the following definition:

Definition 9.6.1. A function

$$\varphi : [G] \longrightarrow \mathbb{C}$$

is **rapidly decreasing** if for any Siegel set as above and all $r \in \mathbb{R}_{>1}$ there is a constant $c \in \mathbb{R}_{>0}$ (depending on $\mathfrak{S}(t)$ and r) such that one has

$$|\varphi(x)| \leq c\alpha(s_x)^{-r}$$

for all $x \in \mathfrak{S}(t)$ and $\alpha \in \Delta$.

Observe that in Definition 9.6.1 the exponent of $\alpha(s_x)$ can be taken to be arbitrarily small, whereas for a moderate growth function (using the definition of moderate growth provided by Lemma 6.3.2) the exponent of $\alpha(s_x)$ is allowed to be any fixed real number. Rapidly decreasing functions go to zero faster than the inverse of any polynomial as one goes to infinity in a Siegel set whereas functions of moderate growth have at most polynomial growth in a Siegel set.

We can also phrase the condition of rapid decrease in terms of the norms introduced in (6.8) above. Using the same argument proving Lemma 6.3.2 one obtains the following lemma:

Lemma 9.6.2 *A function $\varphi : [G] \rightarrow \mathbb{C}$ is rapidly decreasing if and only if for all $r \in \mathbb{R}_{>1}$ there is a constant $c \in \mathbb{R}_{>0}$ depending on r such that*

$$|\varphi(x)| \leq c\|x\|^{-r}$$

for all $x \in G(\mathbb{A}_F)^1$. □

In this lemma we restrict to $x \in G(\mathbb{A}_F)^1$ since φ is invariant under A_G . We recall that the norm $\|g\|$ depends on the choice of a closed immersion $\iota : G \rightarrow \mathrm{SL}_{2n}$ as in (6.3). The notion of rapid decrease is independent of the choice of ι , but different ι may lead to different constants c .

Later in the proof of Proposition 14.3.3 we will require the following mild strengthening of the notion of rapid decrease:

Lemma 9.6.3 *Assume that $\varphi : [G] \rightarrow \mathbb{C}$ is a rapidly decreasing function and that $\Omega_G \subset G(\mathbb{A}_F)$ is a compact set. Then for any Siegel set $\mathfrak{S}(t)$ as above and all $r \in \mathbb{R}_{>1}$ there is a constant $c \in \mathbb{R}_{>0}$ (depending on Ω_G , $\mathfrak{S}(t)$, and r) such that one has*

$$|\varphi(xg)| \leq c\alpha(s_x)^{-r}$$

for all $(x, g) \in \mathfrak{S}(t) \times \Omega_G$ and $\alpha \in \Delta$.

Proof. This follows from Lemma 9.6.2, Exercise 6.2, and (6.9). $\square$

Proposition 9.4.7 implies the following theorem:

Theorem 9.6.4 *If $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ and $\varphi \in L_{\text{cusp}}^2([G])$ then $R(f)\varphi$ is rapidly decreasing.* $\square$

Corollary 9.6.5 *Any smooth $\varphi \in L_{\text{cusp}}^2([G])$ is rapidly decreasing.*

Here when we say that φ is smooth we mean that there is a compact open subgroup $K \leq G(\mathbb{A}_F^\infty)$ such that $\varphi \in L_{\text{cusp}}^2([G])^K$, and φ lies in the set of vectors in $L_{\text{cusp}}^2([G])^K$ that are smooth with respect to the action of $G(F_\infty)$ in the sense of §4.2.

Proof. It suffices to show that such a φ can be written as a finite sum

$$\varphi = \sum_i R(f_i)\varphi_i$$

for some $f_i \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ and $\varphi_i \in L_{\text{cusp}}^2([G])$. This follows from Theorem 4.2.8. $\square$

9.7 Cuspidal automorphic forms

We assume again that G is a reductive group over a global field F and let $K = K_\infty K^\infty \leq G(\mathbb{A}_F)$ be a maximal compact subgroup. We use it to define the space of automorphic forms $\mathcal{A} = \mathcal{A}(G)$ as in definitions 6.3.3 and 6.4.1. In the number field case the definition only depends on K_∞ , not K^∞ . In the function field case it is independent of the choice of K . In this section we prove Theorem 6.5.3, which constructs a natural bijection between isomorphism classes of discrete cuspidal automorphic representations in the sense of §3.7 and cuspidal automorphic representations in the sense of §6.5.

To prove this we first state the following result:

Theorem 9.7.1 *An element of $\mathcal{A}_{\text{cusp}}^{A_G}$ is compactly supported in the function field case and rapidly decreasing in the number field case.* $\square$

In the function field case this follows from Theorem 9.5.1 and for the number field case we refer to [MW95, §1.2.12].

We also require the following lemma:

Lemma 9.7.2 *Suppose that (π, V) is an irreducible unitary representation of $A_G \backslash G(\mathbb{A}_F)$ on a Hilbert space V . The subspace $V_{\text{fin}} < V$ of K -finite vectors is dense. For every $\varphi \in V_{\text{fin}}$, there is an $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ such that $\pi(f)\varphi = \varphi$.*

Proof. If F is a function field, then the density of V_{fin} in V follows from Lemma 5.3.7. In this case every vector $\varphi \in V_{\text{fin}}$ is fixed by a compact open subgroup $K' \leq K$, so $\pi(1_{K'})\varphi = \varphi$.

If F is a number field then by Theorem 6.6.1 we can write $V \cong V_{\infty} \widehat{\otimes} V^{\infty}$ where V_{∞} is an irreducible unitary representation of $A_G \backslash G(F_{\infty})$ and V^{∞} is an irreducible unitary representation of $G(\mathbb{A}_F^{\infty})$. By Theorem 6.6.2 we can then write $V_{\text{fin}} = V_{\infty \text{fin}} \otimes V_{\text{fin}}^{\infty}$ where $V_{\infty \text{fin}} < V_{\infty}$ and $V_{\text{fin}}^{\infty} < V^{\infty}$ are the spaces of K_{∞} -finite and K^{∞} -finite vectors, respectively. Thus the density assertion is implied by Proposition 4.4.7 and Lemma 5.3.7. The fact that every element in V_{fin} is of the form $\pi(f)\varphi$ for some $\varphi \in V$ is implied by the corresponding statements for V_{∞} and V^{∞} . For V^{∞} one proceeds as in the function field case. For V_{∞} we use Proposition 4.5.6. $\square$

We restate Theorem 6.5.3 for the convenience of the reader:

Theorem 9.7.3 *The space of cuspidal automorphic forms $\mathcal{A}_{\text{cusp}}^{A_G}$ is a dense subspace of $L_{\text{cusp}}^2([G])$. If (π, V) is a discrete cuspidal automorphic representation, then the space of K -finite vectors V_{fin} in V is a cuspidal automorphic representation (π, V_{fin}) , and*

$$\mathcal{A}_{\text{cusp}}^{A_G}(\pi) = L_{\text{cusp}}^2(\pi)_{\text{fin}}, \quad (9.40)$$

the space of K -finite vectors in $L_{\text{cusp}}^2(\pi)$. The multiplicity of (π, V_{fin}) in $\mathcal{A}_{\text{cusp}}^{A_G}(\pi)$ is finite and equal to the multiplicity of (π, V) in $L_{\text{cusp}}^2(\pi)$. One has

$$\mathcal{A}_{\text{cusp}}^{A_G} = \bigoplus_{\pi} \mathcal{A}_{\text{cusp}}^{A_G}(\pi) \quad (9.41)$$

where the (algebraic) sum is over isomorphism classes of cuspidal automorphic representations. The association of (π, V_{fin}) to (π, V) defines a bijection between isomorphism classes of discrete cuspidal automorphic representations and isomorphism classes of cuspidal automorphic representations.

In the theorem we have referred to a (discrete) cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ simply as a (discrete) cuspidal automorphic representation. We will continue this practice in the proof.

Proof. By Theorem 9.7.1 elements of $\mathcal{A}_{\text{cusp}}^{A_G}$ are bounded. Since $[G]$ has finite measure we deduce that $\mathcal{A}_{\text{cusp}}^{A_G} \leq L_{\text{cusp}}^2([G])$.

Let (π, V) be a discrete cuspidal automorphic representation. The admissible representation (π, V_{fin}) is an automorphic representation and the isomorphism types of (π, V) and (π, V_{fin}) determine each other by Theorem 6.6.4.

Let π be a cuspidal automorphic representation realized as a subspace (not just subquotient) $W \leq \mathcal{A}_{\text{cusp}}^{A_G}$. Then by Corollary 9.1.3 there is a discrete cuspidal automorphic representation σ of $A_G \backslash G(\mathbb{A}_F)$ such that the projection

$$W \longrightarrow L_{\text{cusp}}^2(\sigma)$$

is nonzero. By our comments in the previous paragraph this implies $\pi = \sigma$ and $W \leq L_{\text{cusp}}^2(\pi)$. Thus

$$\mathcal{A}_{\text{cusp}}^{A_G}(\pi) = \mathcal{A}_{\text{cusp}}^{A_G} \cap L_{\text{cusp}}^2(\pi). \quad (9.42)$$

Let us now prove (9.40). By (9.42) we have $\mathcal{A}_{\text{cusp}}^{A_G}(\pi) \leq L_{\text{cusp}}^2(\pi)_{\text{fin}}$. By Corollary 9.1.3 $L_{\text{cusp}}^2(\pi)$ is a finite direct sum of irreducible representations, all isomorphic to (π, V) . Thus by the proof of Theorem 6.6.4, $L_{\text{cusp}}^2(\pi)_{\text{fin}}$ is contained in $\mathcal{A}^{A_G}(\pi)$. For every $\varphi \in L_{\text{cusp}}^2(\pi)_{\text{fin}}$ we can find an $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ such that $\pi(f)\varphi = \varphi$ by Lemma 9.7.2, so any function $\varphi \in L_{\text{cusp}}^2(\pi)_{\text{fin}}$ is cuspidal in the sense of Definition 6.5.1 by Lemma 9.2.3. Thus (9.40) is valid. The equality (9.40) and the comments in the previous paragraph together imply the statement on equality of multiplicities following (9.40). We moreover deduce the density of $\mathcal{A}_{\text{cusp}}^{A_G}$ in $L_{\text{cusp}}^2([G])$ from (9.40), Corollary 9.1.3 and Lemma 9.7.2.

Assume F is a function field. Then $\mathcal{A}_{\text{cusp}}^{A_G} = \bigcup_{K'} (\mathcal{A}_{\text{cusp}}^{A_G})^{K'}$, where the union is over all compact open subgroups $K' \leq G(\mathbb{A}_F)$. On the other hand, by Theorem 9.5.1, for all compact open subgroups $K' \leq G(\mathbb{A}_F)$ there is a compact subset $\Omega \leq [G]$ such that any element of

$$L_{\text{cusp}}^2([G])^{K'}$$

is supported on Ω . We may take Ω to be K' -invariant, and then Ω/K' is a finite set. It follows that $L_{\text{cusp}}^2([G])^{K'}$ is finite dimensional, and hence $L_{\text{cusp}}^2(\pi)^{K'} = 0$ for all but finitely many π . Thus we have

$$(\mathcal{A}_{\text{cusp}}^{A_G})^{K'} \leq \bigoplus_{\pi} L_{\text{cusp}}^2(\pi)^{K'} \leq \bigoplus_{\pi} \mathcal{A}_{\text{cusp}}^{A_G}(\pi)^{K'}.$$

Upon varying K' we deduce (9.41) in the function field case.

Assume F is a number field. For each finite set Ξ of K_∞ -types, compact open subgroup $K'^\infty \leq G(\mathbb{A}_F^\infty)$ and ideal $J \leq Z(\mathfrak{g})$ with $\dim_{\mathbb{C}} Z(\mathfrak{g})/J < \infty$ let

$$\begin{aligned} \mathcal{A}_{\text{cusp}}^{A_G}(J, \Xi)^{K'^\infty} &= \mathcal{A}_{\text{cusp}}^{A_G} \cap \bigoplus_{\sigma \in \Xi} \mathcal{A}(J, \sigma)^{K'^\infty}, \\ \mathcal{A}_{\text{cusp}}^{A_G}(\pi)(J, \Xi)^{K'^\infty} &= \mathcal{A}_{\text{cusp}}^{A_G}(\pi) \cap \bigoplus_{\sigma \in \Xi} \mathcal{A}(J, \sigma)^{K'^\infty}. \end{aligned}$$

Here we are using notation introduced above (6.16). Then

$$\mathcal{A}_{\text{cusp}}^{A_G} = \bigcup_{J, \Xi, K'^\infty} \mathcal{A}_{\text{cusp}}^{A_G}(J, \Xi)^{K'^\infty} \quad (9.43)$$

where the union is over all J, Ξ, K'^∞ as above. For any closed subspace $V \subset L^2([G])$ let

$$V(\Xi) = \bigoplus_{\sigma \in \Xi} V(\sigma)$$

and let $V(J, \Xi)^{K'^\infty}$ be the subspace of $V(\Xi)^{K'^\infty_{\text{sm}}}$ annihilated by J . Elements of $L^2_{\text{cusp}}([G])(J, \Xi)^{K'^\infty}$ are of moderate growth by Lemma 4.4.13 and they are cuspidal in the sense of Definition 6.5.1 by the same argument used earlier in the proof. Hence

$$L^2_{\text{cusp}}([G])(J, \Xi)^{K'^\infty} \leq \mathcal{A}_{\text{cusp}}^{A_G}(J, \Xi)^{K'^\infty}.$$

The space on the right is finite dimensional by Theorem 6.3.4. This implies that

$$L^2_{\text{cusp}}([G])(J, \Xi)^{K'^\infty} = \bigoplus L^2_{\text{cusp}}(\pi)(J, \Xi)^{K'^\infty} \leq \bigoplus \mathcal{A}_{\text{cusp}}^{A_G}(\pi)(J, \Xi)^{K'^\infty}.$$

Combining this with (9.43) we deduce (9.41) in the number field case.

The final assertion on the bijection follows from what we have already proved. $\square$

Exercises

9.1. Prove that the Hilbert-Schmidt norm of a Hilbert-Schmidt operator is independent of the choice of basis. Show that the trace norm and trace of a trace-class operator are independent of the choice of basis as well.

9.2. Prove that if A and B are Hilbert-Schmidt operators then $A \circ B$ is of trace class.

9.3. Prove that an operator of trace class is Hilbert-Schmidt and a Hilbert-Schmidt operator is compact.

9.4. Let (Y, μ) be a σ -finite measurable space. Then $L^2(Y, d\mu)$ is a separable Hilbert space. Let $K(x, y) \in L^2(Y \times Y, d\mu \times d\mu)$. Prove that the operator

$$L^2(Y, d\mu) \longrightarrow L^2(Y, d\mu) \\ \varphi \longmapsto \left(x \mapsto \int_Y K(x, y) \varphi(y) d\mu(y) \right)$$

is Hilbert-Schmidt.

9.5. Let V be a Hilbert space and let $A : V \rightarrow V$ be a Hilbert-Schmidt operator. Prove that the restriction of A to a closed subspace $W \leq V$ is also a Hilbert-Schmidt operator.

9.6. Let G be a reductive group over a global field F and let $H \leq G$ be a subgroup such that $H(F)(A_G \cap H(\mathbb{A}_F)) \backslash H(\mathbb{A}_F)$ is compact. Prove that $(A_G \cap H(\mathbb{A}_F)) \backslash H(\mathbb{A}_F)$ is unimodular.

9.7. Prove that a Dirac sequence on $A_G \backslash G(\mathbb{A}_F)$ exists.

9.8. Using Theorem 4.2.9 prove that every $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ can be written as

$$f = \sum_{j=1}^r f_{1j} * f_{2j} \quad (9.44)$$

for some integer r and some $f_{ij} \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$.

9.9. Let G be a reductive group over $\mathbb{Q}$ and let $N \leq G$ be a nontrivial unipotent subgroup. Prove that $N(\mathbb{R}) \cap K_\infty$ is trivial for all compact subgroups $K_\infty \leq G(\mathbb{R})$. Conclude that for all compact subgroups $K \leq G(\mathbb{A}_\mathbb{Q})$ one has

$$N(\mathbb{Q})(N(\mathbb{A}_\mathbb{Q}) \cap K) \subsetneq N(\mathbb{A}_\mathbb{Q}),$$

that is, the inclusion is proper.

9.10. For $n \geq 1$, let $J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. For $\mathbb{Q}$ -algebras R , let

$$\mathrm{Sp}_{2n}(R) = \{g \in \mathrm{GL}_{2n}(R) : Jg^{-t}J^{-1} = g\}.$$

Let $\mathbb{H}_n$ be the set of $n \times n$ symmetric complex matrices $Z = X + iY$ whose imaginary part Y is positive definite (this is called **Siegel's upper-half space** of degree n). The group $\mathrm{Sp}_{2n}(\mathbb{R})$ acts on $\mathbb{H}_n$ via

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}.$$

Here A, B, C, D are $n \times n$ matrices.

- (a) Show that $\mathrm{Sp}_{2n}(\mathbb{R})$ acts transitively on $\mathbb{H}_n$.
- (b) Choose a Borel subgroup $B < \mathrm{Sp}_{2n}$ and compute the image of the associated Siegel set $\mathfrak{S}(t)$ under

$$\mathrm{Sp}_{2n}(\mathbb{A}_\mathbb{Q}) \longrightarrow \mathrm{Sp}_{2n}(\mathbb{R}) \longrightarrow \mathbb{H}_n$$

for $n = 1$ and $n = 2$.

Chapter 10

Eisenstein Series



Langlands' calculation of the constant terms of Eisenstein series was one of the most influential computations of the 20th century.

E. Lapid

In this chapter we survey the theory of Eisenstein series. Our main goal is to state Langlands' decomposition of $L^2([G])$.

10.1 Induced representations

In §8.1 we mentioned Harish-Chandra's philosophy of cusp forms [HC70a]. It is a slogan for the assertion that the irreducible representations of a reductive group can all be obtained as subquotients of the parabolic inductions of cuspidal representations on Levi subgroups. This philosophy is in part based on the foundational results of Langlands [Lan76] which we survey in this chapter. We largely follow [Art05, §7] in our exposition. We refer the reader to the standard reference [MW95] for proofs of many statements. Langlands' original proofs are notoriously difficult. Happily, Bernstein and Lapid [BL19] found a much simpler proof of the analytic properties of Eisenstein series given in Theorem 10.3.2. Based on this proof a simpler derivation of the spectral decomposition of $L^2([G])$ given in Theorem 10.5.1 has been obtained by Delorme [Del20].

We begin by setting notation for the induction of automorphic representations from Levi subgroups of G to G itself. Intertwining operators involving such representations are discussed in §10.2. Eisenstein series, defined in §10.3, allow one to prove that these induced representations are again automorphic,

and in §10.5 we use Eisenstein series to state Langlands' decomposition of $L^2([G])$. In the case of GL_n all of this theory simplifies significantly. In particular, one can use it to define an analogue of direct sums in the category of automorphic representations of general linear groups. This is discussed in §10.6 and §10.7. We end the chapter with a statement, in §10.8, of Mœglin and Waldspurger's foundational result which precisely describes the discrete spectrum of $L^2([\mathrm{GL}_n])$ in terms of cuspidal representations of smaller general linear groups.

Let G be a reductive group over a global field F with minimal parabolic subgroup $P_0 \leq G$. As usual, parabolic subgroups containing P_0 are said to be **standard**. We fix a Levi subgroup $M_0 \leq P_0$. Then $P_0 = M_0 N_0$ where $N_0 \leq P_0$ is the unipotent radical. By Proposition 1.9.7 there is then a unique Levi decomposition

$$P = MN$$

for each standard parabolic subgroup P such that M contains M_0 ; we always use this Levi decomposition. We can choose a maximal compact subgroup $K \leq G(\mathbb{A}_F)$ in good position with respect to all (P, M) in the sense of §B.1. Indeed, this follows upon combining Theorem B.2.5, Proposition B.3.1, Theorem B.4.1, and Theorem B.4.3. We normalize the Haar measures dg on $G(\mathbb{A}_F)$, dk on K , and $d_\ell p$ on $P(\mathbb{A}_F)$ so that

$$dg = d_\ell p dk$$

(see Exercise 3.5). Let $\mathfrak{a}_M := \mathrm{Hom}(X^*(M), \mathbb{R})$. Generalizing (9.20), we have a map

$$H_M : M(\mathbb{A}_F) \longrightarrow \mathfrak{a}_M \tag{10.1}$$

defined by

$$\langle H_M(m), \lambda \rangle = \log |\lambda(m)|$$

for $\lambda \in X^*(M)$. We observe that H_M is trivial on $K \cap M(\mathbb{A}_F)$. Thus using the Iwasawa decomposition $G(\mathbb{A}_F) = N(\mathbb{A}_F)M(\mathbb{A}_F)K$, we can define a morphism

$$\begin{aligned} H_P : G(\mathbb{A}_F) &\longrightarrow \mathfrak{a}_M \\ nmk &\longmapsto H_M(m) \end{aligned}$$

for $(n, m, k) \in N(\mathbb{A}_F) \times M(\mathbb{A}_F) \times K$.

The **discrete spectrum** $L^2_{\mathrm{disc}}([M]) \leq L^2([M])$ is the largest closed subspace of $L^2([M])$ that decomposes as a Hilbert space direct sum of irreducible subrepresentations. We denote by R_{disc} the restriction of the regular representation R of $L^2([M])$ to this subspace. Let (σ, V) be an $A_M \backslash M(\mathbb{A}_F)$ -subrepresentation of $L^2_{\mathrm{disc}}([M]) \leq L^2([M])$.

For

$$\lambda \in \mathfrak{a}_{M\mathbb{C}}^* := \mathrm{Hom}(\mathfrak{a}_M, \mathbb{C}) = X^*(M) \otimes_{\mathbb{Z}} \mathbb{C},$$

we then form the global induced representation

$$(I(\sigma, \lambda), \text{Ind}_P^G(V)).$$

Here $\text{Ind}_P^G(V)$ is the space of measurable functions

$$\varphi : N(\mathbb{A}_F)A_MM(F)\backslash G(\mathbb{A}_F) \longrightarrow \mathbb{C}$$

such that for all $x \in G(\mathbb{A}_F)$ the function

$$m \longmapsto \varphi(mx)$$

lies in $V \leq L_{\text{disc}}^2([M])$ and such that

$$\|\varphi\|^2 = \int_K \int_{A_MM(F)\backslash M(\mathbb{A}_F)} |\varphi(mk)|^2 dmdk < \infty.$$

Thus $\text{Ind}_P^G(V)$ is a Hilbert space. The action is given by

$$I(\sigma, \lambda)(g)\varphi(x) = \varphi(xg)e^{\langle H_P(xg), \lambda + \rho_P \rangle} e^{-\langle H_P(x), \lambda + \rho_P \rangle},$$

where $\rho_P \in \mathfrak{a}_M^*$ is the half-sum of the positive roots of a maximal split torus of P .

Suppose now that σ is irreducible, so it is an automorphic representation of $A_M \backslash M(\mathbb{A}_F)$. Since

$$e^{\langle H_M(m), \lambda \rangle} = \prod_v e^{\langle H_M(m_v), \lambda \rangle}$$

for $m \in M(\mathbb{A}_F)$, we have a factorization

$$I(\sigma, \lambda) \cong \otimes'_v I(\sigma_v, \lambda),$$

where the local factors $I(\sigma_v, \lambda)$ are defined as in §4.9 in the Archimedean setting and §8.1 in the non-Archimedean setting. There is a slight difference between our conventions in the global setting and in the local setting in terms of the spaces on which $I(\sigma, \lambda)$ acts. In the global setting, we are normalizing so that the space on which $I(\sigma, \lambda)$ acts is independent of λ . In the local setting, we incorporated λ into the definition of the functions themselves. The two definitions only differ by the character $e^{-\langle H_P(x_v), \lambda + \rho_P \rangle}$ at a place v .

10.2 Intertwining operators

We now need to discuss the relationships between various parabolic subgroups. As in the previous section, we write $P_0 = M_0N_0$ for a minimal parabolic subgroup of G with Levi subgroup M_0 and unipotent radical N_0 . We let P and P' be standard parabolic subgroups (with respect to P_0) and

let

$$P = MN \quad \text{and} \quad P' = M'N'$$

be the unique Levi decompositions such that M and M' contain M_0 . Assume for the moment that

$$P' \leq P$$

so that $M' \leq M$. The inclusions $Z_M \leq Z_{M'} \leq M' \leq M$ induce restriction morphisms

$$X^*(M) \longrightarrow X^*(M') \quad \text{and} \quad X^*(Z_{M'}^t) \longrightarrow X^*(Z_M^t). \quad (10.2)$$

Here for any commutative group Z we denote by Z^t the maximal torus of Z . For a reductive group H over F let

$$\mathfrak{a}_H^* := X^*(H) \otimes_{\mathbb{Z}} \mathbb{R}. \quad (10.3)$$

Thus $\mathfrak{a}_H^* \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{a}_{H\mathbb{C}}^*$. Moreover we have an isomorphism $\mathfrak{a}_H^* \xrightarrow{\sim} \mathfrak{a}_{Z_H^t}^*$ by Exercise 10.1. Thus (10.2) induces an injection and a surjection

$$\mathfrak{a}_M^* \hookrightarrow \mathfrak{a}_{M'}^* \longrightarrow \mathfrak{a}_M^*$$

such that the composite is an isomorphism. Similarly, by duality we obtain

$$\mathfrak{a}_M \hookrightarrow \mathfrak{a}_{M'} \longrightarrow \mathfrak{a}_M, \quad (10.4)$$

where again the composite is an isomorphism. We often identify $\mathfrak{a}_M^*$ and $\mathfrak{a}_M$ with their images in $\mathfrak{a}_{M'}^*$ and $\mathfrak{a}_{M'}$, respectively. We write

$$\mathfrak{a}_{M'}^M := \ker(\mathfrak{a}_{M'} \longrightarrow \mathfrak{a}_M) \quad (10.5)$$

and $(\mathfrak{a}_{M'}^M)^*$ for its dual in $\mathfrak{a}_{M'}^*$. Then the considerations above imply

$$\mathfrak{a}_{M'}^* = \mathfrak{a}_M^* \oplus (\mathfrak{a}_{M'}^M)^* \quad \text{and} \quad \mathfrak{a}_{M'} = \mathfrak{a}_M \oplus \mathfrak{a}_{M'}^M. \quad (10.6)$$

Example 10.1. Let G be GL_3 , let M_0 be the maximal torus of diagonal matrices, and let $B = M_0N_0$ be the Borel subgroup of upper triangular matrices. Take $P' = B$ and let P be the standard parabolic subgroup of type $(2, 1)$. In this case $Z_{M'} = M' = M_0$ and

$$Z_M(R) := \left\{ \begin{pmatrix} aI_2 & \\ & b \end{pmatrix} : a, b \in R^\times \right\}.$$

One has an isomorphism $\mathbb{Z}^3 \xrightarrow{\sim} X^*(M')$ as in (1.39) and an isomorphism

$$\begin{aligned} \mathbb{Z}^2 &\xrightarrow{\sim} X^*(M) \\ (k_1, k_2) &\longmapsto \left(\begin{pmatrix} g & \\ & b \end{pmatrix} \mapsto (\det g)^{k_1} b^{k_2} \right). \end{aligned}$$

These isomorphisms induce a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}^3 \\ \downarrow \sim & & \downarrow \sim \\ X^*(M) & \longrightarrow & X^*(M'), \end{array}$$

where the top arrow is given by $(k_1, k_2) \mapsto (k_1, k_1, k_2)$ and the bottom arrow is given by restriction. Similarly one has an isomorphism

$$\begin{aligned} \mathbb{Z}^2 &\xrightarrow{\sim} X^*(Z_M) \\ (k_1, k_2) &\mapsto \left(\begin{pmatrix} aI_2 & \\ & b \end{pmatrix} \mapsto a^{k_1} b^{k_2} \right). \end{aligned}$$

It induces a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}^2 \\ \downarrow \sim & & \downarrow \sim \\ X^*(Z_{M'}) & \longrightarrow & X^*(Z_M), \end{array}$$

where the bottom arrow is again given by restriction and the top arrow is given by $(k_1, k_2, k_3) \mapsto (k_1 + k_2, k_3)$.

Let $T \leq M_0$ be a maximal split torus. Recall that

$$W(G, T)(F) = W(G, T)(\overline{F})$$

is the Weyl group of T in G (see Lemma 1.7.9 for the second equality). The Weyl group $W(G, T)(F)$ acts on $\mathfrak{a}_{M_0}$. For a pair of standard parabolic subgroups P and P' , we have injections $\mathfrak{a}_M \hookrightarrow \mathfrak{a}_{M_0}$ and $\mathfrak{a}_{M'} \hookrightarrow \mathfrak{a}_{M_0}$ as discussed above and we identify $\mathfrak{a}_M$ and $\mathfrak{a}_{M'}$ with their images as mentioned above. Thus we can define

$$W(\mathfrak{a}_M, \mathfrak{a}_{M'})$$

to be the set of linear isomorphisms from $\mathfrak{a}_M$ onto $\mathfrak{a}_{M'}$ obtained by restricting elements in the Weyl group $W(G, T)(F)$. If $W(\mathfrak{a}_M, \mathfrak{a}_{M'})$ is nonempty then P and P' are said to be **associate**. An **association class** (with respect to a given minimal parabolic subgroup) is a set of standard parabolic subgroups that are all associate to each other.

Lemma 10.2.1 *The standard parabolic subgroups P and P' are associate if and only if their Levi subgroups M and M' are $G(F)$ -conjugate.*

Proof. This is essentially [Sol20, Lemma 1]; we reproduce the proof here. Let Δ be the set of simple roots of T in G corresponding to P_0 . We recall that standard parabolic subgroups are in bijection with subsets $J \subseteq \Delta$ by Theorem 1.9.4. We write P_J for the parabolic subgroup corresponding to J and $\Phi(J) := \mathbb{Z}J \cap \Phi(G, T)$. As in (1.47), let M_J be the group generated by $C_G(T)$ and G_α for $\alpha \in \Phi(J)$. Then by Proposition 1.9.7 (or more precisely

its proof), $M_0 = M_\emptyset$ and M_J is the unique Levi subgroup of P_J containing M_0 .

Now assume that P_J and $P_{J'}$ are associate. We observe that $\mathfrak{a}_{M_J}^*$ is just the $\mathbb{R}$ -span of $\mathfrak{a}_G^*$ and $\Delta - J$, and similarly for $\mathfrak{a}_{M_{J'}}^*$. Thus by Lemma 1.7.9 there exists an element $g \in N_G(T)(F)$ such that $g\Phi(J)g^{-1} = \Phi(J')$. We deduce that $gM_Jg^{-1} = M_{J'}$. Conversely if $gM_Jg^{-1} = M_{J'}$ then gTg^{-1} is a maximal split torus of $M_{J'}$. Therefore there is an $m \in M_{J'}(F)$ such that $mgTg^{-1}m^{-1} = T$ by Theorem 1.7.6. It follows that $mg \in N_G(T)(F)$ provides an element of $W(\mathfrak{a}_{M_J}, \mathfrak{a}_{M_{J'}})$. $\square$

Example 10.2. Let $G = \mathrm{GL}_3$ and let P_0 be the Borel subgroup of upper triangular matrices. In this case there are three association classes of standard parabolic subgroups. One is the set consisting of the Borel subgroup, another is the set consisting of G , and the third is the association class consisting of the standard parabolic subgroup of type $(2, 1)$ and the standard parabolic subgroup of type $(1, 2)$.

In view of the lemma, it is natural to expect a relationship between representations of G induced from P and those induced from P' when P and P' are associate. Assume P and P' are associate and let $w \in W(\mathfrak{a}_M, \mathfrak{a}_{M'})$. Let $V \leq L_{\mathrm{disc}}^2([M])$ be a subrepresentation. The relationship mentioned above is obtained by introducing intertwining operators

$$M(w, \lambda) : \mathrm{Ind}_P^G(V) \longrightarrow \mathrm{Ind}_{P'}^G(V)$$

defined by

$$M(w, \lambda)\varphi(x) := \int \varphi(\tilde{w}^{-1}nx) e^{\langle H_P(\tilde{w}^{-1}nx), \lambda + \rho_P \rangle} e^{\langle H_{P'}(x), -w\lambda + \rho_{P'} \rangle} dn, \quad (10.7)$$

where the integral is over $(N'(\mathbb{A}_F) \cap \tilde{w}N(\mathbb{A}_F)\tilde{w}^{-1}) \backslash N'(\mathbb{A}_F)$. Here $\tilde{w} \in G(F)$ is any representative for w . At the moment, this is just a formal definition. To make it rigorous, let $K \leq G(\mathbb{A}_F)$ be a maximal compact subgroup and assume that V is an internal direct sum of finitely many irreducible subrepresentations of $L_{\mathrm{disc}}^2([M])$. Let

$$\mathrm{Ind}_P^G(V)^0 \leq \mathrm{Ind}_P^G(V) \quad (10.8)$$

be the dense subspace consisting of functions φ that are smooth, K -finite, and have the property that

$$m \longmapsto \varphi(mk)$$

is an element of $\mathcal{A}(M)$ for all $k \in K$ (see [MW95, §I.2.17]). Here the space $\mathcal{A}(M)$ of automorphic forms on $M(\mathbb{A}_F)$ is defined as in (6.13) and §6.4. When restricted to $\mathrm{Ind}_P^G(V)^0$, the integral $M(w, \lambda)$ converges for λ sufficiently large in a sense we now make precise. Let Φ denote the set of roots of Z_M^t in G

and let Δ be the base attached to the set of positive roots defined by P (see §1.9). Set

$$(\mathfrak{a}_P^*)^+ := \{\lambda \in \mathfrak{a}_M^* : \lambda(\alpha^\vee) > 0 \text{ for all } \alpha \in \Delta\}.$$

Here $\alpha^\vee$ is the coroot associated to α (see (1.33)). For the following proposition, see [MW95, §II.1.6].

Proposition 10.2.2 (Langlands) *The intertwining operator $M(w, \lambda)$ converges absolutely for $\operatorname{Re}(\lambda)$ in a suitable cone in $\mathfrak{a}_M^*$. If $V \leq L_{\text{cusp}}^2([M])$ then $M(w, \lambda)$ converges absolutely for $\operatorname{Re}(\lambda) \in \rho_P + (\mathfrak{a}_P^*)^+$. $\square$*

It turns out that the operators $M(w, \lambda)$ admit meromorphic continuations [MW95, §IV.1.8, §IV.1.10] [Art05, Theorem 7.2]:

Theorem 10.2.3 *The operators $M(w, \lambda)$ admit meromorphic continuations to $\mathfrak{a}_{M\mathbb{C}}^*$ that are holomorphic and unitary on $i\mathfrak{a}_M^*$. If P, P', P'' are standard parabolic subgroups with Levi subgroups M, M', M'' containing M_0 then the operators satisfy the functional equation*

$$M(w_1 w_2, \lambda) = M(w_1, w_2 \lambda) \circ M(w_2, \lambda)$$

for $w_1 \in W(\mathfrak{a}_{M'}, \mathfrak{a}_{M''})$, and $w_2 \in W(\mathfrak{a}_M, \mathfrak{a}_{M'})$. $\square$

The subspace $i\mathfrak{a}_M \subseteq \mathfrak{a}_{P\mathbb{C}}^*$ plays a distinguished role because the representations $I(\sigma, \lambda)$ are unitary for $\lambda \in i\mathfrak{a}_M$ (see Exercise 10.4).

10.3 Eisenstein series

Now $I(\sigma, \lambda)$ is a representation of $G(\mathbb{A}_F)$ on a space of functions on $G(\mathbb{A}_F)$, but the functions in $\operatorname{Ind}_P^G(V)$ are not left invariant under $G(F)$. The functions in this space are left invariant under $P(F)$, however. We therefore do the most naïve thing possible to make them invariant under $G(F)$, namely, average. Assume that

$$\varphi \in \operatorname{Ind}_P^G(V)^0,$$

where $V \leq L_{\text{disc}}^2([M])$ is an irreducible subrepresentation and $\operatorname{Ind}_P^G(V)^0$ is defined as in (10.8). Let

$$E(x, \varphi, \lambda) := \sum_{\delta \in P(F) \backslash G(F)} \varphi(\delta x) e^{\langle H_P(\delta x), \lambda + \rho_P \rangle}. \quad (10.9)$$

This is an **Eisenstein series**. When it converges absolutely it provides an intertwining map from $I(\sigma, \lambda)$ to the regular action of $G(\mathbb{A}_F)$ on functions on $G(\mathbb{A}_F)$:

$$E(xg, \varphi, \lambda) = E(x, I(\sigma, \lambda)(g)\varphi, \lambda) \quad (10.10)$$

(see Exercise 10.3).

The problem is that $E(x, \varphi, \lambda)$ only converges for λ sufficiently large [MW95, §II.1.5]:

Proposition 10.3.1 (Langlands) *If $\operatorname{Re}(\lambda)$ lies in a suitable cone in $\mathfrak{a}_M^*$ then $E(x, \varphi, \lambda)$ converges absolutely and uniformly on compact subsets of $G(F) \backslash G(\mathbb{A}_F)$. If φ is cuspidal, $E(x, \varphi, \lambda)$ converges absolutely and uniformly on compact subsets of $G(F) \backslash G(\mathbb{A}_F)$ for $\operatorname{Re}(\lambda) \in \rho_P + (\mathfrak{a}_P^*)^+$. $\square$*

On the other hand, the representations $I(\sigma, \lambda)$ are unitary when $\lambda \in i\mathfrak{a}_M^*$, which is never in the plane of absolute convergence of the Eisenstein series. Instead, one has to analytically continue the Eisenstein series as a complex analytic function of λ to the line $\lambda \in i\mathfrak{a}_M^*$ in order to construct pieces of $L^2([G])$. In the case $G = \operatorname{GL}_2$, this was accomplished by Selberg [Sel56, Sel63].

Langlands was the first to appreciate the difficulties presented by the general case, and was able to overcome them [Lan76]. It is hard to overestimate the impact of Langlands' work on Eisenstein series to automorphic representation theory (and indeed much of mathematics, especially number theory). This work, for example, led to the discovery of Langlands reciprocity as enunciated in Langlands letter to Weil [Lan, Lan70], led to Langlands-Shahidi theory of automorphic L -functions [Lan71, Sha10], and also is required in any general treatment of a trace formula, either in the Arthur-Selberg sense [Art78, Art80], or in Jacquet's sense [Jac05a, JLR93].

If P and P' are associate standard parabolic subgroups with Levi subgroups M and M' containing M_0 (respectively), then the intertwining operators of the previous section give a relationship between representations induced from P and representations induced from P' . This suggests a relationship between the associated Eisenstein series. The precise relationship is contained in the following theorem of Langlands (see [Art05, Theorem 7.2] and [MW95, §IV.1.10, §IV.1.11]):

Theorem 10.3.2 (Langlands) *The Eisenstein series $E(x, \varphi, \lambda)$ admits a meromorphic continuation to $\mathfrak{a}_{MC}^*$ that is holomorphic on $i\mathfrak{a}_M^*$. It satisfies the functional equation*

$$E(x, M(w, \lambda)\varphi, w\lambda) = E(x, \varphi, \lambda)$$

for $w \in W(\mathfrak{a}_M, \mathfrak{a}_{M'})$. $\square$

Sometimes in applications one needs more precise information about the size of Eisenstein series and the intertwining operators. We refer to [GS01] for an important example where this is crucial. This paper refers to the equally important paper [Mül00], which proves that the Eisenstein series $E(x, \varphi, \lambda)$ is a quotient of functions of finite order under suitable assumptions on φ .

10.4 Constant terms

We keep the notation from §10.2 involving minimal and standard parabolic subgroups. Let v be a non-Archimedean place of F . In §8.1 we defined parabolic induction, which is a functor taking admissible representations of $M(F_v)$ to admissible representations of $G(F_v)$. As explained in §8.3, there is a left adjoint to this functor, given by taking the Jacquet module of a representation.

The functor of parabolic induction produces representations of larger groups from representations of particularly nice subgroups. Jacquet modules allow us to analyze which representations can be obtained using parabolic induction and whether or not two different parabolically induced representations share a constituent. One can view this aspect of Jacquet modules as a manifestation of Mackey theory. This is a set of results in finite group theory that analyze intertwining operators between induced representations in terms of the inducing data. Here the **inducing data** of an induced representation is the subgroup and representation of that subgroup used to form the induced representation. This terminology makes sense in any of the contexts we have discussed, e.g. in finite group theory, in the theory of admissible representations, and in automorphic representation theory.

Eisenstein series, defined as in §10.3, are a global analogue of parabolic induction. For smooth functions $\varphi : G(F) \backslash G(\mathbb{A}_F) \rightarrow \mathbb{C}$ let

$$\varphi_P(g) := \int_{[N]} \varphi(ng) dn \quad (10.11)$$

which we may view as a smooth function $\varphi_P : M(F)N(\mathbb{A}_F) \backslash G(\mathbb{A}_F) \rightarrow \mathbb{C}$. This is the **constant term of φ along P** . Constant terms play the same role in the global theory as Jacquet modules play in the local theory.

We remark that the growth rate of an automorphic form is controlled by its constant terms. For a general discussion, see [MW95, §I.2]. As an example, consider cusp forms. These are automorphic forms all of whose constant terms vanish identically. By Theorem 9.7.1 cusp forms are rapidly decreasing when F is a number field and are compactly supported when F is a function field.

We now discuss the computation of certain constant terms of Eisenstein series following [MW95, §II.1.7]. Let $P = MN$ and $P' = M'N'$ be a pair of standard parabolic subgroups. For an irreducible subrepresentation $V \leq L^2_{\text{disc}}([M])$ define $\text{Ind}_P^G(V)^0$ as in (10.8). For $\varphi \in \text{Ind}_P^G(V)^0$ set

$$E_{P'}(x, \varphi, \lambda) := \int_{[N']} E(nx, \varphi, \lambda) dn.$$

This is the constant term of $E(x, \varphi, \lambda)$ along P' .

To analyze this constant term, we recall the Bruhat decomposition [BT65, §5.20]:

$$G(F) = \coprod_w P(F)w^{-1}P'(F), \quad (10.12)$$

where the disjoint union is over

$$w \in W(M', T)(F) \backslash W(G, T)(F) / W(M, T)(F).$$

Let

$$\Phi^+(M, T)$$

be the set of roots of T contained in $M \cap P_0$. Then as a set of representatives for $W(M', T)(F) \backslash W(G, T)(F) / W(M, T)(F)$ we may take

$$W^{M, M'} := \left\{ w \in W(G, T)(F) : \begin{array}{l} w(\Phi^+(M, T)) \subseteq \Phi^+(G, T), \\ w^{-1}(\Phi^+(M', T)) \subseteq \Phi^+(G, T) \end{array} \right\} \quad (10.13)$$

[Ren10, §V.4.6]. By loc. cit. this is the set of representatives obtained by choosing the Weyl element of minimal length in each double class. We observe that

$$W^{M, M'} := \left\{ w \in W(G, T)(F) : \begin{array}{l} w(M \cap P_0)w^{-1} \leq P_0, \\ w^{-1}(M' \cap P_0)w \leq P_0 \end{array} \right\}. \quad (10.14)$$

By Proposition 1.9.8 the intersection $M' \cap P_0$ is a minimal parabolic subgroup of M' . We say that parabolic subgroups of M' are standard if they contain $M' \cap P_0$ and that a Levi subgroup of M' is standard if it contains M_0 and is a Levi subgroup of a standard parabolic subgroup of M' . The following is [Ren10, Lemma V.4.6]:

Lemma 10.4.1 *If $w \in W^{M, M'}$ then $w^{-1}P'w \cap M$ is a standard parabolic subgroup of M with Levi factor $w^{-1}M'w \cap M$ and unipotent radical $w^{-1}N'w \cap M$. Similarly $wPw^{-1} \cap M'$ is a standard parabolic subgroup of M' with Levi factor $wMw^{-1} \cap M'$ and unipotent radical $wN'w^{-1} \cap M'$. Moreover*

$$\begin{aligned} wPw^{-1} \cap P' &= (wPw^{-1} \cap M')(wPw^{-1} \cap N'), \\ wNw^{-1} \cap P' &= (wNw^{-1} \cap M')(wNw^{-1} \cap N'), \\ w^{-1}P'w \cap P &= (w^{-1}P'w \cap M)(w^{-1}Pw \cap N), \\ w^{-1}N'w \cap P &= (w^{-1}N'w \cap M)(wN'w^{-1} \cap N). \end{aligned} \quad (10.15)$$

□

We let

$$W(M, M') := \left\{ w \in W^{M, M'} : wMw^{-1} \leq M' \right\}.$$

Suppose that $w \in W(M, M')$. Then $M' \cap wPw^{-1}$ is a standard parabolic subgroup of M' by Lemma 10.4.1. Thus the summand corresponding to a

given w in the following proposition is again an Eisenstein series, albeit for a Levi subgroup of G instead of G itself:

Proposition 10.4.2 *Assume that V is a cuspidal automorphic representation of $A_M \backslash M(\mathbb{A}_F)$ and $\varphi \in \text{Ind}_P^G(V)^0$. If $\text{Re}(\lambda) \in \rho_P + (\mathfrak{a}_P^*)^+$ then*

$$E_{P'}(x, \varphi, \lambda) = \sum_w \sum_{m'} M(w, \lambda)(\varphi)(m'x) e^{\langle H_{P'}(m'x), w\lambda - \rho_{P'} \rangle},$$

where the sum on w is over $W(M, M')$ and the sum on m' is over

$$(M'(F) \cap wP(F)w^{-1}) \backslash M'(F).$$

Here the Eisenstein series on the left and the double sum on the right converge absolutely and uniformly on compact subsets of $M'(F)N'(\mathbb{A}_F) \backslash G(\mathbb{A}_F)$.

Proof. We assume throughout the proof that $\text{Re}(\lambda) \in \rho_P + (\mathfrak{a}_P^*)^+$. By Proposition 10.3.1 the sum defining the Eisenstein series converges absolutely and uniformly on compact subsets under this assumption. This justifies the exchanges of integrals and sums in the proof below.

By definition

$$\begin{aligned} E_{P'}(x, \varphi, \lambda) &:= \int_{[N']} E(nx, \varphi, \lambda) dn \\ &= \int_{[N']} \sum_{\delta \in P(F) \backslash G(F)} \varphi(\delta nx) e^{\langle H_P(\delta nx), \lambda + \rho_P \rangle} dn. \end{aligned}$$

Using the Bruhat decomposition (10.12) we have

$$\begin{aligned} &\int_{[N']} \sum_{\delta \in P(F) \backslash G(F)} \varphi(\delta nx) e^{\langle H_P(\delta nx), \lambda + \rho_P \rangle} dn \\ &= \sum_w \int_{[N']} \sum_{p'} \varphi(w^{-1}p'nx) e^{\langle H_P(w^{-1}p'nx), \lambda + \rho_P \rangle} dn, \end{aligned} \tag{10.16}$$

where the sum on w is over $W(M', T)(F) \backslash W(G, T)(F) / W(M, T)(F)$ and the sum on p' is over

$$(P'(F) \cap wP(F)w^{-1}) \backslash P'(F).$$

We choose the set of representatives $W^{M, M'}$. By (10.15) if $(m', n') \in M'(F) \times N'(F)$ and $m'n' \in wP(F)w^{-1}$ then $m' \in wP(F)w^{-1}$, which implies in turn that $n' \in m'^{-1}wP(F)w^{-1}m'$. Using this fact we can write (10.16) as

$$\sum_w \sum_{m'} \int_{[N']} \sum_{n'} \varphi(w^{-1}m'n'nx) e^{\langle H_P(w^{-1}m'n'nx), \lambda + \rho_P \rangle} dn, \tag{10.17}$$

where the sum on w is over $W^{M, M'}$, the sum on m' is over

$$(M'(F) \cap wP(F)w^{-1}) \backslash M'(F),$$

and the sum on n' is over

$$(N'(F) \cap m'^{-1}wP(F)w^{-1}m') \backslash N'(F). \quad (10.18)$$

Taking a change of variables $(n', n) \mapsto (m'^{-1}n'm', m'^{-1}nm')$ and unfolding (see Lemma 9.2.5) we see that

$$\begin{aligned} & \int_{[N']} \sum_{n'} \varphi(w^{-1}m'n'nx) e^{\langle H_P(w^{-1}m'n'nx), \lambda + \rho_P \rangle} dn \\ &= \int_{[N']} \sum_{n'} \varphi(w^{-1}n'nm'x) e^{\langle H_P(w^{-1}n'nm'x), \lambda + \rho_P \rangle} dn \\ &= \int_{N'(F) \cap wP(F)w^{-1} \backslash N'(\mathbb{A}_F)} \varphi(w^{-1}nm'x) e^{\langle H_P(w^{-1}nm'x), \lambda + \rho_P \rangle} dn, \end{aligned} \quad (10.19)$$

where the top sum on n' is over (10.18) and the middle sum on n' is over

$$(N'(F) \cap wP(F)w^{-1}) \backslash N'(F).$$

Combining (10.17) and (10.19), we have shown that

$$E_{P'}(x, \varphi, \lambda) = \sum_w \sum_{m'} \int \varphi(w^{-1}nm'x) e^{\langle H_P(w^{-1}nm'x), \lambda + \rho_P \rangle} dn, \quad (10.20)$$

where the sum on w is over $W^{M, M'}$, the sum on m' is again over

$$(M'(F) \cap wP(F)w^{-1}) \backslash M'(F),$$

and the integral is over $N'(F) \cap wP(F)w^{-1} \backslash N'(\mathbb{A}_F)$.

For the remainder of the proof assume $w \in W^{M, M'}$. By Lemma 10.4.1 if $N' \cap wMw^{-1} \neq 1$ then $w^{-1}N'w \cap M$ is the unipotent radical of the proper parabolic subgroup $w^{-1}P'w \cap M$ of M . Since φ is cuspidal, we deduce that (10.19) vanishes unless $N' \cap wMw^{-1} = 1$.

We claim that $w \in W^{M, M'}$ satisfies $N' \cap wMw^{-1} = 1$ if and only if $w \in W(M, M')$. Using Lemma 10.4.1 again, $N' \cap wMw^{-1} = 1$ if and only if $w^{-1}M'w \cap M = wP'w^{-1} \cap M$, which is to say that the parabolic subgroup $wP'w^{-1} \cap M$ of M is equal to its Levi subgroup $wM'w^{-1} \cap M$. This occurs if and only if $w^{-1}M'w \cap M = M$, which is equivalent to $wMw^{-1} \leq M'$, that is, $w \in W(M, M')$. We conclude from the claim and the previous paragraph that the sum in (10.20) is supported in $w \in W(M, M')$.

Now if $w \in W(M, M')$ then $wN'w^{-1} \cap M$ is trivial, hence by (10.15)

$$N'(F) \cap wP(F)w^{-1} = N'(F) \cap wN(F)w^{-1}.$$

Recalling the definition of the intertwining operator $M(w, \lambda)$ from (10.7) we deduce the proposition. $\square$

10.5 Decomposition of the spectrum

Assume for this section that F is a number field. There is a decomposition

$$L^2([G]) = \bigoplus_{\mathcal{P}} L^2_{\mathcal{P}}([G]), \quad (10.21)$$

where the (finite) direct sum is over all association classes $\mathcal{P}$ of parabolic subgroups. In more detail, consider the Hilbert space of families $\mathcal{F}$ of measurable functions

$$F_P : i(\mathfrak{a}_M^G)^* \longrightarrow \text{Ind}_P^G(L^2_{\text{disc}}([M])) \quad (10.22)$$

indexed by $P \in \mathcal{P}$ and satisfying the Weyl invariance condition

$$F_{P'}(w\lambda) = M(w, \lambda)F_P(\lambda)$$

for $w \in W(\mathfrak{a}_M, \mathfrak{a}_{M'})$.

The relevant inner product is

$$(\mathcal{F}_1, \mathcal{F}_2) = \sum_{P \in \mathcal{P}} n_P^{-1} \int_{i(\mathfrak{a}_M^G)^*} (F_{1P}(\lambda), F_{2P}(\lambda)) d\lambda,$$

where $F_{1P} \in \mathcal{F}_1$ and $F_{2P} \in \mathcal{F}_2$ for $P \in \mathcal{P}$. Here

$$n_P := \sum_{P' \in \mathcal{P}} |W(\mathfrak{a}_M, \mathfrak{a}_{M'})|. \quad (10.23)$$

Given such a family of functions $\mathcal{F}$, we can formally define

$$\sum_{P \in \mathcal{P}} n_P^{-1} \int_{i(\mathfrak{a}_M^G)^*} E(x, F_P(\lambda), \lambda) d\lambda \in L^2([G]). \quad (10.24)$$

We say that $\mathcal{F}$ is compactly supported if each $F_P \in \mathcal{F}$ is compactly supported. Assume that $V \leq L^2_{\text{disc}}([M])$ is an irreducible subrepresentation. Then the formal integral (10.24) is absolutely convergent if $\mathcal{F}$ is compactly supported. The closure of the subspace of $L^2([G])$ spanned by the compactly supported $\mathcal{F}$ is denoted

$$L^2_{\mathcal{P}}([G]).$$

We observe that when $M = G$ one has that $(\mathfrak{a}_M^G)^* = \{0\}$. Thus if $\mathcal{G}$ is the association class of G (which just consists of G itself) we have

$$L_G^2([G]) = L_{\text{disc}}^2([G]).$$

Theorem 10.5.1 (Langlands) *One has that*

$$L^2([G]) = \bigoplus_{\mathcal{P}} L_{\mathcal{P}}^2([G]),$$

where the sum is over association classes $\mathcal{P}$ of parabolic subgroups of G . $\square$

This is [Art05, Theorem 7.2], except that Arthur decomposes $L^2(G(F)\backslash G(\mathbb{A}_F))$ whereas we have decomposed $L^2([G])$. The **continuous spectrum** is

$$L_{\text{cont}}^2([G]) := \bigoplus_{\mathcal{P} \neq \{G\}} L_{\mathcal{P}}^2([G]). \quad (10.25)$$

Here we are using the fact that the association class of G is just G . We then have

$$L_{\text{disc}}^2([G]) \oplus L_{\text{cont}}^2([G]) = L^2([G]). \quad (10.26)$$

The **residual spectrum** is the orthogonal complement of $L_{\text{cusp}}^2([G])$ in $L_{\text{disc}}^2([G])$:

$$L_{\text{res}}^2([G]) \oplus L_{\text{cusp}}^2([G]) := L_{\text{disc}}^2([G]). \quad (10.27)$$

We will see in §10.8 that when $G = \text{GL}_n$ the residual spectrum can be described in terms of residues of Eisenstein series attached to cusp forms. This is true in general [Lan76] but we will not discuss the general case further.

Let us end this section by describing the data appearing in Theorem 10.5.1 when $G = \text{GL}_2$. We take our minimal parabolic subgroup B to be the Borel subgroup of upper triangular matrices, and fix the Levi subgroup $M < B$ of diagonal matrices. The discrete spectrum of $L^2([M])$ is the whole spectrum. In fact, by the Peter-Weyl theorem, Theorem 4.3.4, there is an isomorphism

$$L^2([M]) \cong \widehat{\bigoplus_{\chi \in \widehat{[M]}} \mathbb{C}_{\chi}},$$

where the direct sum is over characters of $[M]$ and $\mathbb{C}_{\chi}$ is a 1-dimensional space on which $[M]$ acts through χ . There is only one association class of standard parabolic subgroups that is not GL_2 itself, and it consists of the single element B . The group $W(\mathfrak{a}_M, \mathfrak{a}_M)$ is just $\mathbb{Z}/2$.

10.6 Local preparation for isobaric representations

In this section we collect some local preliminaries for our discussion of isobaric representations in §10.7. Thus for this section we assume that F is a local field.

We have already discussed in §4.9 and §8.4 how to classify admissible representations of $G(F)$ for reductive F -groups G in terms of tempered representations. When $G = \mathrm{GL}_n$ and F is non-Archimedean, in §8.4 we also discussed how to classify the tempered representations in terms of the square integrable representations. We now explain how these results can be combined to classify all admissible representations of $\mathrm{GL}_n(F)$ in terms of square integrable representations.

Let $n = \sum_{i=1}^k n_i$ and let π_i be an irreducible essentially square integrable representation of $\mathrm{GL}_{n_i}(F)$ for each i . Define

$$A_{\mathrm{GL}_{n_i}} = \begin{cases} \mathbb{R}_{>0} I_{n_i} & \text{if } F \text{ is Archimedean,} \\ \varpi^{\mathbb{Z}} I_{n_i} & \text{if } F \text{ is non-Archimedean.} \end{cases} \quad (10.28)$$

Here in the non-Archimedean case ϖ is a fixed uniformizer for F . The group $A_{\mathrm{GL}_{n_i}}$ depends on the choice of uniformizer in this case, but we will not encode it into the notation. We observe that $A_{\mathrm{GL}_{n_i}}$ does not coincide with the groups that were defined in (2.20) for GL_{n_i} viewed as a group over a global field. However, since we will only use the groups in (10.28) in local settings this should not cause confusion.

Let π_i^1 be the unique irreducible representation of $A_{\mathrm{GL}_{n_i}} \backslash \mathrm{GL}_{n_i}(F)$ such that

$$\pi_i(g) = |\det g|^{\lambda_i} \pi_i^1(g) \quad (10.29)$$

for some $\lambda_i \in \mathbb{C}$. After permuting the π_i if necessary we can assume that

$$\mathrm{Re}(\lambda_1) \geq \cdots \geq \mathrm{Re}(\lambda_k). \quad (10.30)$$

Let P be the standard parabolic subgroup of type $(n_1, \dots, n_k)$ with standard Levi subgroup M . Here and below we use the terminology of Example 1.15. We identify

$$\mathfrak{a}_M^* = \mathbb{R}^k \quad (10.31)$$

in such a way that for $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathfrak{a}_M^*$ one has that

$$\left\langle H_M \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_k \end{pmatrix}, \lambda \right\rangle = \sum_{i=1}^k \lambda_i \log |\det x_i|.$$

Then we can form the induced representation

$$I\left(\otimes_{i=1}^k \pi_i^1, \lambda\right).$$

Theorem 10.6.1 *The induced representation $I\left(\otimes_{i=1}^k \pi_i^1, \lambda\right)$ has a unique irreducible quotient $J\left(\otimes_{i=1}^k \pi_i^1, \lambda\right)$. Assume that $\{\pi'_i\}_{i=1}^{k'}$ is another collection of irreducible admissible essentially square integrable representations of $\mathrm{GL}_{n'_i}(F)$ and $\lambda'_1, \dots, \lambda'_{k'} \in \mathbb{C}$ satisfy $\mathrm{Re}(\lambda'_1) \geq \dots \geq \mathrm{Re}(\lambda'_{k'})$. In the non-Archimedean (resp. Archimedean) case there is an equivalence (resp. infinitesimal equivalence)*

$$J\left(\otimes_{i=1}^k \pi_i^1, \lambda\right) \cong J\left(\otimes_{i=1}^{k'} (\pi'_i)^1, \lambda'\right) \quad (10.32)$$

if and only if $k = k'$ and there is a permutation τ of $\{1, \dots, k\}$ such that $\pi'_i \cong \pi_{\tau(i)}$ and $\lambda'_i = \lambda_{\tau(i)}$.

Proof. This is consequence of Theorem 8.4.2 and the Bernstein-Zelevinsky classification discussed in §8.4 for the non-Archimedean case. For the Archimedean case we refer to the discussion and references after [Kna94, Theorem 1 and 4]. $\square$

Let $\pi_1 \boxplus \dots \boxplus \pi_k$ be the irreducible quotient $J\left(\otimes_{i=1}^k \pi_i^1, \lambda\right)$ of the theorem. In this notation we do not assume that the π_i are arranged so that

$$\mathrm{Re}(\lambda_1) \geq \dots \geq \mathrm{Re}(\lambda_k),$$

so by definition

$$\boxplus_{i=1}^k \pi_i \cong \boxplus_{i=1}^k \pi_{\tau(i)}$$

for any permutation τ of $\{1, \dots, k\}$. Because of Theorem 10.6.1, this notation is unambiguous.

Theorem 10.6.2 *In the non-Archimedean (resp. Archimedean) case any irreducible admissible representation of $\mathrm{GL}_n(F)$ is equivalent (resp. infinitesimally equivalent) to $\boxplus_{i=1}^k \pi_i$ for some set of irreducible admissible essentially square integrable representations $\{\pi_i\}_{i=1}^k$.*

Proof. Again, this is consequence of Theorem 8.4.2 and the Bernstein-Zelevinsky classification in §8.4 in the non-Archimedean case and the discussion following [Kna94, Theorem 1 and 4] in the Archimedean case. $\square$

Given Theorem 10.6.1 and Theorem 10.6.2, we can define an irreducible admissible representation $\boxplus_{i=1}^k \pi_i$ of $\mathrm{GL}_n(F)$ for any tuple $\pi_1, \dots, \pi_k$ of irreducible admissible representations of $\mathrm{GL}_{n_i}(F)$ with $\sum_{i=1}^k n_i = n$. In particular, we can remove the assumption that the π_i are essentially square integrable. For each π_i we write $\pi_i = \boxplus_{j=1}^{k_i} \pi'_{ij}$ and then set

$$\boxplus_{i=1}^k \pi_i = \boxplus_{i=1}^k \left(\boxplus_{j=1}^{k_i} \pi'_{ij} \right). \quad (10.33)$$

The representation $\boxplus_{i=1}^k \pi_i$ is called the **isobaric sum** of the π_i . To emphasize that we are working in the setting of admissible representations of general linear groups over local fields we will sometimes refer to $\boxplus_{i=1}^k \pi_i$ as the local isobaric sum of the π_i . The symbol $\boxplus$ behaves something like a formal direct sum on the category of admissible representations, but we emphasize that $\boxplus_{i=1}^k \pi_i$ is always irreducible.

It is useful to explicitly point out the relationship between the classification of unramified representations in terms of the Satake correspondence and the isobaric sum. For this let $T_n \leq \mathrm{GL}_n$ be the maximal torus of diagonal matrices and let $B_n \leq \mathrm{GL}_n$ be the Borel subgroup of upper triangular matrices.

Theorem 10.6.3 *Let $\chi_1, \dots, \chi_n : F^\times \rightarrow \mathbb{C}^\times$ be n unramified quasi-characters and let*

$$\begin{aligned} \chi : T_n(F) &\longrightarrow \mathbb{C}^\times \\ (t_{ij}) &\longmapsto \prod_{i=1}^n \chi_i(t_{ii}). \end{aligned}$$

The isobaric sum $\boxplus_{i=1}^n \chi_i$ is then the unique unramified subquotient of the induced representation $\mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(\chi)$.

Proof. We know that there is a unique unramified subquotient of $\mathrm{Ind}_{B_n}^{\mathrm{GL}_n}(\chi)$ by Lemma 7.6.8. On the other hand, the isobaric sum $\boxplus_{i=1}^n \chi_i$ is unramified by [Mat13, Corollary 1.2]. $\square$

We warn the reader that the operation of taking the local isobaric sum can have some unexpected properties:

Example 10.3. Assume F is a non-Archimedean local field and let $B \leq \mathrm{GL}_2$ be the Borel subgroup of upper triangular matrices. Let 1 denote the trivial representation of $B(F)$. Consider

$$\pi_1 = I(1, (\tfrac{3}{2}, -\tfrac{1}{2})) \quad \text{and} \quad \pi_2 = I(1, (\tfrac{1}{2}, -\tfrac{3}{2})).$$

These representations of $\mathrm{GL}_2(F)$ are irreducible by Theorem 8.4.4 and by the definition above

$$\pi_1 \boxplus \pi_2 = |\cdot|^{3/2} \boxplus |\cdot|^{1/2} \boxplus |\cdot|^{-1/2} \boxplus |\cdot|^{-3/2}$$

is the trivial representation of $\mathrm{GL}_4(F)$ by [Zel80, Example 3.2]. But the trivial representation is not a quotient of $\mathrm{Ind}_P^{\mathrm{GL}_4}(\pi_1 \otimes \pi_2)$, where P is the standard parabolic subgroup of GL_4 of type $(2, 2)$ (see Exercise 10.5).

Thus some care is required in treating local isobaric sums. Fortunately in our global application in §10.7 the local representations involved enjoy additional properties that preclude the behavior in Example 10.3. More specifically, the local components of cuspidal automorphic representations are unitary and generic in the sense of §11.3 (see Theorem 11.3.6). The reader can

safely take the definition of generic as a “black box” to be opened in §11.3 for the discussion that follows.

Assume as above that we are given $n = \sum_{i=1}^k n_i$ and representations π_i . Define λ_i as in (10.29). We assume that the λ_i satisfy (10.30). We say that an irreducible admissible representation π_i of $\mathrm{GL}_{n_i}(F)$ is **essentially unitary** if the representation π_i^1 of $A_{\mathrm{GL}_{n_i}} \backslash \mathrm{GL}_{n_i}(F)$ defined in (10.29) is unitary.

Theorem 10.6.4 *If π_i is an essentially unitary generic representation for $1 \leq i \leq k$ then $\bigoplus_{i=1}^k \pi_i$ is a subquotient of*

$$I\left(\bigotimes_{i=1}^k \pi_i^1, \lambda\right).$$

Proof. We claim that each unitary generic representation π of $\mathrm{GL}_n(F)$ is of the form $I(\sigma, \lambda)$ where σ is a square integrable representation of a Levi subgroup of $\mathrm{GL}_n(F)$. Assuming the claim, the theorem follows from [MW89, §1.2(3)]. If F is non-Archimedean the claim is part of Theorem 8.4.4. If F is Archimedean, then the claim follows from Theorem 10.6.2 and [Jac09, Lemma 2.5]. $\square$

Thus under the assumptions of Theorem 10.6.4 the pathological behavior in Example 10.3 does not occur.

10.7 Isobaric representations

Assume now that F is a number field. So far we have explained how the entire spectrum of $L^2([G])$ can be described in terms of automorphic representations of Levi subgroups M of parabolic subgroups P of G . Let us state a less precise result that applies to all automorphic representations, not just discrete automorphic representations. Given a cuspidal automorphic representation σ of $M(\mathbb{A}_F)$, we form the induced representation

$$I(\sigma, \lambda) \tag{10.34}$$

as in (10.1). For the proof of the following theorem, see the appendix to [BJ79]:

Theorem 10.7.1 (Langlands) *Any irreducible subquotient of $I(\sigma, \lambda)$ is an automorphic representation, and every automorphic representation is of this form.* $\square$

This theorem applies to more general automorphic representations than those implicit in Theorem 10.5.1, but yields less information. We remark that in [BJ79, Appendix, Lemma 1] Langlands observes that the subquotients of $I(\sigma, \lambda)$ can be described as follows. For all but finitely many v the representation σ_v is unramified and $I(\sigma_v, \lambda)$ has a unique irreducible unramified

subquotient π_v° . The irreducible subquotients of $I(\sigma, \lambda)$ are the representations of the form $\otimes'_v \pi_v$, where π_v is an irreducible subquotient of $I(\sigma_v, \lambda)$ for all v and $\pi_v \cong \pi_v^\circ$ for almost every v .

Let us restrict $G = \mathrm{GL}_n$ and discuss refinements of this result. As a first step we discuss an operation on automorphic representations of GL_n that plays the role in automorphic representation theory of the direct sum in ordinary representation theory. In fact, under the conjectural Langlands correspondence (to be discussed in Chapter 12), this operation should be induced by taking direct sums of L -parameters.

Assume that

$$\sum_{i=1}^k n_i = \sum_{j=1}^{k'} n'_j = n$$

and σ_i (resp. σ'_j) are cuspidal automorphic representations of $A_{\mathrm{GL}_{n_i}} \backslash \mathrm{GL}_{n_i}(\mathbb{A}_F)$ (resp. $A_{\mathrm{GL}_{n'_j}} \backslash \mathrm{GL}_{n'_j}(\mathbb{A}_F)$) for $1 \leq i \leq k$ and $1 \leq j \leq k'$. We let

$$\sigma = \otimes_{i=1}^k \sigma_i \quad \text{and} \quad \sigma' = \otimes_{j=1}^{k'} \sigma'_j.$$

We can then form the induced representations $I(\sigma, \lambda)$ and $I(\sigma', \lambda')$ as in §10.1. Here $I(\sigma, \lambda)$ and $I(\sigma', \lambda')$ are formed with respect to the standard parabolics P and P' of type $(n_1, \dots, n_k)$ and $(n'_1, \dots, n'_{k'})$, respectively. The following theorem is [JS81a, Theorem 4.4]:

Theorem 10.7.2 (Jacquet and Shalika) *Let S be a finite set of places of F including the infinite places such that $I(\sigma, \lambda)$ and $I(\sigma', \lambda')$ are unramified outside of S . If the irreducible unramified subquotients of $I(\sigma_v, \lambda)$ and $I(\sigma'_v, \lambda')$ are isomorphic for $v \notin S$ then $k = k'$ and there is a permutation τ of the set of k elements such that*

$$\sigma_i \cong \sigma'_{\tau(i)} \quad \text{and} \quad \lambda_i = \lambda'_{\tau(i)}.$$

□

This immediately implies the following corollary:

Corollary 10.7.3 *If π is an irreducible subquotient of $I(\sigma, \lambda)$ and $I(\sigma', \lambda')$ then the conclusion of Theorem 10.7.2 holds.* □

Thus every automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ is a subquotient of $I(\sigma, \lambda)$, where σ and λ are unique up to a permutation in the sense made precise in Theorem 10.7.2.

Given the Langlands classification of Theorem 4.9.2 and Theorem 8.4.2, it would be aesthetically pleasing if σ_v were tempered for each v . Indeed, if this were the case, then upon rearranging the σ_i we could assume that λ is in the positive Weyl chamber. There would then be a canonical irreducible subquotient of $I(\sigma, \lambda)$, namely,

$$\otimes'_v J(\sigma_v, \lambda)$$

with the local factors $J(\sigma_v, \lambda)$ defined as in Theorem 4.9.1 and Theorem 8.4.1. An important open conjecture is that cuspidal representations always have tempered local factors:

Conjecture 10.7.4 (The Ramanujan conjecture) *If π is a cuspidal automorphic representation of $A_{\mathrm{GL}_n} \backslash \mathrm{GL}_n(\mathbb{A}_F)$, then π_v is tempered for all v .*

A discussion of this conjecture and what is known towards it is contained in [Sar05].

Since the Ramanujan conjecture remains unproven to find a canonical irreducible subquotient of $I(\sigma, \lambda)$ we must proceed differently. It is convenient to adjust our notation slightly. For $1 \leq i \leq k$ let π_i be a cuspidal automorphic representation of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$. Let π_i^1 be the unique cuspidal automorphic representation of $A_{\mathrm{GL}_{n_i}} \backslash \mathrm{GL}_{n_i}(\mathbb{A}_F)$ such that

$$\pi_i(g) = |\det g|^{\lambda_i} \pi_i^1(g)$$

for some $\lambda_i \in \mathbb{C}$. After rearranging the π_i we can assume (10.30) holds. Let P be the standard parabolic subgroup of GL_n of type $(n_1, \dots, n_k)$ and let $\lambda = (\lambda_1, \dots, \lambda_k)$. Then we can form the induced representation

$$I(\otimes_{i=1}^k \pi_i^1, \lambda).$$

By Theorem 10.7.1 any irreducible subquotient of this representation is automorphic. Recall the local isobaric sums

$$\boxplus_{i=1}^k \pi_{iv}$$

defined in §10.6. In the following theorem we make use of the important fact that if π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ then π_v is generic for all v (see Theorem 11.3.6). The notion of a generic representation was previously mentioned in §10.6 and will be defined in §11.3. We again invite the reader to treat this concept as a black box for the time being.

Theorem 10.7.5 *The representation*

$$\boxplus_{i=1}^k \pi_i := \otimes'_v (\boxplus_{i=1}^k \pi_{iv}) \tag{10.35}$$

is an irreducible subquotient of $I(\otimes_{i=1}^k \pi_i^1, \lambda)$, hence automorphic.

Proof. As just mentioned the representations π_{iv} are generic. They are also essentially unitary. Thus for all v the representation $\boxplus_{i=1}^k \pi_{iv}$ is an irreducible subquotient of $I(\otimes_{i=1}^k \pi_{iv}^1, \lambda)$ by Theorem 10.6.4, and it is unramified for all but finitely many v by Theorem 10.6.3. In view of the discussion of the irreducible subquotients of induced representations after Theorem 10.7.1, the proof follows. $\square$

Definition 10.7.6. An automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ is **isobaric** if there are cuspidal automorphic representations π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$ with

$\sum_{i=1}^k n_i = n$ such that

$$\pi \cong \boxplus_{i=1}^k \pi_i.$$

Not all automorphic representations are isobaric. A concrete example is given in Exercise 10.8.

In view of Theorem 10.7.2, we can extend the definition of $\boxplus$ to arbitrary tuples of isobaric automorphic representations in a well-defined manner just as in (10.33). The operation $\boxplus$ on the set of isomorphism classes of isobaric representations behaves something like a direct product, but just as in the local case, isobaric sums are always irreducible representations. In fact, isobaric sums correspond to direct sums via the Langlands correspondence (see §13.2).

10.8 A theorem of Mœglin and Waldspurger

For this section, we assume that F is a number field. Let G be a reductive group over F . The spectral decomposition of $L^2([G])$ proved by Langlands leaves open the important question of how to describe the discrete spectrum of $L^2([M])$ for Levi subgroups M of G . Some information on this question is obtained in the course of Langlands' proof of Theorem 10.5.1, but nothing definitive.

In [Jac84] Jacquet gave a precise conjectural description of $L^2_{\text{disc}}([\text{GL}_n])$ which was later proven by Mœglin and Waldspurger in [MW89]. We state this result in this section. Since the Levi subgroups of GL_n are products of general linear groups, by induction this gives a complete spectral decomposition of $L^2([\text{GL}_n])$.

Fix an integer n . For each factorization $n = md$, there is a unique standard parabolic subgroup P with Levi subgroup M isomorphic to GL_d^m . Suppose we are given a cuspidal automorphic representation π of $A_{\text{GL}_d} \backslash \text{GL}_d(\mathbb{A}_F)$. We can then form the tensor product representation $\pi^{\otimes m}$ of $A_M \backslash M(\mathbb{A}_F)$. Let

$$L^2_{\text{cusp}}(\pi^{\otimes m}) \leq L^2_{\text{cusp}}([M])$$

be the $\pi^{\otimes m}$ -isotypic subspace. We can then consider a form

$$\varphi \in \text{Ind}_P^G(L^2_{\text{cusp}}(\pi^{\otimes m}))^0$$

and the corresponding Eisenstein series

$$E(x, \varphi, \lambda).$$

Jacquet constructed [Jac84] an irreducible subrepresentation (σ, m) of $L^2([\text{GL}_n])$ by taking an iterated residue of $E(x, \varphi, \lambda)$. If we identify $\mathfrak{a}_M^* = \mathbb{R}^m$ as in (10.31) then the iterated residue is taken at

$$\left(\frac{m-1}{2}, \frac{m-3}{2}, \dots, \frac{3-m}{2}, \frac{1-m}{2} \right).$$

The representation (σ, m) is called a **Speh representation**, since Speh constructed the local analogues of these representations (see [Spe83]). In terms of isobaric sums, one has

$$(\sigma, m) = \boxplus_{i=1}^m \left(\sigma \otimes |\cdot|^{(m+1)/2-i} \right)$$

(see [MW89]).

To get a feel for these representations, it is useful to note that if v is a non-Archimedean place of F where σ is unramified then

$$L(s, (\sigma, m)_v) = \prod_{i=1}^m L\left(s - i + \frac{m+1}{2}, \sigma_v\right).$$

Here $L(s, (\sigma, m)_v)$ is the L -function of $(\sigma, m)_v$ (see §11.8). Jacquet then conjectured the following theorem, which was proven in [MW89]:

Theorem 10.8.1 (Mœglin and Waldspurger) *Any irreducible subrepresentation of $L^2([\mathrm{GL}_n])$ is isomorphic to a Speh representation (σ, m) for a unique factorization $md = n$ and a unique cuspidal automorphic representation σ of $A_{\mathrm{GL}_d} \backslash \mathrm{GL}_d(\mathbb{A}_F)$.* $\square$

To reduce possible confusion, we observe that a subrepresentation of $L^2([\mathrm{GL}_n])$ is (by definition) a subspace of $L^2_{\mathrm{disc}}([\mathrm{GL}_n])$. The theorem gives a complete description of $L^2_{\mathrm{disc}}([\mathrm{GL}_n])$. More specifically, it implies that

$$L^2_{\mathrm{disc}}([\mathrm{GL}_n]) \cong \bigoplus_{d|n} \widehat{\bigoplus_{\sigma} V(\sigma, n/d)}, \quad (10.36)$$

where the inner Hilbert space direct sum is over isomorphism classes of cuspidal automorphic representations of $A_{\mathrm{GL}_d} \backslash \mathrm{GL}_d(\mathbb{A}_F)$ and $V(\sigma, n/d)$ is the space of $(\sigma, n/d)$.

Theorem 10.8.1 forms the basis of much of what we know about the discrete spectrum for other groups. For example, it together with the Jacquet-Langlands correspondence is used to describe the discrete spectrum of inner forms of GL_n in [Bad08]. As a consequence of the theory of twisted endoscopy, Arthur [Art13] and Mok [Mok15] have given a description of the discrete spectrum of quasi-split classical groups in terms of the parametrization of Mœglin and Waldspurger. This is explained in §13.8.

Exercises

10.1. Let G be a reductive group and let Z_G^t be the maximal torus of Z_G . Show that the map

$$X^*(G) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow X^*(Z_G^t) \otimes_{\mathbb{Z}} \mathbb{Q}$$

induced by restriction is an isomorphism. Give an example to show that the restriction map

$$X^*(G) \longrightarrow X^*(Z_G^t)$$

need not be surjective.

10.2. Prove that the subspace $\text{Ind}_P^G(V)^0 \leq \text{Ind}_P^G(V)$ of (10.8) is dense.

10.3. Prove the intertwining relation (10.10).

10.4. Prove that $I(\sigma, \lambda)$ is unitary for $\lambda \in \mathfrak{ia}_M$.

10.5. With notation as in Example 10.3 prove that the trivial representation is not a quotient of $\text{Ind}_P^{\text{GL}_4}(\pi_1 \otimes \pi_2)$.

For the remaining problems, let $B \leq \text{GL}_2$ be the Borel subgroup of upper triangular matrices, 1 denote the trivial representation of $B(\mathbb{A}_F)$, and let

$$\pi := I(1, (\tfrac{1}{2}, -\tfrac{1}{2}));$$

it is a representation of $\text{GL}_2(\mathbb{A}_F)$.

10.6. Prove that every irreducible subquotient of $I(1, (\frac{1}{2}, -\frac{1}{2}))$ is automorphic and that the (unique) isobaric subquotient is the trivial representation of $\text{GL}_2(\mathbb{A}_F)$.

10.7. For all places v of F , prove that π_v has a composition series of length 2. The unique irreducible subrepresentation St_v is known as the **Steinberg representation**.

10.8. Prove that the irreducible subquotients of π are in bijection with finite sets of places of F : the set S of places corresponds to the representation $(\otimes_{v \in S} \text{St}_v) \otimes \pi^S$.

Chapter 11

Rankin-Selberg L -functions



Artin and Hecke were together at Göttingen, but neither realized the intimate connection between the two different types of L -functions they were constructing. The moral of the story is to talk with your colleagues.

L. Saper

In this chapter we sketch the theory of generic representations and Rankin-Selberg L -functions.

11.1 Paths to the construction of automorphic L -functions

Let F be a global field and let π and π' be a pair of cuspidal automorphic representations of $A_{\mathrm{GL}_n} \backslash \mathrm{GL}_n(\mathbb{A}_F)$ and $A_{\mathrm{GL}_m} \backslash \mathrm{GL}_m(\mathbb{A}_F)$, respectively. An important analytic invariant of this pair is the **Rankin-Selberg L -function**

$$L(s, \pi \times \pi').$$

This is a meromorphic function on the complex plane satisfying a functional equation whose poles can be explicitly described in terms of π and π' . These L -functions play a crucial role in automorphic representation theory. In particular they are built into the statement of the local Langlands correspondence for GL_n . This will be discussed in §12.5. We refer to §13.6 for the interpretation of Rankin-Selberg L -functions in terms of Langlands L -functions (to be defined in §12.7).

In this chapter we describe one method by which Rankin-Selberg L -functions can be defined, namely, the **Rankin-Selberg method**. A more detailed treatment is contained in [Cog07], which was our primary reference for this chapter. In preparation for the discussion, we define **generic representations** in §11.3. We have concentrated on the case of general linear groups in our treatment, but the Rankin-Selberg method can be applied in much greater generality. To learn about further developments [Bum05] is a place to start.

There are other approaches to defining these L -functions. One is via the so-called **Langlands-Shahidi method**. This method also uses the notion of a generic representation. The idea is that Rankin-Selberg L -functions (and some other L -functions) occur in the constant terms of certain Eisenstein series. This approach was historically important because it suggested both the general definition of a Langlands L -function (discussed in §12.7) and the general formulation of Langlands functoriality. We do not discuss this construction. It is discussed at length in the book [Sha10].

In the special case where π' is the trivial representation 1 of $\mathrm{GL}_1(\mathbb{A}_F)$

$$L(s, \pi) := L(s, \pi \times 1)$$

is known as the **principal** or **standard** L -function of π . There is an alternate approach to defining these L -functions due to Godement and Jacquet [GJ72]. It is a direct generalization of Tate's construction of the L -functions of Hecke characters in his famous thesis [Tat67], which in turn is an adelic formulation of the classical work of Hecke.

11.2 Generic characters

Let G be a quasi-split reductive group over a global or local field F , let $B \leq G$ be a Borel subgroup with unipotent radical N , and let $T \leq B$ be a maximal split torus. Let $\Phi(G, T)$ be the set of roots of T in G and let $\Delta \subset \Phi(G, T)$ be the base associated to B (see §1.9). For each $\alpha \in \Delta$, there is a corresponding root group $N_\alpha \leq N$ (see (1.45)).

Definition 11.2.1. If F is a local field, a character $\psi : N(F) \rightarrow \mathbb{C}^\times$ is called **generic** if $\psi|_{N_\alpha(F)}$ is nontrivial for each simple root $\alpha \in \Delta$. If F is global, a character $\psi : N(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ trivial on $N(F)$ is called **generic** if ψ_v is generic for all places v .

Example 11.1. Take $G = \mathrm{GL}_n$ and let $B_n \leq \mathrm{GL}_n$ be the Borel subgroup of upper triangular matrices. Moreover let $N_n \leq B_n$ be the unipotent radical. If F is a local field, $\psi : F \rightarrow \mathbb{C}^\times$ is a nontrivial character, and $m_1, \dots, m_{n-1} \in F$, then

$$N_n(F) \longrightarrow \mathbb{C}^\times$$

$$\begin{pmatrix} 1 & x_{12} & * & * & * \\ & 1 & x_{23} & * & * \\ & & \ddots & \ddots & * \\ & & & 1 & x_{(n-1)n} \\ & & & & 1 \end{pmatrix} \longmapsto \psi(m_1 x_{12} + \cdots + m_{n-1} x_{(n-1)n}) \quad (11.1)$$

is a character of $N_n(F)$. Every character of $N_n(F)$ is of this form. Comparing this with the description of the root groups from (1.42), we see that the character is generic if and only if all of the m_i are nonzero. If F is global, any character of $N_n(\mathbb{A}_F)$ trivial on $N_n(F)$ is of the same form, where the m_i are in F and $\psi : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times$ is nontrivial. Again, the character is generic if and only if $\prod_{i=1}^{n-1} m_i \neq 0$. If all $m_i = 1$ we call this the **standard character** attached to ψ .

11.3 Generic representations

Let G be a quasi-split reductive group over a local field F , let $B \leq G$ be a Borel subgroup with unipotent radical N , and let $\psi : N(F) \rightarrow \mathbb{C}^\times$ be a character. Let (π, V) be an admissible representation of $G(F)$.

Definition 11.3.1. Assume F is non-Archimedean. A **ψ -Whittaker functional** on V is a linear functional $\lambda : V \rightarrow \mathbb{C}$ such that

$$\lambda(\pi(n)\varphi) = \psi(n)\lambda(\varphi)$$

for $\varphi \in V$ and $n \in N(F)$.

In this non-Archimedean setting, since V has the discrete topology, all linear functionals are continuous.

If F is Archimedean, we assume that V is a Hilbert space. After (4.9) we endowed V_{sm} with the structure of a Fréchet space. It therefore makes sense to discuss continuous linear functionals on V_{sm} .

Definition 11.3.2. Assume F is Archimedean. A **ψ -Whittaker functional** on V is a continuous linear functional $\lambda : V_{\text{sm}} \rightarrow \mathbb{C}$ such that

$$\lambda(\pi(n)\varphi) = \psi(n)\lambda(\varphi)$$

for $\varphi \in V_{\text{sm}}$ and $n \in N(F)$.

Assume, as above, that in the Archimedean setting (π, V) is unitary.

Definition 11.3.3. An irreducible admissible representation (π, V) of $G(F)$ is **ψ -generic** if it admits a nonzero ψ -Whittaker functional.

An irreducible admissible representation (π, V) is simply said to be **generic** if there exists a Borel subgroup $B \leq G$ with unipotent radical N and a generic character $\psi : N(F) \rightarrow \mathbb{C}^\times$ such that (π, V) is ψ -generic. We warn the reader that, in the older literature, generic representations were sometimes referred to as **nondegenerate representations**.

The following theorem [GK73, Sha74] has turned out to be extremely important:

Theorem 11.3.4 *Assume ψ is generic. The space of ψ -Whittaker functionals on an irreducible admissible representation of $G(F)$ is at most one dimensional.* $\square$

The assumption that ψ is generic cannot be removed (see Exercise 11.2).

We now define the related notion of a Whittaker function. The space of ψ -**Whittaker functions** on $G(F)$ is defined as

$$\mathcal{W}(\psi) := \{W \in C^\infty(G(F)) : W(n g) = \psi(n) W(g)\}$$

for all $(n, g) \in N(F) \times G(F)$. This space admits a natural action of $G(F)$:

$$\begin{aligned} G(F) \times \mathcal{W}(\psi) &\longrightarrow \mathcal{W}(\psi) \\ (g, W) &\longmapsto (x \mapsto W(xg)). \end{aligned}$$

Let (π, V) be an irreducible admissible representation of $G(F)$. When F is Archimedean we assume π is unitary. A ψ -**Whittaker model** of π is the image of a nonzero $G(F)$ -intertwining map

$$\Lambda : V_{\text{sm}} \longrightarrow \mathcal{W}(\psi) \tag{11.2}$$

such that the functional

$$\begin{aligned} V_{\text{sm}} &\longrightarrow \mathbb{C} \\ \varphi &\longmapsto \Lambda(\varphi)(1) \end{aligned} \tag{11.3}$$

is continuous. Here we give V_{sm} the Fréchet topology mentioned above when F is Archimedean. When F is non-Archimedean $V = V_{\text{sm}}$ is given the discrete topology so the continuity condition is vacuous. We observe that (11.3) is a Whittaker functional. Conversely, a Whittaker functional λ gives rise to a Whittaker model via

$$\begin{aligned} V_{\text{sm}} &\longrightarrow \mathcal{W}(\psi) \\ \varphi &\longmapsto (g \mapsto \lambda(\pi(g)\varphi)). \end{aligned}$$

Thus a ψ -Whittaker model of π exists if and only if π is ψ -generic. Moreover we obtain the following corollary of Theorem 11.3.4:

Corollary 11.3.5 *If ψ is generic and π is ψ -generic then π admits a ψ -Whittaker model. The model is unique up to scaling the map (11.2) by a nonzero complex number. $\square$*

Motivated by Corollary 11.3.5, if ψ is generic and (π, V) is ψ -generic, we let

$$\mathcal{W}(\pi, \psi) \quad (11.4)$$

be the image of any Λ as in (11.2). We then have a map

$$\begin{aligned} V_{\text{sm}} &\longrightarrow \mathcal{W}(\pi, \psi) \\ \varphi &\longmapsto W_\psi^\varphi := \Lambda(\varphi). \end{aligned} \quad (11.5)$$

By Schur's lemma, this map is well defined up to multiplication by a nonzero complex number. Moreover it is an isomorphism. In the Archimedean case, we are deliberately avoiding putting a topology on $\mathcal{W}(\pi, \psi)$ other than the topology induced by Λ via transport of structure. For one approach to introducing a more natural topology we refer to [Wal92, Chapter 15].

Now assume that F is global, let π be a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ realized in a subspace $V \leq L_{\text{cusp}}^2([G])$, and let $\varphi \in V_{\text{sm}}$. Let $\psi : N(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ be a character trivial on $N(F)$. We then define the **global ψ -Whittaker function**

$$W_\psi^\varphi(g) := \int_{[N]} \varphi(n g) \bar{\psi}(n) dn. \quad (11.6)$$

This definition depends on the choice of invariant Radon measure dn on $[N]$, but only up to scaling by a positive real number. We say that (π, V) is **globally ψ -generic** if

$$W_\psi^\varphi(g) \neq 0$$

for some $\varphi \in V$ and $g \in G(\mathbb{A}_F)$. We remark that a priori this notion depends on the realization of π in $L_{\text{cusp}}^2([G])$ if it does not occur with multiplicity 1. It is not hard to check that if π is globally ψ -generic and $\pi \cong \otimes'_v \pi_v$ then each π_v is ψ_v -generic (see Exercise 11.3). As in the local setting, we simply say (π, V) is **generic** if there is a Borel subgroup $B \leq G$ with unipotent radical N and a generic character $\psi : N(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ trivial on $N(F)$ such that (π, V) is ψ -generic.

We can also define the notion of a global ψ -Whittaker model. Let

$$\mathcal{W}(\psi) := \{W \in C^\infty(A_G \backslash G(\mathbb{A}_F)) : W(n g) = \psi(n) W(g)\}. \quad (11.7)$$

Here $(n, g) \in N(\mathbb{A}_F) \times A_G \backslash G(\mathbb{A}_F)$. Suppose that (π, V) is globally ψ -generic. We have a $G(\mathbb{A}_F)$ -intertwining map

$$V_{\text{sm}} \longrightarrow \mathcal{W}(\psi)$$

$$\varphi \longmapsto W_{\psi}^{\varphi}.$$

The **global ψ -Whittaker model** of π is the image $\mathcal{W}(\pi, \psi)$ of this intertwining map.

The notion of a generic representation is only useful if we have an interesting supply of generic representations. Consider the case of GL_n . Let ψ and ψ' be generic characters of $N(\mathbb{A}_F)$ and $N'(\mathbb{A}_F)$, respectively, where N and N' are the unipotent radicals of two Borel subgroups of GL_n . By Exercise 11.1 a cuspidal automorphic representation of $A_{\mathrm{GL}_n} \backslash \mathrm{GL}_n(\mathbb{A}_F)$ is ψ -generic if and only if it is ψ' -generic. Therefore the following theorem implies that all cuspidal automorphic representations of $A_{\mathrm{GL}_n} \backslash \mathrm{GL}_n(\mathbb{A}_F)$ are ψ -generic with respect to any generic character ψ of the unipotent radical of any Borel subgroup. Moreover they admit an expansion in terms of Whittaker functionals.

Let $N_n \leq \mathrm{GL}_n$ be the unipotent radical of the Borel subgroup of upper triangular matrices.

Theorem 11.3.6 *Let $\varphi \in L_{\mathrm{cusp}}^2([\mathrm{GL}_n])$ be a smooth vector in the space of a cuspidal automorphic representation π of $A_{\mathrm{GL}_n} \backslash \mathrm{GL}_n(\mathbb{A}_F)$. If $\psi : N_n(\mathbb{A}_F) \rightarrow \mathbb{C}^{\times}$ is a generic character then one has that*

$$\varphi(g) = \sum_{\gamma \in N_{n-1}(F) \backslash \mathrm{GL}_{n-1}(F)} W_{\psi}^{\varphi} \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right) \quad (11.8)$$

for a unique choice of invariant measure on $[N_n]$. The sum converges absolutely and uniformly on compact subsets of $[G]$. $\square$

The expression for φ in (11.8) is called its **Whittaker expansion**. It is a generalization (but not the only generalization) of the well-known Fourier expansion of a modular form.

We now prove this theorem in the special case $n = 2$. Consider the function

$$x \longmapsto \varphi \left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g \right). \quad (11.9)$$

This is a continuous function on the compact abelian group $F \backslash \mathbb{A}_F$, and hence admits a Fourier expansion. If we fix a nontrivial character $\psi : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^{\times}$, then all other characters are of the form $\psi_{\alpha}(x) := \psi(\alpha x)$ for $\alpha \in F$ (see Lemma C.1.2). Thus the Fourier expansion of the function (11.9) is

$$\sum_{\alpha \in F} \psi(\alpha x) \int_{F \backslash \mathbb{A}_F} \varphi \left(\begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} g \right) \overline{\psi}(\alpha y) dy \quad (11.10)$$

for a unique Haar measure dy on $F \backslash \mathbb{A}_F$. Since φ is cuspidal, the $\alpha = 0$ term vanishes identically. Taking a change of variables $y \mapsto \alpha^{-1}y$ for each $\alpha \in F^{\times}$ and using the left $\mathrm{GL}_2(F)$ -invariance of φ , we see that (11.10) is equal to

$$\sum_{\alpha \in F^{\times}} \psi(\alpha x) W_{\psi}^{\varphi} \left(\begin{pmatrix} \alpha & \\ & 1 \end{pmatrix} g \right). \quad (11.11)$$

Setting $x = 0$ we obtain Theorem 11.3.6 when $n = 2$. We will not prove Theorem 11.3.6 in the higher rank case because we cannot improve on the exposition in [Cog07, §1.1]. The basic idea is to proceed as in the $n = 2$ case, using abelian Fourier analysis on certain abelian subgroup of $N_n(\mathbb{A}_F)$. However, since $N_n(\mathbb{A}_F)$ itself is no longer abelian for $n \geq 3$ one has to combine this with a clever inductive argument using the fact that φ is cuspidal.

11.4 Formulae for Whittaker functions

In this section we explain how to compute Whittaker functions, starting from the local unramified situation.

Until otherwise specified, let F be a non-Archimedean local field unramified over its prime field. Let ϖ be a uniformizer of F . We say that a character $\psi : F \rightarrow \mathbb{C}^\times$ is **unramified** if $\psi|_{\varpi^{-1}\mathcal{O}_F}$ is nontrivial and $\psi|_{\mathcal{O}_F}$ is trivial. Let $\psi : F \rightarrow \mathbb{C}^\times$ be an unramified character. The results we will state in this section are valid for any unramified reductive group over F , but to simplify our discussion we will assume that G is a split reductive group. Let $B \leq G$ be a Borel subgroup with split maximal torus $T \leq B$. We let $K \leq G(F)$ be a hyperspecial subgroup in good position with respect to (B, T) . We note that by the Iwasawa decomposition one has

$$G(F) = \coprod_{\mu \in X_*(T)} N(F)\mu(\varpi)K, \quad (11.12)$$

where N is the unipotent radical of B .

Using the notation of §7.6, for $\lambda \in \mathfrak{a}_{T\mathbb{C}}^*$ we can form the induced representation $I(\lambda)$, where the induction is with respect to the Borel subgroup B . Its unique unramified subquotient is denoted by $J(\lambda)$. Any unramified representation is equivalent to such a $J(\lambda)$. Assume that $J(\lambda)$ is generic. Then the space

$$\mathcal{W}(J(\lambda), \psi)^K$$

is 1 dimensional by Corollary 5.5.6. An element of the 1-dimensional space $\mathcal{W}(J(\lambda), \psi)^K$ is fixed by K on the right and transforms according to ψ under the action of $N(F)$ on the left. Thus it is determined by its value on $T(F)$.

The choice of T and B , by duality, gives rise to a maximal torus and Borel subgroup $\widehat{T} \leq \widehat{B} \leq \widehat{G}$ as explained in §7.3. We have $X^*(\widehat{T}) = X_*(T)$. For the purpose of Theorem 11.4.1, we say that $\mu \in X_*(T)$ is **dominant** if the corresponding character in $X^*(\widehat{T})$ is a dominant weight with respect to $\widehat{B}$. By Cartan-Weyl theory, if μ is dominant there is a unique isomorphism class $V(\mu)$ of irreducible representation of $\widehat{G}$ with μ as its highest weight. Let χ_μ be the character of $V(\mu)$.

Theorem 11.4.1 *Any nonzero W in the 1-dimensional space $\mathcal{W}(J(\lambda), \psi)^K$ satisfies $W(I) \in \mathbb{C}^\times$, where $I \in G(F)$ is the identity element. If μ is a dominant weight then*

$$\frac{W(\mu(\varpi))}{W(I)} = \delta_B^{1/2}(\mu(\varpi))\chi_\mu(q^{-\lambda}).$$

If μ is not a dominant weight then $W(\mu(\varpi)) = 0$. □

Here $q^{-\lambda} \in \widehat{G}(\mathbb{C})$ is the Langlands class of $J(\lambda)$ as in Theorem 7.6.10.

Theorem 11.4.1 was originally proved in unpublished work of Kato after being conjectured by Langlands. A new proof together with a generalization to unramified groups is contained in [CS80]. The formula is commonly referred to as the **Casselman-Shalika formula**. In [FGKV98, Theorem 5.3] one can find the formula in the form stated in Theorem 11.4.1. To deduce the expression in Theorem 11.4.1 from the expression in [CS80] one uses the Weyl character formula.

It is instructive to explicate Theorem 11.4.1 when $G = \mathrm{GL}_n$. In this case it is due to Shintani [Shi76]. We assume that $B = B_n$ is the Borel subgroup of upper triangular matrices and $T = T_n \leq B_n$ is the maximal torus of diagonal matrices. In this case weights can be identified with tuples $(k_1, \dots, k_n) \in \mathbb{Z}^n$ as (1.39). The cocharacter μ attached to such a tuple is (1.40). The dominant weights are those with

$$k_1 \geq \dots \geq k_n.$$

The associated irreducible representation of GL_n is $\mathbb{S}_{k_1, \dots, k_n}(V_{\mathrm{st}})$, where V_{st} is the standard representation and $\mathbb{S}_{k_1, \dots, k_n}$ is the Schur functor attached to the partition $k_1, \dots, k_n$ of $\sum_{i=1}^n k_i$. If we identify T_n with $\mathbb{G}_m^n$ in the natural manner then the character of $\mathbb{S}_{k_1, \dots, k_n}(V_{\mathrm{st}})$ is the restriction to $\mathbb{G}_m^n$ of the Schur polynomial $\mathbb{S}_{k_1, \dots, k_n} : \mathbb{G}_a^n \rightarrow \mathbb{G}_a$. Normalize $W \in \mathcal{W}(J(\lambda), \psi)^{\mathrm{GL}_n(\mathcal{O}_F)}$ so that $W(I) = 1$. Then in this case Theorem 11.4.1 reads

$$W \begin{pmatrix} \varpi^{k_1} & & \\ & \ddots & \\ & & \varpi^{k_n} \end{pmatrix} = \begin{cases} \delta_{B_n}^{1/2}(\mu(\varpi)) \mathbb{S}_{k_1, \dots, k_n}(q^{-\lambda_1}, \dots, q^{-\lambda_n}) & \text{if } k_1 \geq \dots \geq k_n \\ 0 & \text{otherwise.} \end{cases} \quad (11.13)$$

Here $\{q^{-\lambda_i} : 1 \leq i \leq n\}$ are the eigenvalues of $q^{-\lambda}$.

We now return to the global setting. Let (π, V) be a globally generic cuspidal representation of a quasi-split reductive group G , realized on a closed subspace $V \leq L_{\mathrm{cusp}}^2([G])$. By Flath's theorem (Theorem 5.7.1) and Theorem 6.6.3, we have a factorization

$$V_{\mathrm{sm}} \cong V_{\infty \mathrm{sm}} \otimes (\otimes'_{v \nmid \infty} V_v). \quad (11.14)$$

In the function field case, $V_{\infty \mathrm{sm}} \cong \otimes_{v|\infty} V_v$ by Theorem 5.7.2. In the number field case, by [Fol95, Theorem 7.25] and Theorem 3.10.2 we have

$$V_\infty \cong \widehat{\otimes}_{v|\infty} V_v$$

for some irreducible unitary representations (π_v, V_v) . Here the V_v are Hilbert spaces and the tensor product is the (completed) tensor product of Hilbert spaces. It follows that

$$V_{\infty\text{sm}} \cong \widehat{\otimes}_{v|\infty} V_{v\text{sm}}, \quad (11.15)$$

where the completed tensor product is defined with respect to the Fréchet topology described after (4.9). Here we may take the completed tensor product to be the completion of the algebraic tensor product with respect to either the projective or inductive topology since these two topologies coincide on the algebraic tensor product of two Fréchet spaces [War72, Appendix 2.2]. Thus we have an isomorphism

$$\iota : V_{\text{sm}} \xrightarrow{\sim} (\widehat{\otimes}_{v|\infty} V_{v\text{sm}}) \otimes (\otimes'_{v\nmid\infty} V_v), \quad (11.16)$$

where the hat should be regarded as merely an algebraic tensor product in the function field case.

Suppose that the spaces V_v are subspaces of a space of functions

$$\{\varphi_v : G(F_v) \longrightarrow \mathbb{C}\}.$$

For example, when V_v is unramified it can be realized as functions in a suitable principal series representation by Theorems 7.6.9 and 7.6.10. We point out that the fact that the isomorphism (11.16) exists does not imply that we can write $\varphi(g)$ as $\prod_v \varphi_v(g_v)$ for some collection of φ_v . However, remarkably, if we pass to the Whittaker model, an analogous statement turns out to be true as we now explain.

Fix isomorphisms

$$\begin{aligned} V_{v\text{sm}} &\longrightarrow \mathcal{W}(\pi_v, \psi_v) \\ \varphi_v &\longmapsto W_{\psi_v}^{\varphi_v} \end{aligned}$$

for all v (here $V_v = V_{v\text{sm}}$ if v is non-Archimedean). Let $K = \prod_v K_v \leq G(\mathbb{A}_F)$ be a maximal compact subgroup. For almost all places, K_v is hyperspecial and V_v contains a unique line of K_v -fixed vectors. Hence the same is true of $\mathcal{W}(\pi_v, \psi_v)$. There is then a unique $W_{v0} \in \mathcal{W}(\pi_v, \psi_v)$ such that $W_{v0}(I) = 1$, where $I \in G(F_v)$ is the identity. Here we are using the fact that any nonzero unramified element W of $\mathcal{W}(\pi_v, \psi_v)$ satisfies $W(I) \neq 0$. This was stated in the split case in Theorem 11.4.1 and is still true in the quasi-split case [CS80]. The restricted direct product (11.14) depends on a choice of nonzero vector $\varphi_{v0} \in V_v^{K_v}$ for all but finitely many v . By multiplying by suitable elements of $\mathbb{C}^\times$ we can assume that $W_{\psi_v}^{\varphi_{v0}} = W_{v0}$ whenever K_v is hyperspecial.

Consider the restricted direct product

$$(\widehat{\otimes}_{v|\infty} \mathcal{W}(\pi_v, \psi_v)) \otimes (\otimes'_{v \nmid \infty} \mathcal{W}(\pi_v, \psi_v)) \quad (11.17)$$

with respect to the W_{v0} . Here when F is a number field the tensor product over the infinite places is completed with respect to the topologies on the $\mathcal{W}(\pi_v, \psi_v)$ obtained by transport of structure from $V_{v\text{sm}}$. In the function field case, it should just be understood as an algebraic direct product. Consider the restricted direct product $\otimes'_v \mathcal{W}(\pi_v, \psi_v)$ over all places, where we do not take the completed tensor product at the infinite places. The space $\otimes'_v \mathcal{W}(\pi_v, \psi_v)$ is naturally a subspace of $C^\infty(G(\mathbb{A}_F))$ and one can check that the same is true of (11.17).

Proposition 11.4.2 *One has that*

$$\mathcal{W}(\pi, \psi) = (\widehat{\otimes}_{v|\infty} \mathcal{W}(\pi_v, \psi_v)) \otimes (\otimes'_{v \nmid \infty} \mathcal{W}(\pi_v, \psi_v)).$$

In particular, if $\iota(\varphi) = \otimes'_v \varphi_v$ then there is a $c \in \mathbb{C}^\times$ such that

$$W_\psi^\varphi(g) = c \prod_v W_{\psi_v}^{\varphi_v}(g_v)$$

for all $g \in G(\mathbb{A}_F)$. Here ι the isomorphism of (11.16).

Thus to compute the global Whittaker function W_ψ^φ it suffices to compute the local Whittaker functions $W_{\psi_v}^{\varphi_v}$.

Proof. Given the isomorphism (11.16), to prove the first equality it suffices to prove the second. Thus assume $\iota(\varphi) = \otimes'_v \varphi_v$.

Upon replacing φ by $R(g)\varphi$ for some $g \in G(\mathbb{A}_F)$ we can assume that $W_\psi^\varphi(I)$ is nonzero. Choose a finite set S of places of F large enough that for $v \notin S$ the group K_v is hyperspecial, $g_v \in K_v$, and $\varphi_v = \varphi_{v0}$. We claim that $W_{\psi_v}^{\varphi_v}(I) \neq 0$ for all $v \in S$ and

$$\frac{W_\psi^\varphi(g_S I^S)}{W_\psi^\varphi(I)} = \prod_{v \in S} \frac{W_{\psi_v}^{\varphi_v}(g_v)}{W_{\psi_v}^{\varphi_v}(I)}. \quad (11.18)$$

Assuming (11.18), then

$$\frac{W_\psi^\varphi(g)}{W_\psi^\varphi(I)} = \frac{W_\psi^\varphi(g_S I^S)}{W_\psi^\varphi(I)} = \prod_{v \in S} \frac{W_{\psi_v}^{\varphi_v}(g_v)}{W_{\psi_v}^{\varphi_v}(I)} = \prod_v \frac{W_{\psi_v}^{\varphi_v}(g_v)}{W_{\psi_v}^{\varphi_v}(I)}$$

since φ is fixed by K^S and we have normalized so that $W_{\psi_v}^{\varphi_{v0}}(I) = 1$ for $v \notin S$. Thus the claim implies the proposition.

Consider the linear functionals

$$\lambda_{1S}, \lambda_{2S} : \otimes_{v \in S} V_v \longrightarrow \mathbb{C}$$

given one pure tensors by

$$\begin{aligned}\lambda_{1S}(\otimes_{v \in S} \varphi'_v) &:= W_\psi^{t^{-1}(\varphi'_S \otimes (\otimes'_{v \notin S} \varphi_v))}(I), \\ \lambda_{2S}(\otimes_{v \in S} \varphi'_v) &:= \prod_{v \in S} W_{\psi_v}^{\varphi'_v}(I).\end{aligned}$$

It follows from Theorem 11.3.4 that there is a $c \in \mathbb{C}^\times$ such that $\lambda_{1S} = c\lambda_{2S}$. This implies the claim. $\square$

As an application of Proposition 11.4.2 and Theorem 11.3.6, we obtain the following theorem of Shalika [Sha74, Theorem 5.5]:

Theorem 11.4.3 (Multiplicity one) *An irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_F)$ occurs with multiplicity at most one in $L^2_{\mathrm{cusp}}([\mathrm{GL}_n])$.*

We note that Theorem 11.4.3 is false for essentially every reductive group that is not a general linear group.

Proof. Let (π_1, V_1) and (π_2, V_2) with $V_i \leq L^2_{\mathrm{cusp}}([\mathrm{GL}_n])$ be two realizations of a given cuspidal automorphic representation (π, V) . Choose equivariant maps $L_i : V \rightarrow V_i$. We then obtain Whittaker functionals

$$\begin{aligned}\lambda_i : V_{\mathrm{sm}} &\longrightarrow \mathbb{C} \\ \varphi &\longmapsto W_\psi^{L_i(\varphi)}(I_n),\end{aligned}$$

where I_n is the identity element. They are nonzero by Theorem 11.3.6. By Theorem 11.3.4 and Proposition 11.4.2 one therefore has that $\lambda_1 = c\lambda_2$ for some $c \in \mathbb{C}^\times$. Thus

$$\begin{aligned}L_1(\varphi)(g) &= \sum_{\gamma \in N_{n-1}(F) \backslash \mathrm{GL}_{n-1}(F)} W_\psi^{L_1(\varphi)}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \\ &= c \sum_{\gamma \in N_{n-1}(F) \backslash \mathrm{GL}_{n-1}(F)} W_\psi^{L_2(\varphi)}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right) \\ &= cL_2(\varphi)(g).\end{aligned}$$

This implies that V_1 and V_2 have nonzero intersection and hence they are equal. $\square$

11.5 Local Rankin-Selberg L -functions

Our goal in this section is to define the family of local Rankin-Selberg integrals and state their functional equations. The definition of this family was originally given in [JPSS83] in the non-Archimedean case and the Archimedean case was treated definitively in [Jac09]. We refer to these papers for proofs of the unproved statements we make below. At least in the non-Archimedean case, Rankin-Selberg L -functions are then defined to be the

reciprocals of the smallest polynomial functions in q^{-s} that cancel the poles of the family of local Rankin-Selberg integrals. One takes a slightly different approach to define Rankin-Selberg integrals in the Archimedean case. We will be more precise in both cases below.

Let F be a local field and let $\psi : F \rightarrow \mathbb{C}^\times$ be a nontrivial character. Let π and π' be irreducible admissible generic representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$, respectively. Let $\psi : F \rightarrow \mathbb{C}^\times$ be an additive character. Use it to define a standard generic character of $N_n(F)$ and $N_m(F)$ as in Example 11.1. We denote these characters again by ψ . Let

$$W \in \mathcal{W}(\pi, \psi) \quad \text{and} \quad W' \in \mathcal{W}(\pi', \bar{\psi}).$$

If $m < n$ let

$$\begin{aligned} \Psi(s; W, W') &:= \int W \left(\begin{smallmatrix} h & & \\ & I_{n-m} & \\ & & 1 \end{smallmatrix} \right) W'(h) |\det(h)|^{s-(n-m)/2} dh, \\ \tilde{\Psi}(s; W, W') &:= \int \int W \left(\begin{smallmatrix} h & & & \\ & I_{n-m-1} & & \\ & & x & \\ & & & 1 \end{smallmatrix} \right) dx W'(h) |\det h|^{s-(n-m)/2} dh, \end{aligned} \quad (11.19)$$

where the top integral is over $N_m(F) \backslash \mathrm{GL}_m(F)$, the outer integral on the bottom is over $N_m(F) \backslash \mathrm{GL}_m(F)$, and the inner integral on the bottom is over $(n-m-1) \times m$ matrices with entries in F .

If $m = n$, then for each $\Phi \in \mathcal{S}(F^n)$, the Schwartz space, let

$$\Psi(s; W, W', \Phi) := \int_{N_n(F) \backslash \mathrm{GL}_n(F)} W(g) W'(g) \Phi(e_n g) |\det g|^s dg. \quad (11.20)$$

Here $e_n \in F^n$ is the elementary vector with 0's in the first $n-1$ entries and 1 in the last entry. The expressions in (11.19) and (11.20) are known as **local Rankin-Selberg integrals**. They converge absolutely for $\mathrm{Re}(s)$ sufficiently large.

Proposition 11.5.1 *Assume that F is non-Archimedean.*

- (a) *For π and π' unitary, (11.19) and (11.20) converge absolutely for $\mathrm{Re}(s) \geq 1$. For π and π' tempered, they converge absolutely for $\mathrm{Re}(s) > 0$.*
- (b) *The integrals (11.19) and (11.20) are rational functions of q^{-s} .*
- (c) *There is a unique polynomial $P_{\pi, \pi'} \in \mathbb{C}[x]$ satisfying $P_{\pi, \pi'}(0) = 1$ such that*

$$\begin{aligned} &\langle \Psi(s; W, W') : (W, W') \in \mathcal{W}(\pi, \psi) \times \mathcal{W}(\pi', \bar{\psi}) \rangle \\ &= \langle \tilde{\Psi}(s; W, W') : (W, W') \in \mathcal{W}(\pi, \psi) \times \mathcal{W}(\pi', \bar{\psi}) \rangle \\ &= P_{\pi, \pi'}(q^{-s})^{-1} \mathbb{C}[q^s, q^{-s}] \end{aligned}$$

when $m \neq n$ and

$$\langle \Psi(s; W, W', \Phi) : (W, W', \Phi) \in \mathcal{W}(\pi, \psi) \times \mathcal{W}(\pi', \bar{\psi}) \times \mathcal{S}(F^n) \rangle$$

$$= P_{\pi, \pi'}(q^{-s})^{-1} \mathbb{C}[q^s, q^{-s}]$$

when $m = n$. □

In the proposition the angle brackets denote the $\mathbb{C}$ -linear span of the given functions in $\mathbb{C}(q^{-s})$. One sets

$$L(s, \pi \times \pi') := P_{\pi, \pi'}(q^{-s})^{-1}. \quad (11.21)$$

In the Archimedean case, one actually defines $L(s, \pi \times \pi')$ using the Archimedean local Langlands correspondence, which was established at an early stage in the subject by Langlands himself. We will explain this in more detail in §12.3. At the moment it is enough to know that $L(s, \pi \times \pi')$ is a meromorphic function of s that does not vanish (see §12.2 and 12.3).

We note the following important bound:

Theorem 11.5.2 *If π and π' are unitary, then $L(s, \pi \times \pi')$ is holomorphic and nonzero for $\operatorname{Re}(s) \geq 1$.* □

This follows from Proposition 11.5.1 in the non-Archimedean case. We refer to [BR94a, §2] for a proof in the Archimedean case.

A holomorphic function $f(s)$ of $s \in \mathbb{C}$ is **bounded in vertical strips** if it is bounded in

$$V_{\sigma_1, \sigma_2} := \{s : \sigma_1 \leq \operatorname{Re}(s) \leq \sigma_2\}$$

for all $\sigma_1 < \sigma_2 \in \mathbb{R}$. A meromorphic function $f(s)$ is bounded in vertical strips if for each $\sigma_1 < \sigma_2$ there is a polynomial $P_{\sigma_1, \sigma_2} \in \mathbb{C}[x]$ such that $P_{\sigma_1, \sigma_2}(s)f(s)$ is holomorphic and bounded in vertical strips in the original sense.

Theorem 11.5.3 *Assume F is Archimedean. The local Rankin-Selberg integrals (11.19) and (11.20) admit meromorphic continuations to functions of s that are bounded in vertical strips.* □

Let

$$w_n := \begin{pmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{pmatrix} \quad \text{and} \quad w_{n,m} := \begin{pmatrix} I_m & \\ & w_{n-m} \end{pmatrix}.$$

For functions W, W' on $\operatorname{GL}_n(F), \operatorname{GL}_m(F)$, respectively, let

$$\begin{aligned} \widetilde{W}(g) &:= W(w_n g^{-t}), \\ \widetilde{W}'(g) &:= W'(w_m g^{-t}), \\ R(w_{n,m})\widetilde{W}(x) &:= \widetilde{W}(x w_{n,m}). \end{aligned}$$

Theorem 11.5.4 *There is a meromorphic function $\gamma(s, \pi \times \pi', \psi)$ such that if $m < n$ then*

$$\widetilde{\Psi}(1-s; R(w_{n,m})\widetilde{W}, \widetilde{W}') = \omega'(-1)^{n-1} \gamma(s, \pi \times \pi', \psi) \Psi(s; W, W')$$

and if $m = n$ then

$$\Psi(1-s; \widetilde{W}, \widetilde{W}', \widehat{\Phi}) = \omega'(-1)^{n-1} \gamma(s, \pi \times \pi', \psi) \Psi(s; W, W', \Phi)$$

for all $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \bar{\psi})$. Here ω' is the central quasi-character of π' . If F is non-Archimedean, then $\gamma(s, \pi \times \pi', \psi) \in \mathbb{C}(q^{-s})$. $\square$

Here

$$\widehat{\Phi}(x_1, \dots, x_n) = \int_{F^n} \Phi(y_1, \dots, y_n) \psi \left(\sum_{i=1}^n x_i y_i \right) dy_1 \dots dy_n$$

is the Fourier transform of Φ . Due to alternate notational conventions the functional equations in the Archimedean case presented in [Jac09] are different.

We need one other factor, the local ε -factor:

$$\varepsilon(s, \pi \times \pi', \psi) := \frac{\gamma(s, \pi \times \pi', \psi) L(s, \pi \times \pi')}{L(1-s, \pi^\vee \times \pi'^\vee)}. \quad (11.22)$$

Proposition 11.5.5 *In the non-Archimedean case, $\varepsilon(s, \pi \times \pi', \psi)$ is a function of the form cq^{-fs} for some real number f and $c \in \mathbb{C}^\times$.* $\square$

The ε -factor is a useful arithmetic invariant of the pair (π, π') . Assume that π' is trivial. In this case $f \in \mathbb{Z}_{\geq 0}$ and the ideal $\varpi^f \mathcal{O}_F$ is called the **conductor** of π . One writes $f := f(\pi)$, and calls it the **exponent of the conductor**, or sometimes just the conductor. Let

$$K_1(\varpi^{f(\pi)}) := \left\{ g \in \mathrm{GL}_n(\mathcal{O}_F) : g \equiv \begin{pmatrix} & * \\ * & \vdots \\ 0 \dots 0 & 1 \end{pmatrix} \pmod{\varpi^{f(\pi)}} \right\}.$$

Then one has the following (see [Mat13]):

Theorem 11.5.6 *The number $f(\pi)$ is the smallest nonnegative integer such that the space V of π satisfies*

$$V^{K_1(\varpi^{f(\pi)})} \neq 0.$$

Moreover, $\dim_{\mathbb{C}} V^{K_1(\varpi^{f(\pi)})} = 1$. $\square$

11.6 Unramified Rankin-Selberg L -functions

In this section we assume that F is non-Archimedean and unramified over its prime field, that $\psi : F \rightarrow \mathbb{C}^\times$ is an unramified character, and that π and π' are irreducible unramified generic representations of $\mathrm{GL}_n(F)$ and $\mathrm{GL}_m(F)$, respectively. Let $W \in \mathcal{W}(\pi, \psi)^{\mathrm{GL}_n(\mathcal{O}_F)}$ and $W' \in \mathcal{W}(\pi', \psi)^{\mathrm{GL}_m(\mathcal{O}_F)}$ be the unique vectors satisfying $W(I_n) = W'(I_m) = 1$. Let $T_m \leq \mathrm{GL}_m$ be the maximal torus of diagonal matrices. We assume that $\pi \cong J(\lambda)$ and $\pi' \cong J(\lambda')$, where $\lambda \in \mathfrak{a}_{T_n\mathbb{C}}^*$ and $\lambda' \in \mathfrak{a}_{T_m\mathbb{C}}^*$.

Our aim in this section is to prove the following identity:

Theorem 11.6.1 *For $\mathrm{Re}(s)$ sufficiently large, if $n > m$ one has that*

$$\Psi(s; W, W') = \det \left(I_{mn} - q^{-s} q^{-\lambda} \otimes q^{-\lambda'} \right)^{-1}.$$

If $m = n$ one has that

$$\Psi(s; W, W', \mathbb{1}_{\mathcal{O}_F^n}) = \det \left(I_{n^2} - q^{-s} q^{-\lambda} \otimes q^{-\lambda'} \right)^{-1}.$$

Implicit in the definition of the local Rankin-Selberg integrals is a choice of measure on $\mathrm{GL}_m(F)$ and $N_m(F)$ so that we can form a quotient measure on $N_m(F) \backslash \mathrm{GL}_m(F)$. In order for the equalities above to be valid, we must choose the unique Haar measures that assign volume 1 to $\mathrm{GL}_m(\mathcal{O}_F)$ and $N_m(\mathcal{O}_F)$, respectively.

Proof. For each $m \in \mathbb{Z}_{\geq 1}$ let $T^+(m)$ be the set of the tuples $\mu = (k_1, \dots, k_m) \in \mathbb{Z}^m$ with $k_1 \geq \dots \geq k_m \geq 0$. This can be identified with a subset of the dominant weights of $\hat{T}_m$ (which in turn are cocharacters of T_m , see Example 1.14). If $n > m$ let

$$T^+(m) \hookrightarrow T^+(n)$$

be the injection given by extending the tuple $k_1, \dots, k_m$ by adding zeros.

Assume for the moment that $n > m$. Then by (11.13) we have

$$\begin{aligned} \Psi(s; W, W') &= \sum_{\mu \in T^+(m)} W \left(\begin{pmatrix} \mu(\varpi) \\ I_{n-m} \end{pmatrix} W'(\mu(\varpi)) | \det(\mu(\varpi)) |^{s-(n-m)/2} \delta_{B_m}^{-1}(\mu(\varpi)) \right) \\ &= \sum_{\mu \in T^+(m)} \mathbb{S}_{k_1, \dots, k_m, 0, \dots, 0}(q^{-\lambda_1}, \dots, q^{-\lambda_n}) \mathbb{S}_{k_1, \dots, k_m}(q^{-\lambda'_1}, \dots, q^{-\lambda'_m}) q^{-|\mu|s} \\ &= \det(I_{mn} - q^{-s} q^{-\lambda'} \otimes q^{-\lambda})^{-1}, \end{aligned}$$

where we let $|\mu| = k_1 + \dots + k_m$. This last equality is known as the **Cauchy identity** [Bum13, Chapter 38]. Similarly, when $n = m$ we have

$$\Psi(s; W, W', \mathbb{1}_{\mathcal{O}_F^n})$$

$$\begin{aligned}
&= \sum_{\mu \in T^+(n)} W(\mu(\varpi)) W'(\mu(\varpi)) \mathbb{1}_{\mathcal{O}_F^n}(e_n \mu(\varpi)) |\det(\mu(\varpi))|^s \delta_{B_n}^{-1}(\mu(\varpi)) \\
&= \sum_{\mu \in T^+(n)} \mathbb{S}_{k_1, \dots, k_n}(q^{-\lambda_1}, \dots, q^{-\lambda_n}) \mathbb{S}_{k_1, \dots, k_n}(q^{-\lambda'_1}, \dots, q^{-\lambda'_n}) q^{-|\mu|s} \\
&= \det(I_{n^2} - q^{-s} q^{-\lambda} \otimes q^{-\lambda'})^{-1}.
\end{aligned}$$

Here, as before, $e_n \in F^n$ is the elementary vector with 0's in the first $n-1$ entries and 1 in the last entry. $\square$

Theorem 11.6.1 together with Proposition 11.5.1 implies that

$$\det(I_{mn} - q^{-s} q^{-\lambda} \otimes q^{-\lambda'})^{-1} \in L(s, \pi \times \pi') \mathbb{C}[q^s, q^{-s}].$$

More is true [JPSS83, §9.5]:

Theorem 11.6.2 *One has $L(s, \pi \times \pi') = \det(I_{mn} - q^{-s} q^{-\lambda} \otimes q^{-\lambda'})^{-1}$.* $\square$

Finally, we note the following:

Corollary 11.6.3 *Under the assumptions at the beginning of this section one has*

$$\varepsilon(s, \pi_v \times \pi'_v, \psi_v) = 1.$$

Proof. This follows immediately from Theorems 11.5.4, 11.6.1, and 11.6.2 and the definition of $\varepsilon(s, \pi_v \times \pi'_v, \psi_v)$ in (11.22). $\square$

11.7 Global Rankin-Selberg L -functions

Let π and π' be cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_m(\mathbb{A}_F)$, respectively.

We have previously defined the local Rankin-Selberg L -functions $L(s, \pi_v \times \pi'_v)$. For $\mathrm{Re}(s)$ sufficiently large the global Rankin-Selberg L -function is the product

$$L(s, \pi \times \pi') := \prod_v L(s, \pi_v \times \pi'_v). \quad (11.23)$$

If $\psi : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times$ is a nontrivial character we also define

$$\varepsilon(s, \pi \times \pi') := \prod_v \varepsilon(s, \pi_v \times \pi'_v, \psi_v). \quad (11.24)$$

The reason that ψ is not encoded in to the left-hand side of (11.24) is that the right-hand side is in fact independent of the choice of ψ . In (11.24) the local factor $\varepsilon(s, \pi_v \times \pi'_v, \psi_v)$ is 1 for all but finitely many v by Corollary 11.6.3.

On the other hand if v is a non-Archimedean place then $\varepsilon(s, \pi_v \times \pi'_v, \psi_v)$ converges for all s by Proposition 11.5.5. If v is Archimedean then the same is true [Jac09, §16]. Therefore the product (11.24) converges for all s .

We observe that for $\operatorname{Re}(s)$ sufficiently large

$$L(s, \pi \otimes |\det|^{s_1} \times \pi' \otimes |\det|^{s_2}) = L(s + s_1 + s_2, \pi \times \pi').$$

Thus without loss of generality we can assume that π (resp. π') is trivial on A_{GL_n} (resp. A_{GL_m}). In this case $L(s, \pi_v \times \pi'_v)$ converges absolutely for $\operatorname{Re}(s) \geq 1$ for all v by Theorem 11.5.2. Moreover the product in (11.23) converges absolutely for $\operatorname{Re}(s) > 1$ by the local results just mentioned and [JS81b, Theorem 5.3].

The following theorem collects the basic facts about Rankin-Selberg L -functions [Cog07, Theorems 4.1 and 4.2]:

Theorem 11.7.1 *Assume that π (resp. π') is trivial on A_{GL_n} (resp. A_{GL_m}). The Rankin-Selberg L -function admits a meromorphic continuation to the plane, holomorphic except for possible simple poles at $s = 0, 1$. It is bounded in vertical strips. There are poles at $s = 0, 1$ if and only if $m = n$ and $\pi \cong \pi'^\vee$. One has a functional equation*

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1 - s, \pi^\vee \times \pi'^\vee). \quad (11.25)$$

□

One might ask why we do not define global versions of the γ -functions $\gamma(s, \pi_v \times \pi'_v, \psi_v)$. Heuristically speaking, Theorem 11.7.1 implies that the product of the local γ -factors is 1, and hence there is no reason to define a different notation for this object. However this does not make much literal sense, as the product of the local γ -factors does not converge.

One extremely important consequence of this theorem is the strong multiplicity one theorem:

Theorem 11.7.2 (Strong multiplicity one) *Let π and π' be cuspidal automorphic representations of $\operatorname{GL}_n(\mathbb{A}_F)$ and let S be a finite set of places of F . If $\pi^S \cong \pi'^S$ then $\pi \cong \pi'$.* □

We leave the proof as Exercise 11.6. It is a consequence of Theorem 11.5.2 and Theorem 11.7.1. Despite its name, Theorem 11.7.2 does not (at least directly) imply the multiplicity one result of Theorem 11.4.3.

More generally, one has the following (see Exercise 11.8).

Corollary 11.7.3 *Let π and π' be isobaric automorphic representations of $\operatorname{GL}_n(\mathbb{A}_F)$ and let S be a finite set of places of F . If $\pi^S \cong \pi'^S$ then $\pi \cong \pi'$.* □

For a beautiful generalization of the strong multiplicity one theorem, we refer the reader to the work of Ramakrishnan on automorphic analogues of the Chebotarev density theorem [Ram15].

We outline the proof of Theorem 11.7.1 in the case $m = n - 1$. Let φ be in the space of π and φ' in the space in π' . For positive integers j let $K_{j\infty} = \prod_{v|\infty} K_{jv}$ where K_{jv} is the standard maximal compact subgroup of $\mathrm{GL}_j(F_v)$ of Example 4.2. We assume that φ is $K_{n\infty}$ -finite and φ' is $K_{m\infty}$ -finite. Define

$$I(s; \varphi, \varphi') := \int_{\mathrm{GL}_{n-1}(F) \backslash \mathrm{GL}_{n-1}(\mathbb{A}_F)} \varphi \left(\begin{smallmatrix} h & \\ & 1 \end{smallmatrix} \right) \varphi'(h) |\det h|^{s-1/2} dh. \quad (11.26)$$

The integral $I(s; \varphi, \varphi')$ converges for all s by Proposition 14.3.3. Let $\tilde{\varphi}(g) := \varphi(g^{-t})$ and $\tilde{\varphi}'(g) := \varphi'(g^{-t})$. Taking a change of variables $h \mapsto h^{-t}$ we see that

$$I(s; \varphi, \varphi') = I(1-s; \tilde{\varphi}, \tilde{\varphi}'). \quad (11.27)$$

This simple change of variables is the key to the proof of Theorem 11.7.1.

Define

$$\Psi(s; W_\psi^\varphi, W_\psi^{\varphi'}) = \int_{N_{n-1}(\mathbb{A}_F) \backslash \mathrm{GL}_{n-1}(\mathbb{A}_F)} W_\psi^\varphi \left(\begin{smallmatrix} h & \\ & 1 \end{smallmatrix} \right) W_\psi^{\varphi'}(h) |\det h|^{s-1/2} dh.$$

This is a **global Rankin-Selberg integral**. It does not converge absolutely for all s as one can already see from the unramified calculation in §11.6. However it does converge absolutely for $\mathrm{Re}(s)$ sufficiently large by so-called gauge estimates for Whittaker functions developed in [JPSS79a, JPSS79b].

Theorem 11.7.4 *For $\mathrm{Re}(s)$ sufficiently large $I(s; \varphi, \varphi') = \Psi(s; W_\psi^\varphi, W_\psi^{\varphi'})$.*

In particular, the functional equation (11.27) immediately implies an analogous one for the global Rankin-Selberg integral $\Psi(s; W_\psi^\varphi, W_\psi^{\varphi'})$.

Proof. Replacing φ by its Whittaker expansion as in Theorem 11.3.6 we have that

$$\begin{aligned} I(s; \varphi, \varphi') &= \int_{\mathrm{GL}_{n-1}(F) \backslash \mathrm{GL}_{n-1}(\mathbb{A}_F)} \varphi \left(\begin{smallmatrix} h & \\ & 1 \end{smallmatrix} \right) \varphi'(h) |\det h|^{s-1/2} dh \\ &= \int_{\mathrm{GL}_{n-1}(F) \backslash \mathrm{GL}_{n-1}(\mathbb{A}_F)} \sum_{\gamma} W_\psi^\varphi \left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \begin{smallmatrix} h & \\ & 1 \end{smallmatrix} \right) \varphi'(h) |\det h|^{s-1/2} dh, \end{aligned}$$

where the inner sum is over $\gamma \in N_{n-1}(F) \backslash \mathrm{GL}_{n-1}(F)$. Since $\varphi'(h)$ is left $\mathrm{GL}_{n-1}(F)$ -invariant and $|\det \gamma| = 1$ for $\gamma \in \mathrm{GL}_{n-1}(F)$, we may unfold as in Lemma 9.2.5 to obtain

$$I(s; \varphi, \varphi') = \int_{N_{n-1}(F) \backslash \mathrm{GL}_{n-1}(\mathbb{A}_F)} W_\psi^\varphi \left(\begin{smallmatrix} h & \\ & 1 \end{smallmatrix} \right) \varphi'(h) |\det h|^{s-1/2} dh.$$

This is absolutely convergent for $\operatorname{Re}(s)$ sufficiently large by the proof of [JS81a, Proposition 3.3]. Thus unfolding as above is justified, as are our forthcoming manipulations with the integral.

For $u \in N_n(\mathbb{A}_F)$ one has $W_\psi^\varphi(ug) = \psi(u)W_\psi^\varphi(g)$. Thus

$$\begin{aligned}
 I(s; \varphi, \varphi') &= \int_{N_{n-1}(\mathbb{A}_F) \backslash \operatorname{GL}_{n-1}(\mathbb{A}_F)} \int_{[N_{n-1}]} W_\psi^\varphi \left(\begin{pmatrix} u & \\ & 1 \end{pmatrix} \begin{pmatrix} h & \\ & 1 \end{pmatrix} \right) \varphi'(uh) du |\det h|^{s-1/2} dh \\
 &= \int_{N_{n-1}(\mathbb{A}_F) \backslash \operatorname{GL}_{n-1}(\mathbb{A}_F)} W_\psi^\varphi \left(\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right) \int_{[N_{n-1}]} \psi(u) \varphi'(uh) du |\det h|^{s-1/2} dh \\
 &= \int_{N_{n-1}(\mathbb{A}_F) \backslash \operatorname{GL}_{n-1}(\mathbb{A}_F)} W_\psi^\varphi \left(\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right) W_{\frac{\varphi'}{\psi}}(h) |\det h|^{s-1/2} dh \\
 &= \Psi(s; W_\psi^\varphi, W_{\frac{\varphi'}{\psi}}).
 \end{aligned}$$

□

In the case $m = n - 1$, Theorem 11.7.1 can be deduced from Theorems 11.5.4 and 11.7.4; we leave this as Exercise 11.9.

11.8 The nongeneric case

In the constructions above, crucial use was made of the fact that cuspidal automorphic representations are globally generic. Indeed, our definition of local L -functions was given in terms of the Whittaker model in the non-Archimedean case.

We now explain how to treat the general case. Assume first that F is a local field. When F is Archimedean, local Rankin-Selberg L -functions and ε -factors are defined using the local Langlands correspondence in complete generality (see §12.3), whether or not the representations under consideration are generic.

Assume now that F is a non-Archimedean local field. From §10.6 we know that every pair of irreducible admissible representations π of $\operatorname{GL}_n(F)$ and π' of $\operatorname{GL}_m(F)$ can be written as

$$\pi \cong \boxplus_{i=1}^k \pi_i \quad \text{and} \quad \pi' \cong \boxplus_{i=1}^{k'} \pi'_i$$

for some irreducible admissible essentially square integrable representations π_i and π'_i . Thus each π_i and π'_i is generic by Theorem 8.4.4.

For nontrivial characters $\psi : F \rightarrow \mathbb{C}^\times$ we then define

$$L(s, \pi \times \pi') = \prod_{i=1}^k \prod_{j=1}^{k'} L(s, \pi_i \times \pi'_j),$$

$$\varepsilon(s, \pi \times \pi', \psi) = \prod_{i=1}^k \prod_{j=1}^{k'} \varepsilon(s, \pi_i \times \pi'_j, \psi).$$

Of course, one has to check that this is consistent with our earlier definitions. In other words, one must prove that if π and π' are generic, then the procedure just described for defining local L -functions and ε -factors yields the same result as if we did not decompose π and π' into isobaric sums and simply used our original definitions. This follows from Theorem 8.4.4 and [JPSS83, §3.1, §9.4].

We adopt the analogous conventions in the global case and thereby obtain global Rankin-Selberg L -functions and ε -factors for isobaric automorphic representations. Their analytic properties can be deduced from the case where the isobaric representations are cuspidal.

Let $\pi_0, \pi_1, \dots, \pi_k$ be a collection of cuspidal automorphic representations of $A_{\mathrm{GL}_{n_i}} \backslash \mathrm{GL}_{n_i}(\mathbb{A}_F)$. The **multiplicity of** π_0 in $\boxplus_{i=1}^k \pi_i$ is

$$|\{1 \leq i \leq k : \pi_i \cong \pi_0\}|.$$

The reason we are being explicit about this definition is that $\boxplus_{i=1}^k \pi_i$ is irreducible, although it is induced by the direct sum in a sense made precise in §13.2. Applying Theorem 11.7.1 we deduce the following analytic formula for the multiplicity:

Lemma 11.8.1 *Assume that $\pi_0, \pi_1, \dots, \pi_k$ are a collection of cuspidal automorphic representations of $A_{\mathrm{GL}_{n_i}} \backslash \mathrm{GL}_{n_i}(\mathbb{A}_F)$. The order of the pole of*

$$L(s, \pi_0 \times (\boxplus_{i=1}^k \pi_i))$$

at $s = 1$ is the multiplicity of $\pi_0^\vee$ in $\boxplus_{i=1}^k \pi_i$. □

11.9 The converse theorem

Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. We have seen in Theorem 11.7.1 that for all cuspidal automorphic representations π' of $\mathrm{GL}_m(\mathbb{A}_F)$, the L -function $L(s, \pi \times \pi')$ is analytically well behaved. The converse theorem asserts conversely that if an admissible representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ has the property that $L(s, \pi \times \pi')$ is well behaved for enough cuspidal automorphic representations π' then π is automorphic.

Let $\psi : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times$ be a nontrivial character and let π (resp. π') be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_F)$ (resp. $\mathrm{GL}_m(\mathbb{A}_F)$). We can then define

$$\varepsilon(s, \pi \times \pi', \psi) := \prod_v \varepsilon(s, \pi_v \times \pi'_v, \psi_v). \quad (11.28)$$

We observe that π_v, π'_v, ψ_v are unramified for all but finitely many v and hence all but finitely many factors are 1 by Corollary 11.6.3. If π and π' are isobaric automorphic representations then $\varepsilon(s, \pi \times \pi', \psi) = \varepsilon(s, \pi \times \pi')$, and in particular in this case $\varepsilon(s, \pi \times \pi', \psi)$ is independent of ψ .

Theorem 11.9.1 (The converse theorem) *Let π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with central quasi-character invariant under $Z_{\mathrm{GL}_n}(F)$ such that π_v is generic for all v . Assume that $L(s, \pi)$ converges for $\mathrm{Re}(s)$ sufficiently large. Assume moreover that, for all cuspidal automorphic representations π' of $\mathrm{GL}_m(\mathbb{A}_F)$ with $1 \leq m \leq \max(n-2, 1)$,*

- (a) *the L -functions $L(s, \pi \times \pi')$ and $L(s, \pi^\vee \times \pi'^\vee)$ have analytic continuations to the plane that are holomorphic and bounded in vertical strips and*
- (b) *$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi', \psi) L(1-s, \pi^\vee \times \pi'^\vee)$.*

Then π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$. □

We refer to [CPS99] for a proof of Theorem 11.9.1 and for variants. By loc. cit., the assumption that $L(s, \pi)$ converges for $\mathrm{Re}(s)$ sufficiently large implies that $L(s, \pi \times \pi')$ and $L(s, \pi^\vee \times \pi'^\vee)$ converge for $\mathrm{Re}(s)$ sufficiently large. Hence assumption (a) in Theorem 11.9.1 is meaningful.

The converse theorem has been used to establish cases of Langlands functoriality [CKPSS04]. In addition, it is philosophically important because it asserts, roughly, that an L -function that satisfies reasonable assumptions has to come from an automorphic representation. The reader is encouraged to reflect on this comment after reading §12.8, which explains the conjectural relationship between Galois representations and automorphic representations.

We end this chapter with the following intriguing conjecture [CPS99, §8, Conjecture 2]:

Conjecture 11.9.2 (Piatetski-Shapiro) *Let π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_F)$ with central quasi-character invariant under $Z_{\mathrm{GL}_n}(F)$ such that π_v is generic for all v . Assume that $L(s, \pi)$ converges for $\mathrm{Re}(s)$ sufficiently large. Assume moreover that for all characters*

$$\chi : A_{\mathbb{G}_m} F^\times \backslash \mathbb{A}_F^\times \longrightarrow \mathbb{C}^\times$$

- (a) *the L -functions $L(s, \pi \times \chi)$ and $L(s, \pi^\vee \times \chi^{-1})$ have analytic continuations to the plane that are holomorphic and bounded in vertical strips and*
- (b) *$L(s, \pi \times \chi) = \varepsilon(s, \pi \times \chi, \psi) L(1-s, \pi^\vee \times \chi^{-1})$.*

Then there is an automorphic representation π_0 of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $\pi_0^S \cong \pi^S$ for any finite set S of places of F including the infinite places and the places where π_0 and π are ramified. Moreover,

$$L(s, \pi \times \chi) = L(s, \pi_0 \times \chi) \quad \text{and} \quad \varepsilon(s, \pi \times \chi) = \varepsilon(s, \pi_0 \times \chi)$$

for all χ as above.

This conjecture is open even for $n = 4$.

Exercises

11.1. Let B_1 and $B_2 \leq \mathrm{GL}_n$ be Borel subgroups defined over a local or global field F . For $1 \leq i \leq 2$, let N_i be the unipotent radical of B_i . If F is a local (resp. global) field let ψ_i be a generic character of $N_i(F)$ (resp. $N_i(\mathbb{A}_F)$). When F is a local field prove that an irreducible admissible representation π of $\mathrm{GL}_n(F)$ is ψ_1 -generic if and only if it is ψ_2 -generic. When F is a global field prove that a cuspidal automorphic representation π of $A_{\mathrm{GL}_n} \backslash \mathrm{GL}_n(\mathbb{A}_F)$ is ψ_1 -generic if and only if it is ψ_2 -generic.

11.2. Let $T \leq \mathrm{SL}_2$ be the maximal torus of diagonal matrices, $N_2 \leq \mathrm{SL}_2$ the unipotent radical of the Borel subgroup of upper triangular matrices, and let $\lambda \in \mathfrak{a}_{T\mathbb{C}}^*$. Consider the unramified principal series representation $I(\lambda)$ of SL_2 (see §7.6). Observe that $\ell(\varphi) := \varphi(I_2)$ is an $N_2(F)$ -invariant functional. Prove that for $\mathrm{Re}(\lambda)$ sufficiently large the functional

$$\varphi \longmapsto \int_{N_2(F)} \varphi \left(\begin{pmatrix} 1 & \\ t & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \right) dt$$

on the space (7.58) of $I(\lambda)$ is absolutely convergent, $N_2(F)$ -invariant, and not a complex multiple of ℓ . Conclude that the assumption that ψ is generic in Theorem 11.3.4 cannot be removed.

11.3. Let G be a quasi-split reductive group over a global field F and assume that π is a cuspidal globally ψ -generic representation of $G(\mathbb{A}_F)$. If $\pi \cong \otimes'_v \pi_v$ prove that π_v is ψ_v -generic for all v .

11.4. Prove Theorem 11.4.1 when $G = \mathrm{GL}_2$.

11.5. Let F be a non-Archimedean local field, let $\psi : F \rightarrow \mathbb{C}^\times$ be an unramified character, and let $\lambda \in \mathfrak{a}_{T_n\mathbb{C}}^*$. Assume that $\pi = J(\lambda)$ is unitary and generic. Let $W \in \mathcal{W}(\pi, \psi)^{\mathrm{GL}_n(\mathcal{O}_F)}$ and $W' \in \mathcal{W}(\pi^\vee, \overline{\psi})^{\mathrm{GL}_n(\mathcal{O}_F)}$ be the unique Whittaker functions satisfying $W(I_n) = W'(I_n) = 1$. Using the identity

$$\det(I_{n^2} - q^{-s} q^\lambda \otimes q^{-\lambda}) \Psi(s; W, W', \mathbb{1}_{\mathcal{O}_F^\times}) = 1$$

and Proposition 11.5.1, prove that every eigenvalue of $q^{-\lambda}$ lies in $(q^{-1/2}, q^{1/2})$.

11.6. If π and π' are cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $\pi^S \cong \pi'^S$ for some finite set S of places of F , prove that $\pi \cong \pi'$.

11.7. Prove Theorem 6.7.4.

11.8. If π and π' are isobaric automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $\pi^S \cong \pi'^S$ for some finite set S of places of F , prove that $\pi \cong \pi'$.

11.9. Prove Theorem 11.7.1 in the case $m = n - 1$ using Theorem 11.7.4 and the local functional equations of Theorem 11.5.4.

Chapter 12

Langlands Functoriality



The local Langlands conjecture is one of those hydra-like conjectures which seems to grow as it gets proved.

M. Harris and R. Taylor

Langlands functoriality has been a driving force in number theory, representation theory, and beyond since it was first posed in a famous letter of Langlands to Weil. We survey progress towards the local Langlands conjecture and state the global Langlands functoriality conjectures in this chapter.

12.1 The Weil group

In this chapter we state the Langlands reciprocity and functoriality conjectures. Though the conjectures are mostly open, extremely important progress has been made.

We begin with reciprocity, deferring a discussion of functoriality to §12.6. Let G be a reductive group which is unramified over a non-Archimedean local field F . By Corollary 7.5.2 one has a bijection

$$\left\{ (\widehat{G} \rtimes \mathrm{Fr}_{\mathfrak{p}} / \sim)(\mathbb{C}) \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{isomorphism classes of irreducible} \\ \text{unramified representations of } G(F) \end{array} \right\}.$$

This is the archetypal example of Langlands reciprocity, or, more precisely, a Langlands correspondence. Langlands reciprocity is the idea that representations of generalized Galois groups with image in ${}^L G$ correspond to representations of the points of G in locally compact rings. One refers informally to the Galois side of the correspondence and the automorphic side of the

correspondence. This terminology is standard, but a bit loose, because the Galois side really consists of representations of generalized Galois groups and the representations on the other side of the correspondence are only automorphic in the global setting. It is not difficult to see how the bijection above fits into this framework but we will postpone a discussion to §12.5. The local Langlands correspondence, which will be discussed in §12.3, §12.4, and §12.5, is a set of theorems and conjectures that enunciate Langlands reciprocity in the local setting. Global Langlands reciprocity is much deeper. We refer to conjectures 12.6.2 and 12.6.3 for one manner of making the concept of global Langlands reciprocity precise.

Before introducing the generalized Galois groups that play a role in the theory we indicate why working with traditional Galois groups is not sufficient. Let F be a global or local field, let F^{sep} be a separable closure of F , and let

$$\text{Gal}_F := \text{Gal}(F^{\text{sep}}/F)$$

be the absolute Galois group of F . It is endowed with the profinite topology. For a prime $\mathfrak{p}$ of $\mathcal{O}_F$ we denote by $\text{Fr}_{\mathfrak{p}} \in \text{Gal}_F$ a choice of geometric Frobenius element. It is well defined up to conjugation and characterized by the fact that it induces the automorphism $x \mapsto x^{|\mathcal{O}_F/\mathfrak{p}|^{-1}}$ on the algebraic closure of $\mathcal{O}_F/\mathfrak{p}$.

A continuous homomorphism $\text{Gal}_F \rightarrow \text{GL}_n(\mathbb{C})$ necessarily has finite image (see Exercise 12.1). On the other hand, for any prime ℓ of $\mathbb{Z}$ there are many continuous homomorphisms $\text{Gal}_F \rightarrow \text{GL}_n(\mathbb{Q}_{\ell})$ with infinite image.

Example 12.1. If E is an elliptic curve over $\mathbb{Q}$ without CM, then the Tate module of E gives a representation with image isomorphic to $\text{GL}_2(\mathbb{Z}_{\ell})$ for almost all primes ℓ [Ser72].

Example 12.2. A more elementary example is given by the cyclotomic character, which is the character

$$\chi_{\ell} : \text{Gal}_{\mathbb{Q}} \longrightarrow \mathbb{Z}_{\ell}^{\times}$$

defined as follows. If $\sigma \in \text{Gal}_{\mathbb{Q}}$ and ζ_n is a primitive ℓ^n -th root of unity then $\sigma(\zeta_n) = \zeta_n^{a_{\sigma,n}}$ for some $a_{\sigma,n} \in (\mathbb{Z}/\ell^n)^{\times}$. We then define

$$\chi_{\ell}(\sigma) = \varprojlim_n a_{\sigma,n}.$$

One can check that if $p \neq \ell$ then $\chi_{\ell}(\text{Fr}_p) = p^{-1}$.

It would be nice to view these examples as complex valued representations of a single group. At least in the special case where we are only considering representations into $\text{GL}_1(\mathbb{C})$, Weil was able to accomplish this by constructing a modification of the Galois group. We discuss this modification following the exposition of [Tat79]. For field extensions E/F of finite degree let

$$C_E := \begin{cases} E^\times & \text{if } F \text{ is local,} \\ E^\times \backslash \mathbb{A}_E^\times & \text{if } F \text{ is global.} \end{cases}$$

By class field theory one has an **Artin reciprocity map**

$$C_E \longrightarrow \mathrm{Gal}_E^{\mathrm{ab}}.$$

Here and below ab denotes the maximal abelian Hausdorff quotient, that is, the quotient by the closure of the commutator subgroup. The Artin reciprocity map is not canonically defined. It depends on a choice of sign corresponding locally to whether uniformizers at a prime $\mathfrak{p}$ are sent to $\mathrm{Fr}_{\mathfrak{p}}$ or its inverse. To differentiate between these two choices, the morphism $\mathrm{Fr}_{\mathfrak{p}}$ is referred to as the **geometric Frobenius** and the morphism $\mathrm{Fr}_{\mathfrak{p}}^{-1}$ is referred to as the **arithmetic Frobenius**. We normalize the Artin reciprocity map so that uniformizers at $\mathfrak{p}$ are sent to $\mathrm{Fr}_{\mathfrak{p}}$, the geometric Frobenius. This agrees with the conventions of [Tat79] and [HT01].

Definition 12.1.1. Let F be a local or global field. Then a **Weil group** of F^{sep}/F is a tuple $(W_F, \phi, \{\mathrm{Art}_E\})$ where W_F is a topological group,

$$\phi : W_F \longrightarrow \mathrm{Gal}_F$$

is a continuous homomorphism with dense image, and for each finite extension E/F

$$\mathrm{Art}_E : C_E \longrightarrow (\phi^{-1}(\mathrm{Gal}_E))^{\mathrm{ab}} \quad (12.1)$$

is an isomorphism. Setting

$$W_E := \phi^{-1}(\mathrm{Gal}_E),$$

these data are required to satisfy the following assumptions:

(a) For each finite extension E/F , the composite

$$C_E \xrightarrow{\mathrm{Art}_E} W_E^{\mathrm{ab}} \xrightarrow{\text{induced by } \phi} \mathrm{Gal}_E^{\mathrm{ab}}$$

is the reciprocity map of class field theory.

(b) Let $w \in W_F$ and $\sigma = \phi(w) \in \mathrm{Gal}_F$. For each E , the diagram

$$\begin{array}{ccc} C_E & \xrightarrow{\mathrm{Art}_E} & W_E^{\mathrm{ab}} \\ \sigma \downarrow & & \downarrow \\ C_{\sigma(E)} & \xrightarrow{\mathrm{Art}_{\sigma(E)}} & W_{\sigma(E)}^{\mathrm{ab}} \end{array}$$

commutes. Here the right arrow is induced by conjugation by w .

(c) For $E' \leq E$, the diagram

$$\begin{array}{ccc}
C_{E'} & \xrightarrow{\text{Art}_{E'}} & W_{E'}^{\text{ab}} \\
\text{inclusion} \downarrow & & \downarrow \text{transfer} \\
C_E & \xrightarrow{\text{Art}_E} & W_E^{\text{ab}}
\end{array}$$

commutes.

(d) The map

$$W_F \longrightarrow \varprojlim W_{E/F}$$

is an isomorphism, where $W_{E/F} := W_F / \overline{W_E}$, the bar denoting closure.

For the proof of the following theorem, we refer to [AT09]:

Theorem 12.1.2. *Let F be a local or global field. Then a Weil group of F^{sep}/F exists.* $\square$

Weil groups are unique up to essentially unique isomorphism in a sense made precise by the following proposition [Tat79, (1.3.1), (1.5.2)]:

Proposition 12.1.3 *Given two Weil groups $(W_{iF}, \phi_i, \{\text{Art}_{iE}\})$, $i \in \{1, 2\}$, of F^{sep}/F there is an isomorphism $\theta : W_{1F} \xrightarrow{\sim} W_{2F}$ such that the diagrams*

$$\begin{array}{ccc}
W_{1F} & \xrightarrow{\phi_1} & \text{Gal}_F \\
\downarrow \theta & \nearrow \phi_2 & \\
W_{2F} & &
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
C_E & \xrightarrow{\text{Art}_{1E}} & W_{1E}^{\text{ab}} \\
& \searrow \text{Art}_{2E} & \downarrow \theta \\
& & W_{2E}^{\text{ab}}
\end{array}$$

commute for all finite field extensions E/F . Here the right vertical arrow is induced by θ . The isomorphism θ is unique up to inner automorphism by an element of $\ker \phi_1$. $\square$

In many cases the Weil group can be described explicitly:

Example 12.3. Let F be a non-Archimedean local field with residue field κ and for all finite degree field extensions E/F let κ_E be the residue field of E . Put $\bar{\kappa} = \bigcup_E \kappa_E$. Let q be the cardinality of κ . The group W_F is the dense subgroup of Gal_F generated by the $\sigma \in \text{Gal}_F$ such that on $\bar{\kappa}$, σ acts as $x \mapsto x^{q^n}$ for some $n \in \mathbb{Z}$. The topology on W_F is the subspace topology with respect to the inclusion $W_F \rightarrow \text{Gal}_F$. The automorphism $\text{Art}_F(a)$ acts as $x \mapsto x^{|a|}$ on $\bar{\kappa}$.

Example 12.4. In the global function field case, the Weil group is defined as in the previous example, but one replaces the residue field above with the constant field.

Example 12.5. If $F = \mathbb{C}$ then W_F is just $\mathbb{C}^\times$ viewed as a topological group in the usual manner, ϕ is the trivial map, and Art_F is the identity map.

Example 12.6. If $F = \mathbb{R}$ then W_F is $\mathbb{C}^\times \cup j\mathbb{C}^\times$ where $j^2 = -1$ and $jcj^{-1} = \bar{c}$ for $c \in \mathbb{C}^\times$. Here the bar denotes complex conjugation. We give the subgroup $\mathbb{C}^\times < W_F$ the usual topology and declare that it is open in W_F . This uniquely determines the topology on the topological group W_F . The homomorphism ϕ takes $\mathbb{C}^\times$ to 1 and $j\mathbb{C}^\times$ to the nontrivial element of $\text{Gal}(\mathbb{C}/\mathbb{R})$.

For number fields no description of the Weil group as explicit as in the examples above is known. However, for all global fields, one has local-global compatibility analogous to local-global compatibility for Galois groups. In more detail, let v be a place of F and let $\iota : F^{\text{sep}} \rightarrow F_v^{\text{sep}}$ be an F -homomorphism. Let E/F be a finite degree field extension and let $E_v := \iota(E)F_v$. Then there are commutative diagrams

$$\begin{array}{ccc} W_{F_v} & \xrightarrow{\phi_v} & \text{Gal}_{F_v} \\ \downarrow \theta_v & & \downarrow \\ W_F & \xrightarrow{\phi} & \text{Gal}_F \end{array} \quad \text{and} \quad \begin{array}{ccc} E_v^\times & \xrightarrow{\text{Art}_{E_v}} & W_{E_v}^{\text{ab}} \\ \downarrow & & \downarrow \\ E^\times \backslash \mathbb{A}_E^\times & \xrightarrow{\text{Art}_E} & W_E^{\text{ab}} \end{array} \quad (12.2)$$

where the right horizontal arrows are both induced by ι and the map $E_v^\times \rightarrow E^\times \backslash \mathbb{A}_E^\times$ sends t to the class of the idele that is t in the v th place and 1 elsewhere. The morphism θ_v is unique in the function field case and unique up to conjugation by an element of $\ker(\phi : W_F \rightarrow \text{Gal}_F)$ in the number field case [Tat79, Proposition 1.6.1].

Almost by definition of the Weil group, one obtains the following:

Theorem 12.1.4 (The Langlands correspondence for GL_1) *There is a canonical bijection between isomorphism classes of automorphic representations of $\text{GL}_1(\mathbb{A}_F)$ and continuous representations $W_F \rightarrow \text{GL}_1(\mathbb{C})$.*

Proof. The equivalence class of an automorphic representation of $\text{GL}_1(\mathbb{A}_F)$ can be identified with a quasi-character $\chi : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$. On the other hand we have an isomorphism $\text{Art}_F : F^\times \backslash \mathbb{A}_F^\times \xrightarrow{\sim} W_F^{\text{ab}}$ by definition of the Weil group. Thus we associate to χ the representation $\chi \circ \text{Art}_F^{-1} : W_F^{\text{ab}} \rightarrow \mathbb{C}^\times$. $\square$

12.2 The Weil-Deligne group and L -parameters

The Weil group has more complex representations than the Galois group. However it still does not have enough complex representations to allow for a generalization of Theorem 12.1.4 (or even a local analogue) from GL_1 to general reductive groups G . The correct replacement of W_F when F is a number field should be the as yet hypothetical Langlands group $\mathcal{L}_F$ (see §12.6 for a discussion of $\mathcal{L}_F$). On the other hand, the correct replacement of W_F when F is a local field is known. It is the Weil-Deligne group. In this section we define this group, what we mean by a representation of it, and

then define local L -functions, ε -factors, and γ -factors of a representation. We then introduce the key notion of an L -parameter. Our primary references are [Bor79, Tat79, GR10]. In particular, [GR10] discusses the relationship between various equivalent definitions of the objects we discuss.

The **Weil-Deligne group** of a local field F is

$$W'_F := \begin{cases} W_F & \text{if } F \text{ is Archimedean,} \\ W_F \times \mathrm{SL}_2(\mathbb{C}) & \text{if } F \text{ is non-Archimedean.} \end{cases} \quad (12.3)$$

If F is non-Archimedean we identify W_F with the subgroup $W_F \times \{1\} \leq W'_F$. We have a homomorphism $\phi : W_F \rightarrow \mathrm{Gal}_F$ by definition of the Weil group. We again denote by ϕ the map

$$\phi : W'_F \longrightarrow \mathrm{Gal}_F \quad (12.4)$$

that is just ϕ in the Archimedean case and is the composite of the natural quotient map $W'_F \rightarrow W_F$ with ϕ in the non-Archimedean case.

Definition 12.2.1. Let G be a reductive group over $\mathbb{C}$ and let H be a complex Lie group whose neutral component is $G(\mathbb{C})$. A **representation of W'_F** into H is a continuous homomorphism

$$\rho : W'_F \longrightarrow H$$

such that when F is non-Archimedean $\rho|_{\mathrm{SL}_2(\mathbb{C})}$ is induced by a morphism of group schemes $\mathrm{SL}_2 \rightarrow G$ over $\mathbb{C}$. Such a representation is **Frobenius semisimple** if for all continuous homomorphisms $H \rightarrow \mathrm{GL}_n(\mathbb{C})$ whose restriction to $G(\mathbb{C})$ is induced by a homomorphism of affine $\mathbb{C}$ -groups $G \rightarrow \mathrm{GL}_n$, the homomorphism

$$W_F \xrightarrow{\rho|_{W_F}} H \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

has image in the set of semisimple elements of $\mathrm{GL}_n(\mathbb{C})$.

By a representation of W'_F we mean a representation of W'_F into $\mathrm{GL}_n(\mathbb{C})$ for some integer n . We point out that the terminology “Frobenius semisimple representation” is usually reserved for the non-Archimedean case, but it is convenient and natural to extend it as in the definition above.

Assume until stated otherwise that F is non-Archimedean. Let κ be the residue field of F , $\bar{\kappa}$ a choice of algebraic closure, and $\mathrm{Gal}_\kappa := \mathrm{Gal}(\bar{\kappa}/\kappa)$. The action of Gal_F preserves the ring of integers and the (unique) prime ideal of the ring of integers of any finite degree extension field E/F contained in F^{sep} . There is thus an exact sequence

$$1 \longrightarrow I_F \longrightarrow \mathrm{Gal}_F \longrightarrow \mathrm{Gal}_\kappa \longrightarrow 1$$

where I_F is the **inertia subgroup**, which can be defined as the kernel of the map $\text{Gal}_F \rightarrow \text{Gal}_\kappa$.

The Weil group W_F is a subgroup of Gal_F containing I_F . We thus have an exact sequence

$$1 \longrightarrow I_F \longrightarrow W_F \longrightarrow \text{Gal}_\kappa.$$

The map $W_F \rightarrow \text{Gal}_\kappa$ is not surjective. Its image is isomorphic to $\mathbb{Z}$, whereas Gal_κ is isomorphic to $\widehat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$. We deduce that

$$W_F \cong I_F \rtimes \langle \text{Fr} \rangle,$$

where Fr is a geometric Frobenius element. A representation ρ of W'_F into H is Frobenius semisimple if and only if $\rho(\text{Fr})$ is semisimple [Tat79, §4.1.3]. This explains the terminology.

We now define L -functions when F is non-Archimedean. Let V be a finite dimensional complex vector space and let $\rho : W'_F \rightarrow \text{GL}_V(\mathbb{C})$ be a Frobenius semisimple representation. An irreducible representation of SL_2 is equivalent to the n th symmetric power Sym^n of the standard representation for some n . A Frobenius semisimple representation of $W'_F = W_F \times \text{SL}_2(\mathbb{C})$ decomposes as a direct sum of exterior tensor products of representations of W_F and $\text{SL}_2(\mathbb{C})$, so we have that

$$V = \bigoplus_{n=0}^{\infty} V_n \otimes \text{Sym}^n.$$

Here V_n is a representation of W_F (that is zero for all but finitely many n). Thus $V_n \otimes \text{Sym}^n$ is the Sym^n -isotypic component of V , viewed as a representation of $\text{SL}_2(\mathbb{C})$ by restriction. Define

$$L(s, \rho) := \prod_{n=0}^{\infty} \det \left(1 - \rho(\text{Fr}) q^{-(s+n/2)}|_{V_n^{I_F}} \right)^{-1}. \quad (12.5)$$

Assume that E/F is a finite degree field extension of either Archimedean or non-Archimedean fields. Let $\rho : W'_E \rightarrow \text{GL}_V(\mathbb{C})$ be a Frobenius semisimple representation. It is not hard to see that if we define the induction

$$\text{Ind}_E^F(\rho) := \text{Ind}_{W'_E}^{W'_F}(\rho) : W'_F \longrightarrow \text{GL}_{V \oplus [W'_F : W'_E]}(\mathbb{C}) \quad (12.6)$$

in the usual manner then it is a Frobenius semisimple representation of W'_F into $\text{GL}_{V \oplus [W'_F : W'_E]}(\mathbb{C})$. When F is non-Archimedean, L -functions are invariant under induction in the following sense:

$$L(s, \rho) = L(s, \text{Ind}_E^F(\rho)). \quad (12.7)$$

Moreover, if $\rho_i : W'_F \rightarrow \text{GL}_{V_i}(\mathbb{C})$, $i \in \{1, 2\}$, are a pair of Frobenius semisimple representations then the direct sum, defined in the usual manner, is a Frobenius semisimple representation into $\text{GL}_{V_1 \oplus V_2}(\mathbb{C})$. When F is non-

Archimedean, L -functions are additive in the following sense:

$$L(s, \rho_1 \oplus \rho_2) = L(s, \rho_1)L(s, \rho_2). \quad (12.8)$$

We now assume until otherwise stated that F is Archimedean. Part of the data of the Weil group is an isomorphism $\text{Art}_F : F^\times \xrightarrow{\sim} W_F^{\text{ab}}$. Thus we can identify quasi-characters of W_F and quasi-characters of $F^\times$. Define the quasi-character

$$\begin{aligned} \chi_{s,m} : F^\times &\longrightarrow \mathbb{C}^\times \\ z &\longmapsto |z|^s \frac{z^m}{(z\bar{z})^{m/2}}. \end{aligned} \quad (12.9)$$

Here if F is complex the bar denotes complex conjugation and m is an integer. If F is real, the bar is the identity map and we assume $m \in \{0, 1\}$. In particular $\chi_{0,1}$ is the sign character. We recall that $|\cdot|$ is the usual norm when F is real and the square of the usual norm when F is complex, i.e. $|x| = (x\bar{x})^{[F:\mathbb{R}]/2}$ for $x \in F$. It is for this reason that we have not written the denominator in the definition of $\chi_{s,m}$ as an absolute value.

We leave the proof of the following lemma as Exercise 12.6:

Lemma 12.2.2 *Every quasi-character of $F^\times$ is of the form $\chi_{s,m}$ for a unique $s \in \mathbb{C}$ and $m \in \mathbb{Z}$ where we assume $m \in \{0, 1\}$ if F is real. $\square$*

Unless otherwise specified, when we discuss the character $\chi_{s,m}$ we will always assume that $s \in \mathbb{C}$, that $m \in \mathbb{Z}$ and if F is real that $m \in \{0, 1\}$.

Let

$$\Gamma_F(s) := \begin{cases} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) & \text{if } F = \mathbb{R}, \\ 2(2\pi)^{-s} \Gamma(s) & \text{if } F = \mathbb{C}. \end{cases} \quad (12.10)$$

Here $\Gamma(s)$ is the usual Γ -function. We then define

$$L(s, \chi_{s',m}) = \Gamma_F\left(s + s' + \frac{|m|}{[F:\mathbb{R}]}\right). \quad (12.11)$$

Now $W_{\mathbb{C}}$ is just $\mathbb{C}^\times$ and $W_{\mathbb{R}}$ contains $W_{\mathbb{C}}$ as a normal subgroup of index 2 (see Examples 12.5 and 12.6). Using this fact the following proposition is not hard to prove (see Exercise 12.7):

Proposition 12.2.3 *Irreducible representations of $W_{\mathbb{C}}$ are 1 dimensional and irreducible representations of $W_{\mathbb{R}}$ are 1 dimensional or 2 dimensional. The irreducible 2 dimensional representations of $W_{\mathbb{R}}$ are the induced representations*

$$\text{Ind}_{\mathbb{C}}^{\mathbb{R}}(\chi_{s,m}),$$

where $m \neq 0$. For $m \neq 0$ one has that

$$\text{Ind}_{\mathbb{C}}^{\mathbb{R}}(\chi_{s,m}) \cong \text{Ind}_{\mathbb{C}}^{\mathbb{R}}(\chi_{s',m'}) \quad (12.12)$$

if and only if $s = s'$ and $m = \pm m'$. $\square$

Motivated by (12.7) we define

$$L(s, \text{Ind}_{\mathbb{C}}^{\mathbb{R}}(\chi_{s', m})) := L(s, \chi_{s', m}).$$

Thus we have defined L -functions for all 1 dimensional and 2 dimensional representations of ρ . By Proposition 12.2.3, a semisimple representation $\rho : W_F \rightarrow \text{GL}_V(\mathbb{C})$ is a direct sum of 1 dimensional or 2 dimensional subrepresentations. Motivated by (12.8) we define the L -function of ρ to be the product of the L -functions of its irreducible subrepresentations. This completes the definition of $L(s, \rho)$ in the Archimedean case.

Now we allow F to be either Archimedean or non-Archimedean and discuss ε -factors. Let $\psi : F \rightarrow \mathbb{C}^\times$ be a nontrivial character, $d^\times x$ be a Haar measure on $F^\times$, dx be a Haar measure on F , and χ be a quasi-character, i.e. a homomorphism

$$\chi : W_F \longrightarrow \text{GL}_1(\mathbb{C}).$$

We identify χ with the character $\chi \circ \text{Art}_F : F^\times \rightarrow \mathbb{C}^\times$. In his thesis [Tat67] Tate proved the existence of a nonzero complex number

$$\varepsilon(s, \chi, \psi) := \varepsilon(s, \chi, \psi, dx)$$

such that

$$\frac{\int_{F^\times} \widehat{f}(x) \chi^{-1}(x) |x|^{1-s} d^\times x}{L(1-s, \chi^{-1})} = \varepsilon(s, \chi, \psi) \frac{\int_{F^\times} f(x) \chi(x) |x|^s d^\times x}{L(s, \chi)} \quad (12.13)$$

for all $f \in \mathcal{S}(F)$, the Schwartz space of F . Here

$$\widehat{f}(y) := \int_F f(x) \psi(xy) dx$$

is the Fourier transform of f . In certain cases $\varepsilon(s, \chi, \psi)$ can be explicitly computed. When F is non-Archimedean and unramified over its prime field, χ and ψ are unramified, and dx is normalized so that $dx(\mathcal{O}_F) = 1$, then $\varepsilon(s, \chi, \psi) = 1$.

Langlands and Deligne defined an ε -factor $\varepsilon(s, \rho, \psi)$ for arbitrary Frobenius semisimple representations $\rho : W'_F \rightarrow \text{GL}_V(\mathbb{C})$ [Del73b, Del75]. In more detail, Langlands constructed these factors via local methods in a long manuscript that he did not complete and publish after learning of Deligne's much simpler global proof. The ε -factor encodes important arithmetic information about the representation ρ , including its Artin conductor [Tat79, §3.4.5] in the non-Archimedean case. We point out that ε -factors are additive. In other words, if we are given Frobenius semisimple representations $\rho_i : W'_F \rightarrow \text{GL}_{V_i}(\mathbb{C})$ for $i \in \{1, 2\}$ then

$$\varepsilon(s, \rho_1 \oplus \rho_2, \psi) = \varepsilon(s, \rho_1, \psi) \varepsilon(s, \rho_2, \psi) \quad (12.14)$$

(see [Tat79, §3.4.2]). Finally we set

$$\gamma(s, \rho, \psi) := \frac{\varepsilon(s, \rho, \psi)L(1-s, \rho^\vee)}{L(s, \rho)}. \quad (12.15)$$

It is now time to discuss L -parameters. Recall that the L -group ${}^L G$ of a reductive group G over F admits a canonical map ${}^L G \rightarrow \mathrm{Gal}_F$ given by taking the quotient by the neutral component. This map occurs in the following definition:

Definition 12.2.4. An L -parameter is a Frobenius semisimple representation ρ of W'_F into ${}^L G$ such that the composite

$$W'_F \xrightarrow{\rho} {}^L G \longrightarrow \mathrm{Gal}_F$$

is the map ϕ of (12.4). Two L -parameters are **equivalent** if they are conjugate by an element of $\widehat{G}(\mathbb{C})$.

In [Bor79, §8.2(ii)] an additional condition is required of L -parameters. We will not make it precise because it is irrelevant for us. Indeed, we will only state a version of the local Langlands conjecture for quasi-split groups, and when G is quasi-split this additional condition is automatic [Bor79, §3.3].

Given an L -parameter $\rho : W'_F \rightarrow {}^L H$ and an L -map ${}^L H \rightarrow {}^L G$, we note that the composite homomorphism

$$W'_F \longrightarrow {}^L G$$

is an L -parameter.

When G is split, recall that the L -group of G is a direct product:

$${}^L G := \widehat{G}(\mathbb{C}) \times \mathrm{Gal}_F.$$

In particular in this case an L -parameter $\rho : W'_F \rightarrow {}^L G$ induces a representation into $\widehat{G}(\mathbb{C})$

$$\rho : W'_F \longrightarrow {}^L G \longrightarrow \widehat{G}(\mathbb{C}).$$

Here the second arrow is the canonical projection. Every Frobenius semisimple representation of W'_F into $\widehat{G}(\mathbb{C})$ arises in this manner. Thus when G is split L -parameters may be identified with Frobenius semisimple representations into $\widehat{G}(\mathbb{C})$. Still assuming G is split, two L -parameters are equivalent if and only if their associated representations into $\widehat{G}(\mathbb{C})$ are $\widehat{G}(\mathbb{C})$ -conjugate.

For this paragraph let F be an arbitrary field and let V be a finite dimensional complex vector space. A **representation** ${}^L G \rightarrow \mathrm{GL}_V(\mathbb{C})$ is a continuous homomorphism whose restriction to $\widehat{G}(\mathbb{C})$ is induced by a morphism $\widehat{G} \rightarrow \mathrm{GL}_V$ of algebraic groups over $\mathbb{C}$.

Returning to the case where F is a local field, given an L -parameter $\rho : W'_F \rightarrow {}^L G$ and a representation

$$r : {}^L G \longrightarrow \mathrm{GL}_V(\mathbb{C})$$

we obtain a representation $r \circ \rho : W'_F \rightarrow \mathrm{GL}_V(\mathbb{C})$. We then define

$$\begin{aligned} L(s, \rho, r) &:= L(s, r \circ \rho), \\ \varepsilon(s, \rho, r, \psi) &:= \varepsilon(s, r \circ \rho, \psi), \\ \gamma(s, \rho, r, \psi) &:= \gamma(s, r \circ \rho, \psi). \end{aligned} \tag{12.16}$$

In the case where $G = \mathrm{GL}_n$ and the map $r : {}^L \mathrm{GL}_n \rightarrow \mathrm{GL}_n(\mathbb{C})$ is just the projection to the neutral component, r is omitted from notation, e.g.

$$L(s, \rho) := L(s, \rho, r).$$

12.3 The Archimedean Langlands correspondence

For this section, F is an Archimedean local field. Over Archimedean local fields the Langlands correspondence between L -packets of representations of $G(F)$ and L -parameters $W_F \rightarrow {}^L G$ was proved by Langlands [Lan89]. There is a substantial literature refining his result in various important ways, but we will not describe it.

Instead we will focus on the $G = \mathrm{GL}_n$ setting and describe the Langlands correspondence explicitly in this case. We recall from §10.6 that any irreducible admissible representation π of $\mathrm{GL}_n(F)$ is infinitesimally equivalent to an isobaric sum

$$\pi \cong \boxplus_{i=1}^k \pi_i,$$

where the π_i are irreducible, admissible, and essentially square integrable. It turns out that the possibilities for essentially square integrable π are very limited. If G is a semisimple group over $\mathbb{R}$ we say that G is **equal rank** if there is a maximal torus $T \leq G$ that is anisotropic. In other words, G is equal rank if and only if there is a maximal torus $T \leq G$ such that $T(\mathbb{R})$ is compact.

Harish-Chandra's fundamental theorem on the existence of discrete series representations follows:

Theorem 12.3.1 (Harish-Chandra) *Let G be a reductive group over $\mathbb{R}$. There are irreducible essentially square integrable representations of $G(\mathbb{R})$ if and only if G^{der} is equal rank. $\square$*

We refer to [Wal88, §7.7.1] for the proof.

By the theorem, if F is complex then $\mathrm{GL}_n(F)$ admits irreducible essentially square integrable representations if and only if $n = 1$. In fact, 1-dimensional representations are always essentially square integrable. If F is real then $\mathrm{GL}_n(F)$ admits irreducible admissible essentially square integrable representations if and only if $1 \leq n \leq 2$ (see Exercise 12.10).

We now explicitly define the Langlands correspondence. The admissible representation corresponding to a 1-dimensional representation $W_F \rightarrow \mathbb{C}^\times$ is the quasi-character defined by the composite

$$F^\times \xrightarrow{\text{Art}_F} W_F^{\text{ab}} \longrightarrow \mathbb{C}^\times.$$

If $W_F \rightarrow \text{GL}_n(\mathbb{C})$ is irreducible and $n > 1$ then $n = 2$ by Proposition 12.2.3. In this case the representation is of the form $\text{Ind}_{\mathbb{C}}^{\mathbb{R}}(\chi_{s,m})$ for some $m > 0$ by the same proposition. The irreducible admissible representation associated to $\text{Ind}_{W_{\mathbb{C}}}^{W_{\mathbb{R}}}(\chi_{s,m})$, by definition, is $\pi_{m+1,2s} = |\det|^s \pi_{m+1,0}$, where $\pi_{m+1,s}$ is the discrete series representation of §4.7.

In general, if $\rho : W_F \rightarrow \text{GL}_n(\mathbb{C})$ is a representation, it is isomorphic to $\oplus_{i=1}^k \rho_i$ where the ρ_i are all irreducible of dimension 1 or 2, and we let the corresponding irreducible admissible representation be $\pi(\rho) := \boxplus_{i=1}^k \pi(\rho_i)$, where $\pi(\rho_i)$ is the irreducible admissible representation corresponding to ρ_i . Using Proposition 12.2.3 and the discussion of isobaric representations given in §10.6, we deduce the following special case of the Archimedean local Langlands correspondence:

Theorem 12.3.2 *The map $\rho \mapsto \pi(\rho)$ described above is a bijection between isomorphism classes of irreducible admissible representations of $\text{GL}_n(F)$ and equivalence classes of semisimple representations $W_F \rightarrow \text{GL}_n(\mathbb{C})$. $\square$*

Now suppose that $m, n \in \mathbb{Z}_{\geq 1}$ and we are given an irreducible admissible representation π of $\text{GL}_n(F)$ and an irreducible admissible representation π' of $\text{GL}_m(F)$ with associated representations $\rho(\pi) : W_F \rightarrow \text{GL}_n(\mathbb{C})$ and $\rho(\pi') : W_F \rightarrow \text{GL}_m(\mathbb{C})$. For any nontrivial character $\psi : F \rightarrow \mathbb{C}^\times$ we let

$$\begin{aligned} L(s, \pi \times \pi') &:= L(s, \rho(\pi) \otimes \rho(\pi')), \\ \varepsilon(s, \pi \times \pi', \psi) &:= \varepsilon(s, \rho(\pi) \otimes \rho(\pi'), \psi), \\ \gamma(s, \pi \times \pi', \psi) &:= \gamma(s, \rho(\pi) \otimes \rho(\pi'), \psi). \end{aligned}$$

This is the definition of the Archimedean Rankin-Selberg L -function, ε -factor, and γ -factor. Given this indirect definition, it is remarkable that Jacquet, Piatetski-Shapiro, and Shalika were able to prove that these factors enjoy essentially the same analytic properties as their non-Archimedean analogues (see Theorem 11.5.4). The fact that they were able to prove this indicates that the Archimedean Langlands correspondence given above has meaning. Another indication of the utility of the correspondence is that it can be used to compute characters of representations. We refer to the foundational work of Shelstad and Langlands [LS87, She82] for more on this matter.

12.4 The local Langlands correspondence for GL_n

Let F be a local field. Let $\Pi(\mathrm{GL}_n)$ be the set of isomorphism classes of irreducible admissible representations of $\mathrm{GL}_n(F)$ and let $\Phi(\mathrm{GL}_n)$ be the set of $\mathrm{GL}_n(\mathbb{C})$ -conjugacy classes of L -parameters $\rho : W'_F \rightarrow {}^L\mathrm{GL}_n$.

By a **local Langlands correspondence** for GL_n over F one means a collection of bijections

$$\mathrm{rec} := \mathrm{rec}_F : \Pi(\mathrm{GL}_n) \xrightarrow{\sim} \Phi(\mathrm{GL}_n) \quad (12.17)$$

such that

- (a) If $\pi \in \Pi(\mathrm{GL}_1)$ then $\mathrm{rec}(\pi) = \pi \circ \mathrm{Art}_F^{-1}$.
- (b) If $\pi_1 \in \Pi(\mathrm{GL}_{n_1})$ and $\pi_2 \in \Pi(\mathrm{GL}_{n_2})$ then

$$L(s, \pi_1 \times \pi_2) = L(s, \mathrm{rec}(\pi_1) \otimes \mathrm{rec}(\pi_2))$$

and

$$\varepsilon(s, \pi_1 \times \pi_2, \psi) = \varepsilon(s, \mathrm{rec}(\pi_1) \otimes \mathrm{rec}(\pi_2), \psi).$$

- (c) If $\pi \in \Pi(\mathrm{GL}_n)$ and $\chi \in \Pi(\mathrm{GL}_1)$ then

$$\mathrm{rec}(\pi \otimes (\chi \circ \det)) = \mathrm{rec}(\pi) \otimes \mathrm{rec}(\chi).$$

- (d) If $\pi \in \Pi(\mathrm{GL}_n)$ and π has central quasi-character χ then

$$\det(\mathrm{rec}(\pi)) = \mathrm{rec}(\chi).$$

- (e) If $\pi \in \Pi(\mathrm{GL}_n)$ then $\mathrm{rec}(\pi^\vee) = \mathrm{rec}(\pi)^\vee$ where $\vee$ denotes the contragredient.

Theorem 12.4.1 (The local Langlands correspondence for GL_n) *Let F be a local field. Then there is a unique local Langlands correspondence for GL_n over F .* $\square$

When $n = 1$, this is essentially local class field theory and when F is Archimedean the correspondence was constructed in §12.3. When F has positive characteristic (and hence is the completion of a global function field), the theorem is due to Laumon, Rapoport, and Stuhler [LRS93]. In the mixed characteristic case, this was proved by Harris-Taylor first [HT01], then a simplified proof was found by Henniart [Hen00].

There are many additional compatibility properties that the local Langlands correspondence for GL_n enjoys. We mention a few. First, the correspondence sends isobaric sums to direct sums. In other words,

$$\mathrm{rec}(\pi_1 \boxplus \pi_2) = \mathrm{rec}(\pi_1) \oplus \mathrm{rec}(\pi_2). \quad (12.18)$$

Second, supercuspidal representations of $\mathrm{GL}_n(F)$ correspond bijectively to irreducible representations of W'_F that are trivial on the $\mathrm{SL}_2(\mathbb{C})$ -factor.

Finally, a representation π is essentially square integrable if and only if $\text{rec}(\pi)$ is irreducible. Indeed, in the Archimedean case, this is clear from our explicit description of the correspondence in §12.3. In the non-Archimedean case, Theorem 8.4.3 asserts that an irreducible admissible representation is essentially square integrable if and only if it is of the form $Q(\sigma^a, \lambda_a)$ with notation as in loc. cit. (here σ is supercuspidal). Such representations correspond to representations of $W'_F = W_F \times \text{SL}_2(\mathbb{C})$ of the form $\text{rec}(\sigma) \otimes \text{Sym}^a$. All of these requirements are more or less built into the classification. In the non-Archimedean setting, the first paper to discuss how the local Langlands correspondence for GL_n could be reduced to the supercuspidal case was [Zel80].

12.5 The local Langlands conjecture

The (conjectural) Langlands correspondence for arbitrary reductive groups G over local fields F is far more complicated. The general shape involves a relationship between

$$\Pi(G) := \left\{ \begin{array}{l} \text{(infinitesimal) equivalence classes of irreducible} \\ \text{admissible representations of } G(F) \end{array} \right\} \quad (12.19)$$

and

$$\Phi(G) := \{\widehat{G}(\mathbb{C})\text{-conjugacy classes of } L\text{-parameters into } {}^L G\}. \quad (12.20)$$

Here in (12.19) the adjective “infinitesimal” should be omitted unless F is Archimedean. We will state the conjecture for quasi-split reductive groups G in this section. Our exposition follows that of Kaletha [Kal16]. We warn the reader that Kaletha does not assume that the groups he considers are quasi-split, and hence the conjectures in loc. cit. are often more intricate than those we state below.

The conjectural correspondence is simplest to state in the tempered case and we will restrict to this setting. We have already defined the notion of a tempered representation in Definition 4.8.1. An L -parameter

$$\rho : W'_F \longrightarrow {}^L G$$

is **tempered** if the image of $\rho(W_F)$ under the projection to $\widehat{G}(\mathbb{C})$ is bounded, or equivalently has compact closure. Note that the projection ${}^L G \rightarrow \widehat{G}(\mathbb{C})$ is just a set-theoretic projection; it is only a homomorphism if G is split. It is clear the notion of being tempered depends only on the equivalence class of the L -parameter. We denote by

$$\Phi_t(G) \quad (12.21)$$

the set of equivalence classes of tempered L -parameters. We denote by

$$\Pi_t(G)$$

the set of isomorphism classes of tempered irreducible admissible representations of $G(F)$. The basic form of the local Langlands conjecture for quasi-split reductive groups is the following:

Conjecture 12.5.1 (The local Langlands correspondence)

(a) *There is a map*

$$\mathrm{LL} : \Pi_t(G) \longrightarrow \Phi_t(G)$$

with finite nonempty fibers $\Pi(\rho) := \mathrm{LL}^{-1}(\rho)$.

(b) *If one element of $\Pi(\rho)$ is essentially square integrable then they all are. An element of $\Pi(\rho)$ is essentially square integrable if and only if the image of ρ is not contained in a proper parabolic subgroup of ${}^L G$.*

(c) *If $\rho \in \Phi_t(G)$ is the image of $\rho_M \in \Phi_t(M)$ for a Levi subgroup $M \leq G$, then $\Pi(\rho)$ consists of the irreducible subquotients of the representations that are parabolically induced from elements of $\Pi(\rho_M)$.*

For the notion of a parabolic subgroup of ${}^L G$, we refer to §7.4. In the last assertion, we are implicitly using the fact that one can construct a morphism ${}^L M \rightarrow {}^L G$ for every Levi subgroup $M \leq G$ as in §7.4. We will discuss the compatibility of the formulation above with the known case of $G = \mathrm{GL}_n$ at the end of this section. Of course, the conjecture as stated above is too imprecise to have much meaning without refinement. It is meant as a template into which one adds various desiderata. At bare minimum one should require that the correspondence satisfies certain character identities; one reference is [Kal16, §1.3–1.4].

For a general tempered L -parameter ρ , the isomorphism classes of representations in $\Pi(\rho)$ are by definition a **local L -packet**, or simply an L -packet. We let

$$\mathrm{LP}_t(G) := \{\Pi(\rho) : \rho \in \Phi_t(G)\} \tag{12.22}$$

denote the set of all tempered L -packets of representations of $G(F)$.

Let us describe the structure of L -packets in an important special case.

Definition 12.5.2. An L -parameter is **unramified** if F is non-Archimedean, G is unramified, and the parameter is trivial on $I_F \times \mathrm{SL}_2(\mathbb{C})$.

Let $\rho : W'_F \rightarrow {}^L G$ be an unramified parameter. The class

$$\rho(\mathrm{Fr}) \in (\widehat{G} \rtimes \mathrm{Fr} / \sim)(\mathbb{C})$$

and a choice of hyperspecial subgroup $K \leq G(F)$ give rise to a representation $J(\lambda, K)$ by (7.31), Lemma 7.5.7, and Theorem 7.6.10. In Theorem 7.6.10 we

did not incorporate K into the notation for simplicity. We observe that if K and K' are $G(F)$ -conjugate then $J(\lambda, K) \cong J(\lambda, K')$, but the converse is not in general true. We let $\Pi(\rho)$ be the set of isomorphism classes of irreducible admissible representations of $G(F)$ isomorphic to $J(\lambda, K)$ for some hyperspecial subgroup K of $G(F)$. The set $\Pi(\rho)$ is an **unramified L -packet**.

The conjecture, as stated, leaves open the question of the structure of the L -packet $\Pi(\rho)$. To describe a little of what is expected, let

$$S_\rho := C_{\widehat{G}(\mathbb{C})}(\rho(W'_F)) \quad (12.23)$$

be the centralizer of $\rho(W'_F)$ in $\widehat{G}(\mathbb{C})$ and let

$$\overline{S}_\rho := S_\rho / Z_{\widehat{G}(\mathbb{C})}^{\text{Gal}_F}. \quad (12.24)$$

Here we are using the fact that $\widehat{G}(\mathbb{C})$ comes equipped with an action of Gal_F , the same action used to define ${}^L G$.

Lemma 12.5.3 *For $G = \text{GL}_n$ the group $\overline{S}_\rho$ is connected.*

Proof. It suffices to prove that S_ρ is connected. The fact that ρ is Frobenius semisimple implies that it is completely reducible. Thus we have the usual canonical isomorphism

$$V_\rho = \bigoplus_{\rho'} V_{\rho'} \otimes \text{Hom}_{\rho(W'_F)}(V_{\rho'}, V_\rho)$$

of representations of W'_F , where the sum is over the finite set of isomorphism classes of irreducible subrepresentations $(\rho', V_{\rho'})$ of (ρ, V_ρ) . Then

$$S_\rho = \prod_{\rho'} \text{Aut}(\text{Hom}_{\rho(W'_F)}(V_{\rho'}, V_\rho)) \cong \prod_{\rho'} \text{GL}_{m(\rho')}(\mathbb{C}),$$

where $m(\rho') = \dim \text{Hom}_{\rho(W'_F)}(V_{\rho'}, V_\rho)$ is the multiplicity of ρ' in ρ . $\square$

Conjecture 12.5.4 *Assume that G is quasi-split. For a tempered L -parameter ρ , there is an injection*

$$\iota : \Pi(\rho) \longrightarrow \text{Irr}(\pi_0(\overline{S}_\rho))$$

that is bijective if F is non-Archimedean.

Here $\text{Irr}(\pi_0(\overline{S}_\rho))$ is the set of isomorphism classes of irreducible representations of the finite group $\pi_0(\overline{S}_\rho)$, the group of components of $\overline{S}_\rho$. Just as in the case of Conjecture 12.5.1, there is not so much content to Conjecture 12.5.4 until after the bijection is refined by stipulating compatibility with various character identities. We refer to [Kal16, §1.4] for a discussion.

The theory of generic representations gives a means of isolating a basepoint in $\Pi(\rho)$. Consider the set of pairs (B, ψ) where $B \leq G$ is a Borel subgroup

and ψ is a generic character of the F -points of its unipotent radical. There is a natural action of $G(F)$ on this set by conjugation and a **Whittaker datum** is a $G(F)$ -orbit. Let $\mathfrak{w}$ be a Whittaker datum and say that an admissible irreducible representation π is **$\mathfrak{w}$ -generic** if it is ψ -generic for some (hence all) pairs $(B, \psi) \in \mathfrak{w}$.

The following is a version of a conjecture of Shahidi [Sha90, §9]:

Conjecture 12.5.5 (Shahidi) *Every tempered L -packet $\Pi(\rho)$ contains a unique $\mathfrak{w}$ -generic element.*

Assume for this paragraph that conjectures 12.5.1, 12.5.4, and 12.5.5 are valid. Given a Whittaker datum $\mathfrak{w}$, we assume that the injection $\iota : \Pi(\rho) \rightarrow \text{Irr}(\pi_0(\overline{S}_\rho))$ can be normalized so that the generic element of the packet (which is unique by Conjecture 12.5.5) is sent to the trivial representation. We call this normalized injection $\iota_{\mathfrak{w}}$. Thus we have a pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : \Pi(\rho) \times \pi_0(\overline{S}_\rho) &\longrightarrow \mathbb{C} \\ (\pi, s) &\longmapsto \text{tr}(\iota_{\mathfrak{w}}(\pi)(s)). \end{aligned} \tag{12.25}$$

It will play a crucial role in Conjecture 12.6.3.

Important progress towards Conjectures 12.5.1, 12.5.4, and 12.5.5 has been made. We briefly indicate some results and refer to [Kal16] for a more detailed discussion. In the Archimedean case, Conjectures 12.5.1, 12.5.4, and 12.5.5 are completely known. They are due to Langlands and Shelstad. Reference [LS87] is a place to begin, though Shelstad has many other papers on the subject. We refer to [Kal16] for a list. Her work and her joint work with Langlands provided crucial insight motivating the formulation of the general conjectures.

There is also a great deal known for classical groups. Consider the following assumptions:

- (i) The field F is of characteristic zero.
- (ii) Arthur's local claims are valid.
- (iii) The stabilization of the twisted trace formula is valid.

Arthur's local claims are several local results that were claimed by Arthur in his book [Art13]. Arthur has stated that he will prove them in a series of forthcoming papers listed as [A24], [A25], [A26], and [A27] in the references of loc. cit.

We will not say precisely what is meant by the stabilization of the twisted trace formula. Mœglin and Waldspurger [MW16a, MW16b] have proven this stabilization under the assumption of the twisted weighted fundamental lemma. There does not seem to be any theoretical obstacle to obtaining this last result, but it has not been completed as of this writing.

The following is Arthur's main theorem on the local Langlands correspondence [Art13, Theorem 1.5.1].

Theorem 12.5.6 *Suppose that (i), (ii), and (iii) above are satisfied. Parts (a) and (b) of Conjecture 12.5.1 and Conjecture 12.5.4 are valid for the split groups Sp_{2n} and SO_{2n+1} . A slightly weaker version of these conjectures is valid for the quasi-split (or split) special orthogonal groups of even dimensional vector spaces.* $\square$

Let $G = \mathrm{SO}_{2n}^*$, which we take to be either the split form of SO_{2n} or a quasi-split inner form, and let O_{2n}^* be the corresponding orthogonal group. In the theorem, the slightly weaker version of the correspondence is the correspondence obtained by replacing equivalence classes in $\Phi_t(G)$ and $\Pi_t(G)$ by equivalence classes up to conjugation by $\widehat{\mathrm{O}}_{2n}^*(\mathbb{C})$ and $\mathrm{O}_{2n}^*(F)$, respectively. We point out that technically speaking we have only defined L -groups for reductive groups, not disconnected algebraic groups with reductive neutral component like O_{2n}^* . The definition must be extended to make sense of the expression $\widehat{\mathrm{O}}_{2n}^*(\mathbb{C})$.

There is an analogous statement for quasi-split unitary groups over local fields [Mok15, Theorem 2.5.1]:

Theorem 12.5.7 *Assume (i), (ii), and (iii) above. Let E/F be a quadratic extension and let G be the corresponding quasi-split unitary group. Parts (a) and (b) of Conjecture 12.5.1 and Conjecture 12.5.4 are valid for G .* $\square$

Key identities that enter into the proof of Theorem 12.5.6 and Theorem 12.5.7 together with work of Konno [Kon02] imply cases of the existence assertion of Conjecture 12.5.5. The uniqueness assertion has also been established in these cases; we refer to [Ato17] for a short proof.

Theorem 12.5.8 *Assume (i), (ii), and (iii) above. If G is the split group Sp_{2n} or SO_{2n+1} , the quasi-split or split special orthogonal group of an even dimensional vector space, or a quasi-split unitary group then Conjecture 12.5.5 is true.* $\square$

Finally we mention that Kaletha has constructed a set of supercuspidal representations from L -parameters trivial on $\mathrm{SL}_2(\mathbb{C})$ under certain assumptions on the group G [Kal19]. For example, it is sufficient that G splits over a tamely ramified extension and the order of the residue field does not divide the order of the Weyl group of $G_{\overline{F}}$. He then verifies that the construction satisfies many of the desiderata of the local Langlands correspondence. One expects that Kaletha's work will lead to an explicit local Langlands correspondence provided that G is, roughly, not too ramified. For any G , Fargues and Scholze have constructed a less explicit candidate for the restriction of LL to the set of supercuspidal representations [FS21]. They moreover prove many of its expected properties. These works indicate that the local Langlands conjecture will be proved in complete generality in the near future.

In this section we have limited our focus to the tempered case. In practice one needs to refine this in two qualitatively different manners. First, one often has to enlarge the set of equivalence classes of tempered L -parameters to a

set of **almost tempered** L -parameters $\Phi_{\text{at}}(G)$ and define a set of **almost tempered** L -packets $\text{LP}_{\text{at}}(G)$ such that there is a local Langlands correspondence relating the two sets. This enlargement is necessary because we do not know that the local components of automorphic representations that are expected to be tempered are in fact tempered. This point will come up again in §12.6. The second refinement involves representations that are far from tempered, but still play a role in the global theory. This refinement requires the introduction of Arthur's A -packets (see [Art89, Art90]).

We close this section by discussing the compatibility of conjectures 12.5.1, 12.5.4, and 12.5.5 and our discussion of the unramified correspondence with the local Langlands correspondence for GL_n (Theorem 12.4.1). By Exercise 12.9 we have a well-defined bijection

$$\text{LL} := \text{rec}|_{\Pi_t(\text{GL}_n)} : \Pi_t(\text{GL}_n) \longrightarrow \Phi_t(\text{GL}_n)$$

and LL satisfies part (a) of Conjecture 12.5.1. The fact that the Langlands correspondence in this case is compatible with the unramified correspondence discussed above is a consequence of Theorem 10.6.3, the fact that rec sends isobaric sums to direct sums, and the fact that all hyperspecial subgroups of $\text{GL}_n(F)$ are $\text{GL}_n(F)$ -conjugate to $\text{GL}_n(\mathcal{O}_F)$ as recalled in Example 2.2. Assertion (b) in Conjecture 12.5.1 is valid by the discussion at the end of §12.4. To check consistency with condition (c) of Conjecture 12.5.1, let M be a Levi subgroup of GL_n , so $M \cong \prod_{i=1}^k \text{GL}_{n_i}$ is a product of general linear groups. If

$$\rho_M \in \Phi_t(M) = \prod_{i=1}^k \Phi_t(\text{GL}_{n_i})$$

then the parabolic induction of the unique representation

$$\pi = \otimes_{i=1}^k \pi_i \in \Pi(\rho_M)$$

to $\text{GL}_n(F)$ is irreducible by Exercise 12.8 and equal to $\boxplus_{i=1}^k \pi_i$. Given that the local Langlands correspondence sends isobaric sums to direct sums, we deduce compatibility of the local Langlands correspondence with requirement (c). To check compatibility with Conjecture 12.5.4, we use Lemma 12.5.3. Finally, to check compatibility with Conjecture 12.5.5, we recall that every irreducible tempered representation of $\text{GL}_n(F)$ is generic by Theorems 8.4.4 and 8.4.5.

12.6 Global Langlands functoriality

Let F be a global field and let G be a reductive group over F . As before, let $\text{Gal}_F := \text{Gal}(F^{\text{sep}}/F)$ and for all places v of F let

$$\text{Gal}_{F_v} := \text{Gal}(F_v^{\text{sep}}/F_v).$$

Then for each place v of F there is a Gal_F -conjugacy class of injective homomorphisms $\text{Gal}_{F_v} \rightarrow \text{Gal}_F$. Choosing such a homomorphism, one obtains an injection

$${}^L G_{F_v} := \widehat{G}(\mathbb{C}) \rtimes \text{Gal}_{F_v} \longrightarrow \widehat{G}(\mathbb{C}) \rtimes \text{Gal}_F =: {}^L G.$$

Motivated by this, define

$$\Phi(G_{\mathbb{A}_F}) := \left\{ (\rho_v) \in \prod_v \Phi(G_{F_v}) : \rho_v \text{ is unramified for almost all } v \right\}. \quad (12.26)$$

Let $\Phi_t(G_{\mathbb{A}_F}) \subset \Phi(G_{\mathbb{A}_F})$ be the subset of $\rho = (\rho_v)$ such that $\rho_v \in \Phi_t(G_{F_v})$ for all v . We call $\Phi(G_{\mathbb{A}_F})$ the set of **adelic L -parameters** into ${}^L G$ and call $\Phi_t(G_{\mathbb{A}_F})$ the set of **tempered adelic L -parameters** into ${}^L G$.

Let H be another reductive group over F and let

$$r : {}^L H \longrightarrow {}^L G \quad (12.27)$$

be an L -map. We then obtain a commutative diagram

$$\begin{array}{ccc} {}^L H_{F_v} & \xrightarrow{r} & {}^L G_{F_v} \\ \downarrow & & \downarrow \\ {}^L H & \xrightarrow{r} & {}^L G \end{array} \quad (12.28)$$

for all places v of F . Given any $\rho \in \Phi(H_{\mathbb{A}_F})$ we can define

$$r \circ \rho = (r \circ \rho_v) \in \Phi(G_{\mathbb{A}_F}). \quad (12.29)$$

Thus given the L -map (12.27) we obtain a commutative diagram

$$\begin{array}{ccc} \Phi_t(H_{\mathbb{A}_F}) & \longrightarrow & \Phi_t(G_{\mathbb{A}_F}) \\ \downarrow & & \downarrow \\ \Phi(H_{\mathbb{A}_F}) & \longrightarrow & \Phi(G_{\mathbb{A}_F}). \end{array} \quad (12.30)$$

For the rest of this section we assume that G and H are quasi-split and that conjectures 12.5.1, 12.5.4, and 12.5.5 are known for G_{F_v} and H_{F_v} for all places v . This is a reasonable working assumption, as the local Langlands correspondence is known in many cases as explained in §12.5. Given a tempered L -parameter $\rho \in \Phi_t(G_{\mathbb{A}_F})$ we can form the **global tempered L -packet**

$$\Pi(\rho) := \{ \pi = \otimes'_v \pi_v : \pi_v \in \Pi(\rho_v) \text{ for all } v \}. \quad (12.31)$$

This is a set of irreducible admissible representations of $G(\mathbb{A}_F)$. Two L -packets $\Pi(\rho)$ and $\Pi(\rho')$ are said to be **equivalent** if there is a bijection

$$j : \Pi(\rho) \longrightarrow \Pi(\rho')$$

such that $j(\pi)^\infty$ is isomorphic to π^∞ and $j(\pi)_\infty$ is infinitesimally equivalent (resp. isomorphic) to π_∞ if F is a number field (resp. a function field). In general L -packets are infinite. The set of all tempered L -packets of representations of $G(\mathbb{A}_F)$ up to equivalence is denoted $\text{LP}_t(G)$. We say a global tempered L -packet is **automorphic** if one of its elements is an automorphic representation. The set of tempered automorphic L -packets is denoted $\text{LP}_{t,\text{aut}}(G)$. There is a commutative diagram

$$\begin{array}{ccc} \text{LP}_{t,\text{aut}}(G) & \longrightarrow & \text{LP}_{w,\text{aut}}(G) \\ \downarrow & & \downarrow \\ \text{LP}_t(G) & \longrightarrow & \text{LP}_w(G), \end{array}$$

where the subscript w denotes the weak L -packets of (7.62). All of the arrows are the canonical maps. The vertical arrows above are injective, but the horizontal arrows are not in general. Let

$$\Pi_t(G) \quad \text{and} \quad \Pi_t(G(\mathbb{A}_F))$$

denote the set of isomorphism classes of automorphic (resp. admissible) representations $\pi = \otimes'_v \pi_v$ of $G(\mathbb{A}_F)$ such that π_v is tempered for all places v of F . We observe that we have a commutative diagram

$$\begin{array}{ccc} \Pi_t(G) & \longrightarrow & \Pi_t(G(\mathbb{A}_F)) \\ & \searrow & \downarrow \\ & & \Phi_t(G_{\mathbb{A}_F}) \end{array} \tag{12.32}$$

where the horizontal arrow is the inclusion and the diagonal and vertical arrows are induced by the local Langlands correspondence.

The remainder of this section is devoted to using the local Langlands correspondence to formulate the global Langlands functoriality conjecture. Given (12.27), we obtain (12.30), which under our assumptions on the local Langlands correspondence above allows us to define the right arrow in the following:

$$\text{LP}_{t,\text{aut}}(H) \longrightarrow \text{LP}_t(H) \longrightarrow \text{LP}_t(G). \tag{12.33}$$

The left arrow is the inclusion. Global Langlands functoriality describes when the image of a given global tempered automorphic L -packet under (12.33) is again automorphic. In other words it describes the intersection of the image of (12.33) with $\text{LP}_{t,\text{aut}}(G)$.

In an important special case it is straightforward to make this precise. Assume that $G = \text{Res}_{E/F} \text{GL}_n$ for some field extension E/F of finite degree.

In this case local L -packets are singletons (see §12.5), hence global L -packets are singletons. The following is the Langlands functoriality conjecture in this setting:

Conjecture 12.6.1 *Suppose that $G = \text{Res}_{E/F} \text{GL}_n$. Let π be an everywhere tempered cuspidal automorphic representation of $H(\mathbb{A}_F)$ with L -parameter $\rho \in \Phi_t(H_{\mathbb{A}_F})$. The unique admissible representation in $\Pi(r \circ \rho)$ is automorphic.*

For general G , we could proceed by refining the weaker Conjecture 7.7.2 by replacing weak L -packets with adelic L -packets. Even if we did this, it is unclear how to isolate automorphic representations within an automorphic L -packet. To treat this problem for arbitrary quasi-split reductive groups currently requires the introduction of the conjectural **Langlands group** $\mathcal{L}_F$. This first appeared in [Lan79a] and is also discussed in [Art02], where a conjectural construction of the group is given. The theory of Tannakian categories also indicates that such a group ought to exist if we assume Langlands functoriality. For a discussion of this point of view and the obstacles one runs into if one pursues it, we refer to [Ram94].

The Langlands group $\mathcal{L}_F$ of a global field F is conjecturally an extension of the Weil group W_F by the complex points of a proalgebraic group. For reductive F -groups G , one should be able to define global L -parameters $\rho : \mathcal{L}_F \rightarrow {}^L G$ as in the local case; they ought to be continuous homomorphisms commuting with the maps $\mathcal{L}_F \rightarrow W_F \rightarrow \text{Gal}_F$ and ${}^L G \rightarrow \text{Gal}_F$. We require these homomorphisms to be semisimple in the sense that for all L -maps ${}^L G \rightarrow {}^L \text{GL}_n$ the representation

$$\mathcal{L}_F \xrightarrow{\rho} {}^L G \longrightarrow {}^L \text{GL}_n \longrightarrow \text{GL}_n(\mathbb{C})$$

is completely reducible, where the rightmost arrow is the canonical projection. Two global L -parameters are **equivalent** if they are $\widehat{G}(\mathbb{C})$ -conjugate.

For every place v one should have homomorphisms

$$W'_{F_v} \longrightarrow \mathcal{L}_F, \quad (12.34)$$

unique up to conjugacy. The union over all v of the conjugacy classes of the images of (12.34) is conjectured to be dense in $\mathcal{L}_F$. This is an analogue of the density of Frobenius conjugacy classes in Gal_F . The homomorphisms (12.34) allow us to define a map

$$\begin{aligned} \Phi(G) &\longrightarrow \Phi(G_{\mathbb{A}_F}) \\ \rho &\longmapsto (\rho|_{W'_{F_v}}). \end{aligned} \quad (12.35)$$

We let $\Phi_t(G)$ be the inverse image of $\Phi_t(G_{\mathbb{A}_F})$ under (12.35). A global parameter is **tempered** if it is in $\Phi_t(G)$. A global parameter $\rho \in \Phi_t(G)$ is **discrete generic** if its image is not contained in any Levi subgroup of ${}^L G$. This definition will be used in the statement of Conjecture 12.6.3.

One is supposed to have a global analogue of the local parametrization of Conjecture 12.5.1:

Conjecture 12.6.2 (The global Langlands correspondence) *There is a surjective map*

$$\Pi_t(G) \longrightarrow \Phi_t(G) \quad (12.36)$$

such that the diagram

$$\begin{array}{ccc} \Pi_t(G) & \longrightarrow & \Pi_t(G(\mathbb{A}_F)) \\ \downarrow & & \downarrow \\ \Phi_t(G) & \longrightarrow & \Phi_t(G_{\mathbb{A}_F}) \end{array}$$

commutes. Here the top horizontal arrow is the inclusion, the bottom horizontal arrow is given by (12.35), and the right vertical arrow is given by the local Langlands correspondence.

Given a $\rho \in \Phi_t(G)$ we define

$$\Pi(\rho) := \Pi((\rho|_{W_{F_v'}})). \quad (12.37)$$

The L -packet $\Pi(\rho)$ contains the fiber of (12.36) over ρ . The containment is proper in general. As in the local setting, one defines

$$S_\rho := C_{\widehat{G}(\mathbb{C})}(\rho(\mathcal{L}_F)),$$

the centralizer of $\rho(\mathcal{L}_F)$ in $\widehat{G}(\mathbb{C})$, and sets

$$\overline{S}_\rho := S_\rho / Z_{\widehat{G}(\mathbb{C})}^{\text{Gal}_F}.$$

The set of automorphic elements of $\Pi(\rho)$ ought to be controlled by $\overline{S}_\rho$ in a sense made precise in Conjecture 12.6.3.

Let $B \leq G$ be a Borel subgroup with unipotent radical N and let $\psi : N(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ be a generic character trivial on $N(F)$. This provides us with a Whittaker datum $\mathfrak{w}_v$ for all v and hence pairings

$$\langle \cdot, \cdot \rangle_v : \Pi(\rho_v) \times \pi_0(\overline{S}_{\rho_v}) \longrightarrow \mathbb{C}$$

as in (12.25). We assume that for every v there is a homomorphism

$$\text{loc}_v : \pi_0(\overline{S}_\rho) \rightarrow \pi_0(\overline{S}_{\rho_v})$$

as in [Kal16, §5.3]. For $(\pi, s) \in \Pi(\rho) \times \pi_0(\overline{S}_\rho)$ we then set

$$\langle \pi, s \rangle := \prod_v \langle \pi_v, \text{loc}_v(s) \rangle.$$

We say that G satisfies the **Hasse principle** if the map on Galois cohomology sets

$$H^1(F, G) \longrightarrow \prod_v H^1(F_v, G)$$

is injective. For more on Galois cohomology, we refer to §17.4.

Conjecture 12.6.3 *Assume that G satisfies the Hasse principle. If $\rho_0 \in \Phi_t(G)$ is a discrete generic global L -parameter then the multiplicity of $\pi \in \Pi(\rho_0)$ in $L^2_{\text{disc}}([G])$ is*

$$\sum_{\rho} \frac{1}{|\pi_0(\overline{S}_{\rho})|} \sum_{s \in \pi_0(\overline{S}_{\rho})} \langle \pi, s \rangle. \quad (12.38)$$

Here the outer sum is over discrete generic global L -parameters $\rho \in \Phi_t(G)$ such that $\pi \in \Pi(\rho)$.

This conjecture was first stated in [Kot84, §12], although we have followed the exposition of [Kal16].

This long excursion on $\mathcal{L}_F$ and the global Langlands correspondence provides the tools necessary to understand when the functorial transfer of packets of representations (12.33) attached to the L -map (12.27) takes automorphic tempered L -packets to automorphic tempered L -packets. Assume that π is a given tempered automorphic representation of $H(\mathbb{A}_F)$ (which is to say that π_v is tempered for all v). Then there is a $(\rho_v) \in \Phi_t(H_{\mathbb{A}_F})$ such that $\pi \in \Pi((\rho_v))$. If assume Conjecture 12.6.2, then in fact there is a $\rho \in \Phi_t(G)$ such that $\rho|_{W_F} = \rho_v$ for all v . In other words, $\Pi((\rho_v)) = \Pi(\rho)$. Conjecture 12.6.3 then tells us precisely when there is an automorphic representation in $\Pi(r \circ \rho)$. This conjectural relationship between the automorphic representations in $\Pi(\rho)$ and $\Pi(r \circ \rho)$ is the **Langlands functoriality conjecture**, at least under the simplifying assumption that we are working with tempered representations of quasi-split groups.

There is an obvious problem with all this in that it is heavily reliant on the hypothetical group $\mathcal{L}_F$. Happily, at least for classical groups, one can use the ideas above to produce unconditional definitions of all of the objects appearing in (12.38). In particular the alternate definitions do not rely on the existence of the Langlands dual group [Art13, Mok15]. This allows one to make precise predictions relating the multiplicities (12.38) on different groups. Remarkably, one can then prove these predictions using the trace formula or other methods (see Chapter 13), establishing relationships between automorphic representations on different groups. In other words, we can state and prove cases of Langlands functoriality without knowing that $\mathcal{L}_F$ exists.

Theoretically, one should be able to reduce whatever one wants to know about automorphic representations to the tempered case. The A -packets of Arthur and his conjectures regarding them were introduced to make this precise [Art89, Art90] (see also [Clo07] and [Sha11] where the relationship

between A -packets and Conjecture 10.7.4, the Ramanujan Conjecture, is discussed). However, even when we expect representations to be tempered, in most cases, we can only prove bounds that establish that they are close to being tempered. Moreover, the only means currently known to prove that they are indeed tempered seems to be using Langlands functoriality (see [Lan70, Sar05]). To overcome this problem, as in the local case, instead of just treating tempered L -parameters and representations, one enlarges the set of parameters involved in the local Langlands correspondence to a set of almost tempered parameters and representations. This is supposed to be a set that is restrictive enough that the local Langlands correspondence is still correct, but general enough to include all representations that can occur as local components of the most tempered part of $L^2_{\text{disc}}([G])$. For examples of such an enlargement, we refer to [Art13].

12.7 Langlands L -functions

Assume that we are given a quasi-split reductive group G over a local field F such that the local Langlands correspondence is known for G . Given a representation

$$r : {}^L G \longrightarrow \text{GL}_V(\mathbb{C}),$$

an irreducible tempered representation π of $G(F)$ and a nontrivial character $\psi : F \rightarrow \mathbb{C}^\times$, one can then use (12.16) to define the local Langlands L -function, ε -factor, and γ -factor as follows:

$$\begin{aligned} L(s, \pi, r) &:= L(s, \text{LL}(\pi), r), \\ \varepsilon(s, \pi, r, \psi) &:= \varepsilon(s, \text{LL}(\pi), r, \psi), \\ \gamma(s, \pi, r, \psi) &:= \gamma(s, \text{LL}(\pi), r, \psi). \end{aligned} \tag{12.39}$$

By definition, admissible representations in the same L -packet necessarily have the same L -functions, ε -factors, and γ -factors. This is the reason for the terminology “ L -packet.” Sometimes one says that elements in the same L -packet are **L -indistinguishable**. When $G = \text{GL}_n \times \text{GL}_m$ and

$${}^L G \longrightarrow \text{GL}_{nm}(\mathbb{C})$$

is the tensor product, Rankin-Selberg theory as discussed in Chapter 11 already furnishes us with a definition of these factors. Part of the content of the local Langlands correspondence for GL_n (Theorem 12.4.1) is that the two definitions coincide.

Now assume that F is a global field, $\psi : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times$ is a nontrivial character, and G is a quasi-split reductive F -group such that the local Langlands correspondence for G_{F_v} is known for every place v . Assume moreover that

we are given a representation

$$r : {}^L G \longrightarrow \mathrm{GL}_V(\mathbb{C}).$$

The **global Langlands L -function** and **global ε -factor** are then

$$\begin{aligned} L(s, \pi, r) &:= \prod_v L(s, \pi_v, r), \\ \varepsilon(s, \pi, r) &:= \prod_v \varepsilon(s, \pi_v, r, \psi_v). \end{aligned} \tag{12.40}$$

As the notation indicates, the global ε -factor is (conjecturally) independent of the choice of ψ_v . If S is a finite set of places of F (containing the infinite places or not) one often writes

$$L^S(s, \pi, r) := L(s, \pi^S, r) := \prod_{v \notin S} L(s, \pi_v, r). \tag{12.41}$$

The basic expectations for these L -functions are recorded in the following conjecture:

Conjecture 12.7.1 *The L -function $L(s, \pi, r)$ is meromorphic as a function of s , is bounded in vertical strips, and satisfies the functional equation*

$$L(s, \pi, r) = \varepsilon(s, \pi, r) L(1 - s, \pi^\vee, r).$$

The notion of a meromorphic function that is bounded in vertical strips was introduced above Theorem 11.5.3. In particular $L(s, \pi, r)$ may have poles.

Conjecture 12.7.1 is in fact a consequence of Conjecture 12.6.1 and Theorem 11.7.1 together with some additional results and conjectures regarding the behavior of the Langlands correspondence under taking contragredients [AV16, Kal13]. Conversely, Conjecture 12.7.1 and the converse theorem, Theorem 11.9.1, can be combined to prove cases of Conjecture 12.6.1. Remarkably this strategy has been successfully executed in important examples. We refer to §13.6, §13.7, [CKPSS04, CFGK19], and Kim's article in [CKM04].

In most cases Conjecture 12.7.1 is open. The most that is known in general is that $L(s, \pi, r)$ converges for $\mathrm{Re}(s)$ sufficiently large (see [Lan71] and [Sha10, §2.5]).

12.8 Algebraic representations

Let F be a number field and let G be a reductive group over F . The hypothetical Langlands group $\mathcal{L}_F$ is expected to admit a morphism

$$\mathcal{L}_F \longrightarrow \mathrm{Gal}_F.$$

In particular, suitable morphisms

$$\mathrm{Gal}_F \longrightarrow {}^L G$$

should give rise to L -packets of automorphic representations of $G(F)$. It is natural to try and describe classes of L -packets that are of this form.

In [Clo90b], Clozel isolated a class of automorphic representations on $G = \mathrm{GL}_n$ that he called “algebraic” and then conjectured that they are associated to particular types of Galois representations (not just representations of $\mathcal{L}_F$). Buzzard and Gee [BG14] later extended Clozel’s conjectures to the case of arbitrary reductive G . We describe briefly Buzzard and Gee’s conjectures in this section, following the exposition in [BG14] closely.

Until otherwise specified, we work locally at an Archimedean place v of F which we omit from notation, writing $F := F_v$. We identify $\overline{F} = \mathbb{C}$. Let B be a Borel subgroup and $T \leq B$ a maximal torus of $G_{\mathbb{C}}$. The local Langlands correspondence is known in the Archimedean case [Lan89]. It provides us with a set-theoretic map

$$\begin{aligned} \mathrm{LL} : \Pi(G) &\longrightarrow \Phi(G) \\ \pi &\longmapsto \mathrm{LL}(\pi) \end{aligned}$$

with notation as in (12.19) and (12.20).

The group $W_{\mathbb{C}}$ is naturally a subgroup of W_F , which is in fact equal to W_F if $F = \mathbb{C}$. We have an isomorphism

$$\mathrm{Art}_{\mathbb{C}} : \mathbb{C}^{\times} \xrightarrow{\sim} W_{\mathbb{C}}$$

which we use to identify $\mathbb{C}^{\times}$ and $W_{\mathbb{C}}$. Fix a maximal torus $\widehat{T}$ in $\widehat{G}$ with an identification $X_*(\widehat{T}) = X^*(T)$. We can assume that $\mathrm{LL}(\pi)(\mathbb{C}^{\times}) \leq \widehat{T}(\mathbb{C})$. Let

$$\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) = \{\sigma, \tau\}.$$

There exist $\lambda_{\sigma}, \lambda_{\tau} \in X^*(T) \otimes \mathbb{C}$ satisfying $\lambda_{\sigma} - \lambda_{\tau} \in X^*(T)$ such that $\mathrm{LL}(\pi)(z) = \sigma(z)^{\lambda_{\sigma}} \tau(z)^{\lambda_{\tau}}$ for all $z \in \mathbb{C}^{\times}$. This follows from the fact that all quasi-characters of $\mathbb{C}^{\times}$ are of the form (12.9) (see Lemma 12.2.2).

The group

$$(X^*(T) \otimes \mathbb{C})^{\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})} \tag{12.42}$$

has a diagonal action of the Weyl group $W(G, T)(\mathbb{C})$. The action of Gal_F on $X^*(T) \otimes \mathbb{C}$ given in the construction of the Langlands dual group in §7.3 is $W(G, T)(\mathbb{C})$ -semilinear. In other words, for

$$(\xi, w, x) \in \mathrm{Gal}_F \times W(G, T)(\mathbb{C}) \times X^*(T) \otimes \mathbb{C}$$

one has $\xi(wx) = \xi(w)\xi(x)$. If $F = \mathbb{R}$ and $\xi \in \mathrm{Gal}_F$ is the nontrivial element then $(\xi(\lambda_{\tau}), \xi(\lambda_{\sigma}))$ is in the same $W(G, T)(\mathbb{C})$ -orbit as $(\lambda_{\sigma}, \lambda_{\tau})$. The group

(12.42) also has an action of Gal_F defined using both the action of Gal_F on $X^*(T)$ and the action on $\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})$. Explicitly, if we think of an element of (12.42) as a morphism

$$\phi : \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C}) \longrightarrow X^*(T) \otimes \mathbb{C}$$

then $\xi\phi := (t \mapsto \xi\phi(t \circ \xi))$. These considerations imply that $(\lambda_\sigma, \lambda_\tau)$ gives us a well-defined element of

$$\left((X^*(T) \otimes \mathbb{C})^{\mathrm{Hom}_{\mathbb{R}}(\mathbb{C}, \mathbb{C})} / W(G, T)(\mathbb{C}) \right)^{\mathrm{Gal}_F}.$$

Definition 12.8.1. An L -parameter $\rho : W_F \rightarrow {}^L G$ is **L -algebraic** if $\lambda_\sigma \in X^*(T)$. An irreducible admissible representation π of $G(F)$ is L -algebraic if $\mathrm{LL}(\pi)$ is L -algebraic.

Recall that we have fixed $T \leq B \leq G_{\mathbb{C}}$. As usual, denote by ρ_B the half-sum of the positive roots of T in B .

Definition 12.8.2. An L -parameter $\rho : W_F \rightarrow {}^L G$ is **C -algebraic** if $\lambda_\sigma - \rho_B \in X^*(T)$. An irreducible admissible representation π of $G(F)$ is C -algebraic if $\mathrm{LL}(\pi)$ is C -algebraic.

The assertion $\lambda_\sigma \in X^*(T)$ is independent of the choice of B , T and of the isomorphism $\bar{F} \cong \mathbb{C}$. Likewise the assertion $\lambda_\sigma - \rho_B \in X^*(T)$ is also independent of these choices. If $\rho_B \in X^*(T)$, then the notions of C -algebraic and L -algebraic coincide.

The expectation (which will be made precise below) is that L -algebraic automorphic representations should correspond to representations of Gal_F with image in $\hat{G}(\overline{\mathbb{Q}}_p)$ (what we mean by this last symbol will be explained below). By Corollary 15.5.4 cohomological automorphic representations in the sense of Definition 15.5.2 are C -algebraic (hence the terminology C -algebraic). The

known techniques for proving that L -algebraic representations correspond to representations of Gal_F invariably involve cohomological representations, so L -algebraic and C -algebraic representations ought to be studied in tandem. The two definitions differ by the half-sum of positive roots. For $G = \mathrm{GL}_n$, the notion of C -algebraic coincides in the isobaric case with Clozel's notation of algebraic used in [Clo90b, Clo16].

One can give a necessary and sufficient condition for an irreducible admissible representation π to be L -algebraic (resp. C -algebraic) in terms of the infinitesimal character of π . To make this precise, assume $F = \mathbb{R}$. Let $T_0 \leq G$ be a maximal torus. Upon changing our choice of B and T above if necessary we can assume that $T_{0\mathbb{C}} = T$. Let

$$\mathfrak{t} := \mathrm{Lie} T.$$

We identify $X_*(T)_{\mathbb{C}} = \mathfrak{t}$ as in Exercise 7.8, and hence identify $X^*(T)_{\mathbb{C}} = \mathfrak{t}^\vee$.

Lemma 12.8.3 *Assume $F = \mathbb{R}$. An irreducible admissible representation π of $G(\mathbb{R})$ is L -algebraic (resp. C -algebraic) if and only if the infinitesimal character χ_λ of π satisfies $\lambda \in X^*(T)$ (resp. $\lambda - \rho_B \in X^*(T)$).*

Proof. In the notation of §4.6, the infinitesimal character of π is χ_λ with $\lambda \in \mathfrak{t}^\vee$, uniquely determined up to the action of $W = W(G, T)(\mathbb{C})$. Using the $W(G, T)(\mathbb{C})$ -equivariant identification

$$\mathfrak{t}^\vee = X^*(T)_\mathbb{C}$$

we may regard λ as an element of $X_*(T)_\mathbb{C}$. The $W(G, T)(\mathbb{C})$ -orbit of λ contains λ_σ by [BG14, §2.3] and [Vog93, Proposition 7.4]. $\square$

For the proof of Corollary 15.5.4 we record [Kna02, (5.43)]:

Lemma 12.8.4 *Assume $F = \mathbb{R}$. If V is a finite dimensional representation of G , viewed as an algebraic group over $\mathbb{R}$, then the infinitesimal character χ_λ of V satisfies $\lambda - \rho_B \in X^*(T)$.* $\square$

The restriction to $F = \mathbb{R}$ in the previous two lemma is only for convenience. It is no loss of generality by an application of Weil restriction of scalars.

To define algebraic automorphic representations, let G be a reductive group defined over a number field F . Fix an algebraic closure $\bar{F}$ of F and form the L -group ${}^L G = \hat{G} \rtimes \text{Gal}_F$. For each place v of F , fix an algebraic closure $\bar{F}_v$ of F_v and an embedding $\bar{F} \hookrightarrow \bar{F}_v$. Let $\pi = \otimes'_v \pi_v$ be an automorphic representation of $G(\mathbb{A}_F)$.

Definition 12.8.5. We say that π is L -algebraic (resp. C -algebraic) if π_v is L -algebraic (resp. C -algebraic) for all infinite places v of F .

We now prepare to state an important conjecture on the existence of Galois representations attached to L -algebraic automorphic representations. It is in fact the primary motivation for introducing the definitions above. Let p be a prime. Fix a choice of algebraic closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and isomorphism $\iota : \mathbb{C} \rightarrow \bar{\mathbb{Q}}_p$.

We note that in the construction of $\hat{G}$ and the action of Gal_F used to form ${}^L G$, we can replace the base field $\mathbb{C}$ by $\bar{\mathbb{Q}}_p$, or in fact by any algebraically closed field. Thus ι induces an isomorphism

$$\iota : \hat{G}(\mathbb{C}) \longrightarrow \hat{G}(\bar{\mathbb{Q}}_p).$$

We construct a semidirect product

$$\hat{G}(\bar{\mathbb{Q}}_p) \rtimes \text{Gal}_F \tag{12.43}$$

using the same Gal_F action as in the construction of ${}^L G$. We have an isomorphism

$$\iota : {}^L G \longrightarrow \hat{G}(\bar{\mathbb{Q}}_p) \rtimes \text{Gal}_F.$$

This isomorphism is used in the following conjecture due to Buzzard and Gee. It is a generalization of a conjecture of Clozel in the case $G = \mathrm{GL}_n$.

Conjecture 12.8.6 *If π is L -algebraic then there is a continuous Galois representation*

$$\rho_\pi = \rho_{\pi, \iota} : \mathrm{Gal}_F \longrightarrow \widehat{G}(\overline{\mathbb{Q}}_p) \rtimes \mathrm{Gal}_F$$

commuting with the projections to Gal_F and a finite set S of places of F such that

- (a) *S contains all infinite places, places dividing p , and all places where π is ramified,*
- (b) *if $v \notin S$ then the Frobenius semisimplification of $\rho_\pi|_{W_{F_v}}$ is $\widehat{G}(\overline{\mathbb{Q}}_p)$ -conjugate to $\iota \circ \mathrm{LL}(\pi_v)$.*

We refer to [Tat79, §4.1.3] for the definition of the Frobenius semisimplification. There is additional important information about π at the infinite places and the places dividing p predicted in the full version of Conjecture 12.8.6 given in [BG14].

Though in general Conjecture 12.8.6 is open, there are important special cases that are known.

Theorem 12.8.7 *Suppose that F is CM or totally real, that $G = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$, and that π has the same infinitesimal character as an irreducible representation of $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}_n$ and in particular is L -algebraic. Then Conjecture 12.8.6 is true. $\square$*

There are in fact two different proofs of Theorem 12.8.7 available, one due to Harris, Lan, Taylor, and Thorn [HLTT16] and another due to Scholze [Sch15].

We close this section by reflecting on the fact that L -algebraic and C -algebraic representations are not the same for general groups. This is a minor irritant. There are proposed modifications of the Langlands dual group that allow one to satisfactorily treat this problem, as well as other related issues including integrality of the Satake isomorphism. We refer to [Ber20, BG14, Zhu20] for more details.

Exercises

12.1. Prove that any continuous homomorphism from a profinite group to a connected Lie group has finite image.

12.2. With notation as in Example 12.2, prove that if $p \neq \ell$ then

$$\chi_\ell(\mathrm{Fr}_p) = p^{-1}.$$

12.3. Given a map $G \rightarrow H$ of reductive groups with normal image, prove that there is an induced map $\widehat{H} \rightarrow \widehat{G}$.

12.4. Prove (12.7).

12.5. Prove (12.8).

12.6. Prove Lemma 12.2.2.

12.7. Prove Proposition 12.2.3.

12.8. For $1 \leq i \leq k$, let π_i be an irreducible tempered representation of $\mathrm{GL}_{n_i}(F)$. Let $\pi = \mathrm{Ind}(\otimes_{i=1}^k \pi_i, 0)$. Prove that π is irreducible.

12.9. Let F be a local field and let π be an irreducible admissible representation of $\mathrm{GL}_n(F)$. Prove that π is tempered if and only if $\mathrm{rec}(\pi)$ is tempered.

12.10. Let G be a semisimple group over $\mathbb{C}$. Show that $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} G$ is equal rank if and only if G is trivial. Show that SL_n (as a group over $\mathbb{R}$) is equal rank if and only if $n \leq 2$ (here we interpret SL_1 as the trivial group).

Chapter 13

Known Cases of Global Langlands Functoriality



...I had not recognized in 1966, when I discovered after many months of unsuccessful search a promising definition of automorphic L -function, what a fortunate, although, and this needs to be stressed, unforeseen by me, or for that matter anyone else, blessing it was that it lay in the theory of Eisenstein series.

R. P. Langlands

Over number fields the global Langlands functoriality conjecture is wide open. Despite this, several important cases have been established. We survey some of the hard earned progress in this chapter.

13.1 Introduction

What is now known as the Langlands functoriality conjecture was first posed in a seventeen page handwritten letter [Lan] that Langlands wrote in 1967 to André Weil (see [Lan70]). In Chapter 12 we stated the conjecture with some degree of precision. There has been decisive progress in the local setting in both equal and mixed characteristic (see §12.4 and §12.5) and in the global setting over function fields (see §13.9). However for number fields the conjecture remains mostly open.

Despite this, there has been hard earned progress. We will survey some of this progress in the current chapter. Throughout F is a global field and $\mathbb{A}_F$ is its ring of adeles. The symbol G will denote a reductive group over F .

13.2 Parabolic induction

Let P be a parabolic subgroup of G . A choice of Levi subgroup $M \leq P$ yields a Levi decomposition $P = MN$ where N is the unipotent radical of P . As in §7.4 we can realize ${}^L M$ as a subgroup of ${}^L P$ and thereby obtain an L -map

$${}^L M \longrightarrow {}^L G. \quad (13.1)$$

In the special case where $G = \mathrm{GL}_n$, conjugacy classes of Levi subgroups are indexed by tuples $n_1, \dots, n_d \in \mathbb{Z}_{>0}$ such that $\sum_{i=1}^d n_i = n$ as explained in (1.15). The map (13.1) is given by the identity on the Galois factor and the natural block diagonal embedding

$$\prod_{i=1}^d \mathrm{GL}_{n_i}(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

$$(x_1, \dots, x_d) \longmapsto \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_d \end{pmatrix}$$

on the neutral components. Thus given a collection of L -parameters into the ${}^L \mathrm{GL}_{n_i}$ composition with (13.1) yields the direct sum of the parameters. The functorial transfer attached to the L -map (13.1) sends the automorphic representation $\otimes_{i=1}^d \pi_i$ of $\prod_{i=1}^d \mathrm{GL}_{n_i}(\mathbb{A}_F)$ to the isobaric sum $\boxplus_{i=1}^d \pi_i$.

For general G and M , Langlands functoriality for the L -map (13.1) was essentially fully established by Langlands himself before Langlands even posed his functoriality conjecture. Making this precise is a matter of isolating the correct family of subquotients of the induced representations of Theorem 10.7.1.

13.3 L -maps into general linear groups

Let G be a quasi-split reductive group over F and let E/F be a finite separable extension. Suppose that we are given an L -map

$$r : {}^L G \longrightarrow {}^L \mathrm{Res}_{E/F} \mathrm{GL}_n$$

for some n . Assume that we know the local Langlands correspondence for G . If π is a cuspidal automorphic representation of $G(\mathbb{A}_F)$ then for all v we have an L -parameter

$$\mathrm{LL}(\pi_v) : W'_{F_v} \longrightarrow {}^L G,$$

at least if π_v is tempered. Here we use the notation of Conjecture 12.5.1. We therefore obtain an L -parameter

$$r \circ \mathrm{LL}(\pi_v) : W'_{F_v} \longrightarrow {}^L\mathrm{Res}_{E/F}\mathrm{GL}_n$$

for each v . By the local Langlands correspondence for $\mathrm{Res}_{E/F}\mathrm{GL}_n(F_v)$ (which is known, see §12.3 and §12.4), there exists an irreducible admissible representation $r(\pi_v)$ of $\mathrm{Res}_{E/F}\mathrm{GL}_n(F_v)$ such that

$$\mathrm{rec}(r(\pi_v)) = \mathrm{LL}(r(\pi_v)) = r \circ \mathrm{LL}(\pi_v). \quad (13.2)$$

Here we are using the notation of Theorem 12.4.1. The representation π_v is unique up to infinitesimal equivalence when v is Archimedean and unique up to equivalence when v is non-Archimedean. We denote by

$$r(\pi) := \otimes'_v r(\pi_v).$$

This is an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A}_E)$. By the version of Langlands functoriality explicated in Conjecture 12.6.1, at least if π is tempered, $r(\pi)$ ought to be automorphic. In practice one does not want to assume that π is tempered, but merely that it is “almost tempered” in the sense that the matrix coefficients of π_v are suitably close to being essentially square integrable. See the end of §12.6 for more details.

The following definition is often useful:

Definition 13.3.1. An isobaric automorphic representation π' of $\mathrm{Res}_{E/F}\mathrm{GL}_n(\mathbb{A}_F)$ is a **weak transfer** of π with respect to r if

$$r(\pi_v) \cong \pi'_v$$

for all v not in some finite set S of places of F .

Weak transfers are also often called **weak lifts**. If one wishes to be more specific, one can say that the functorial transfer of π with respect to r is compatible with the local Langlands correspondence outside of S . If a weak transfer exists then it is unique by strong multiplicity one for GL_n (Corollary 11.7.3). In practice one constructs a weak transfer first and then checks the compatibility of the transfer with the local Langlands correspondence at the places in S .

13.4 Base change

Let E/F be a finite degree extension of number fields. As in Example 7.18 we have

$${}^L\mathrm{Res}_{E/F}\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})^{\mathrm{Hom}_F(E, F^{\mathrm{sep}})} \rtimes \mathrm{Gal}_F$$

where Gal_F acts via its action on $\mathrm{Hom}_F(E, F^{\mathrm{sep}})$. There is an L -map

$$r_{E/F} : {}^L\mathrm{GL}_n \longrightarrow {}^L\mathrm{Res}_{E/F}\mathrm{GL}_n \quad (13.3)$$

defined by stipulating that it is the diagonal embedding on the neutral component and the identity on the Galois factor. Using the notation of (12.20), suppose that $\rho \in \Phi(\mathrm{GL}_{nF_v})$ for some place v of F . Then

$$L(s, r_{E/F} \circ \rho) = \prod_{w|v} L(s, \rho|_{W'_{E_w}}) \quad (13.4)$$

where the product is over all places w of E dividing v . Here W'_{E_w} is the Weil-Deligne group of (12.3). In other words, $r_{E/F}$ corresponds to restriction of Galois representations. The L -map $r_{E/F}$ and the functorial transfers it induces (conjectural or not) are known as **base change**. It conjecturally allows us to relate automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{Res}_{E/F}\mathrm{GL}_n(\mathbb{A}_F) := \mathrm{GL}_n(\mathbb{A}_E)$. It is customary to write

$$\pi_E := r_{E/F}(\pi).$$

We have a left action

$$\begin{aligned} \mathrm{GL}_n(\mathbb{C})^{\mathrm{Hom}_F(E, F^{\mathrm{sep}})} \times (\mathbb{C}^n)^{\mathrm{Hom}_F(E, F^{\mathrm{sep}})} &\longrightarrow (\mathbb{C}^n)^{\mathrm{Hom}_F(E, F^{\mathrm{sep}})} \\ ((g_\xi), (v_\xi)) &\longmapsto (g_\xi v_\xi). \end{aligned}$$

On the other hand we have a left $\mathbb{C}$ -linear action of $\mathrm{Gal}(E/F)$ on $(\mathbb{C}^n)^{\mathrm{Hom}_F(E, F^{\mathrm{sep}})}$ via its action on $\mathrm{Hom}_F(E, F^{\mathrm{sep}})$. These two actions together give us an action of ${}^L\mathrm{Res}_{E/F}\mathrm{GL}_n$ on $(\mathbb{C}^n)^{\mathrm{Hom}_F(E, F^{\mathrm{sep}})}$. Upon choosing an isomorphism $\mathrm{Aut}_{\mathbb{C}}((\mathbb{C}^n)^{\mathrm{Hom}_F(E, F^{\mathrm{sep}})}) \cong \mathrm{GL}_{n[E:F]}(\mathbb{C})$ this yields an L -map

$$\mathrm{AI}_{E/F} : {}^L\mathrm{Res}_{E/F}\mathrm{GL}_n \longrightarrow {}^L\mathrm{GL}_{n[E:F]}. \quad (13.5)$$

This L -map and the functorial transfers it induces (conjectural or not) are called **automorphic induction**. It allows us to relate automorphic representations of $\mathrm{GL}_n(\mathbb{A}_E) = \mathrm{Res}_{E/F}\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{GL}_{n[E:F]}(\mathbb{A}_F)$. We note that

$$L(s, \pi, \mathrm{AI}_{E/F}) = L(s, \pi). \quad (13.6)$$

Assume that E/F is Galois. Using the fact that the regular representation of $\mathrm{Gal}(E/F)$ decomposes as

$$\bigoplus_{\rho} \rho^{\oplus \dim(\rho)}$$

where the sum is over the isomorphism classes of irreducible representations ρ of $\mathrm{Gal}(E/F)$ and $\dim(\rho)$ is the dimension of the space of ρ , one has that

$$L(s, \pi, r_{E/F}) = L(s, \pi, \mathrm{AI}_{E/F} \circ r_{E/F}) = \prod_{\rho} L(s, \pi, r_{\mathrm{st}} \otimes r_{\rho})^{\dim(\rho)}. \quad (13.7)$$

Here $r_{\mathrm{st}} : {}^L\mathrm{GL}_n \rightarrow \mathrm{GL}_n(\mathbb{C})$ is the representation given by projection to the neutral component and

$$r_\rho : {}^L\mathrm{GL}_1 \longrightarrow \mathrm{GL}_{\dim(\rho)}(\mathbb{C})$$

is given by tensoring ρ with the trivial representation of ${}^L\mathrm{GL}_1^\circ = \mathrm{GL}_1(\mathbb{C})$.

If E/F is a prime degree cyclic extension, then the functorial transfers attached to $r_{E/F}$ and $\mathrm{AI}_{E/F}$ are known to exist. The case $n = 2$ is due to Langlands [Lan80], who followed a method discovered by Saito [Sai75] that was placed in a representation theoretic framework by Shintani [Shi79]. Besides placing the theory in the correct level of generality, Langlands was able to use it and the converse theorem to establish the strong Artin conjecture for 2-dimensional Galois representations in many solvable cases (see Theorem 13.5.5).

For GL_n the existence of the functorial transfers was established by Arthur and Clozel [AC89]. To be more precise, let E/F be a prime degree Galois extension. Let θ be a generator of $\mathrm{Gal}(E/F)$. It acts on the set of automorphic representations π' of $\mathrm{GL}_n(\mathbb{A}_E)$ via

$$\pi'^\theta(g) := \pi'(\theta(g)).$$

Let $N_{E/F} : \mathbb{A}_E^\times \rightarrow \mathbb{A}_F^\times$ be the norm map.

Theorem 13.4.1 (Base change) *Let E/F be a cyclic prime degree extension of number fields and let θ be a generator of $\mathrm{Gal}(E/F)$. For every cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ the base change π_E is an isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ such that $\pi_E^\theta \cong \pi_E$. The base change π_E is cuspidal if and only if $\pi \not\cong \pi \otimes \eta$ for all characters*

$$\eta : F^\times \backslash \mathbb{A}_F^\times / N_{E/F}(\mathbb{A}_E^\times) \longrightarrow \mathbb{C}^\times.$$

Conversely, if a cuspidal automorphic representation π' of $\mathrm{GL}_n(\mathbb{A}_E)$ satisfies $\pi' \cong \pi'^\theta$ then $\pi' = \pi_E$ for some cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$. $\square$

Theorem 13.4.2 (Automorphic induction) *Let E/F be a cyclic prime degree extension of number fields and let θ be a generator of $\mathrm{Gal}(E/F)$. If π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, then the automorphic induction $\mathrm{AI}(\pi)$ is an isobaric automorphic representation of $\mathrm{GL}_{n[E:F]}(\mathbb{A}_F)$. The representation $\mathrm{AI}(\pi)$ is cuspidal if and only if $\pi \not\cong \pi^\theta$. A cuspidal automorphic representation of $\mathrm{GL}_{n[E:F]}(\mathbb{A}_F)$ is the automorphic induction of a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ if and only if $\pi \cong \pi \otimes \eta$ for some character $\eta \in F^\times \backslash \mathbb{A}_F^\times / N_{E/F}(\mathbb{A}_E^\times) \rightarrow \mathbb{C}^\times$.* $\square$

Implicitly in this theorem we are using the local Langlands correspondence for the general linear group to define base change and automorphic induction. This was not available when Arthur and Clozel proved their result. Arthur and Clozel instead used the representation theoretic description of the base change isolated by Shintani [AC89, Definition 6.1]. The compatibility of the two notions is checked in [HT01, Lemma VII.2.6].

By breaking an arbitrary solvable extension E/F into a sequence of prime degree cyclic extensions, one deduces the existence of base changes of isobaric automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ along E/F . Characterizing the image of the base change in this case is more subtle, however, see [LR98, Raj02].

The case of an arbitrary extension E/F is open. The first author has described an approach in [Get12, Get20]. The work in [Get12, Get20] can be seen as a refinement and explication of Langlands' beyond endoscopy idea [Lan04] in a special case.

13.5 The strong Artin conjecture

Let E/F be a (nontrivial) finite degree Galois extension of number fields. For any homomorphism

$$\rho : \mathrm{Gal}(E/F) \longrightarrow \mathrm{GL}_n(\mathbb{C}) \quad (13.8)$$

one can define the Artin L -function $L(s, \rho)$. Artin and Brauer proved that this L -function admits a meromorphic continuation to the plane, satisfies a functional equation, and has a finite number of poles [MM97]. Artin conjectured the following refinement of this statement:

Conjecture 13.5.1 (Artin) *If ρ is nontrivial and irreducible then $L(s, \rho)$ is holomorphic.*

If ρ is the trivial representation then $L(s, \rho) = \Lambda_F(s)$, the completed Dedekind zeta function of F . It is certainly not holomorphic; in fact it has simple poles at $s = 0, 1$. Artin's conjecture implies that this is the only irreducible Galois representation whose Artin L -function has a pole.

Let us reinterpret Artin L -functions from the point of view of Langlands functoriality. The representation ρ defines a representation

$$r_\rho : {}^L\mathrm{GL}_1 \longrightarrow \mathrm{GL}_n(\mathbb{C}) \quad (13.9)$$

by tensoring with the trivial representation of ${}^L\mathrm{GL}_1^\circ = \mathrm{GL}_1(\mathbb{C})$. Temporarily denote by 1 the trivial representation of $\mathrm{GL}_1(\mathbb{A}_F)$, which is obviously automorphic. The Langlands functoriality conjecture predicts that the admissible representation $r_\rho(1)$ is automorphic. If this is the case we say that ρ is **automorphic**. One has that

$$L(s, r_\rho(1)) = L(s, 1, r_\rho) = L(s, \rho).$$

Thus the Artin conjecture is implied by the Theorem 11.7.1 and the following conjecture, a refinement of the assertion that ρ is automorphic:

Conjecture 13.5.2 (Strong Artin) *If ρ is irreducible then there exists a cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that*

$$\pi_v = r_\rho(1)_v$$

for all places v .

This is in fact a much deeper conjecture than the Artin conjecture. One can make precise the difference between Conjecture 13.5.1 and Conjecture 13.5.2 using Theorem 11.9.1.

Let E/F be a cyclic prime degree extension and let $\chi : E^\times \backslash \mathbb{A}_E^\times \rightarrow \mathbb{C}^\times$ be a finite order character that is not invariant under $\text{Gal}(E/F)$. Then

$$\rho = \text{Ind}_E^F(\chi) : \text{Gal}_F \longrightarrow \text{GL}_{[E:F]}(\mathbb{C})$$

is an irreducible Galois representation. The strong Artin conjecture is known for this representation by Theorem 13.4.2. A profitable strategy in addressing cases of Conjecture 13.5.2 has been to use cyclic prime degree base change (Theorem 13.4.1) to reduce the conjecture to this case. For example, this strategy can be used to prove the following [AC89]:

Theorem 13.5.3 (Arthur-Clozel) *Assume that E/F is a Galois extension of number fields with nilpotent Galois group $\text{Gal}(E/F)$. Then if ρ is any irreducible complex representation of $\text{Gal}(E/F)$, the strong Artin conjecture is true for ρ . $\square$*

Given that cyclic prime degree base change is known, and every solvable extension may be obtained from a sequence of cyclic prime degree extensions, it is striking that even when $\text{Gal}(E/F)$ is solvable the strong Artin conjecture is still not known in general.

Rather than studying the Artin conjecture for particular types of Galois groups, one could fix an integer n and ask for results towards the strong Artin conjecture for representations $r : \text{Gal}(E/F) \rightarrow \text{GL}_n(\mathbb{C})$. The conjecture is of course true for $n = 1$ by class field theory. However even the case $n = 2$ is not known in general. To explain what is known, it is convenient to recall the following theorem of Klein:

Theorem 13.5.4 (Klein) *A finite subgroup of $\text{PGL}_2(\mathbb{C})$ is isomorphic to one of the following:*

- a cyclic group,*
- a dihedral group,*
- the tetrahedral group A_4 ,*
- the octahedral group S_4 ,*
- the icosahedral group A_5 .*

$\square$

A subgroup of $\text{GL}_2(\mathbb{C})$ mapping to a cyclic group in $\text{PGL}_2(\mathbb{C})$ acts reducibly on $\mathbb{C}^2$, so we can ignore these groups. We say a representation

$$\rho : \text{Gal}(E/F) \longrightarrow \text{GL}_2(\mathbb{C})$$

is **dihedral** if the image of the composite map

$$\rho : \mathrm{Gal}(E/F) \longrightarrow \mathrm{GL}_2(\mathbb{C}) \longrightarrow \mathrm{PGL}_2(\mathbb{C})$$

is a dihedral group. We use the analogous convention when dihedral is replaced by **tetrahedral**, **octahedral**, or **icosahedral**. In all but the icosahedral case, the strong Artin conjecture is known to be true:

Theorem 13.5.5 (Langlands-Tunnell) *The strong Artin conjecture is true for dihedral, tetrahedral, or octahedral representations ρ .* $\square$

In [Lan80] Langlands derived the dihedral and tetrahedral cases from the theory of base change reviewed in the previous section. Tunnell completed the octahedral case in [Tun81]. Interestingly this theorem was used in Wiles' proof of Fermat's last theorem.

The key difference with the icosahedral case is that it is the only case where the $\mathrm{Gal}(E/F)$ is nonsolvable. If one had the theory of base change for nonsolvable extensions, one could make decisive progress on this last case [Get12].

Though the theory of base change for nonsolvable extensions is unavailable at the time of this writing, one can still make progress towards the strong Artin conjecture under certain assumptions. Namely, assume that the base field F is totally real. For every infinite place v of F , complex conjugation defines a conjugacy class $c_v \subset \mathrm{Gal}(\bar{F}/F)$. Any representation $\rho : \mathrm{Gal}(E/F) \rightarrow \mathrm{GL}_2(\mathbb{C})$ extends to $\mathrm{Gal}(\bar{F}/F)$, and we say it is **odd** if $\det(\rho(c_v)) = -1$ for all infinite places v .

Theorem 13.5.6 *If F is a totally real field and $\rho : \mathrm{Gal}(E/F) \rightarrow \mathrm{GL}_2(\mathbb{C})$ is an odd irreducible representation then the strong Artin conjecture is true for ρ .* $\square$

This theorem was proven by Pilloni and Stroth [PS16b] after the case $F = \mathbb{Q}$ was settled by Khare and Wintenberger [KW09a, KW09b] as a consequence of their proof of Serre's conjecture (Kisin made a decisive contribution [Kis09]). There was a body of work prior to this spearheaded by Taylor; see [PS16b] for references. The oddness assumption allows one to apply the powerful machinery of Galois deformation theory that played so pivotal a role in Wiles' proof of Fermat's last theorem.

13.6 The Langlands-Shahidi method

As explained in §10.5, Langlands proved that $L^2([G])$ can be completely described in terms of cuspidal automorphic representations of Levi subgroups of G by means of Eisenstein series. Some time after completing this, he realized that the constant terms of certain Eisenstein series (which naturally

depend on a complex parameter) admit functional equations and could be described in terms of a new type of L -function [Lan71]. This led him to the definition of the Langlands L -functions discussed in §12.7 and to posing his functoriality conjecture.

Langlands obtained coarse functional equations for L -functions that appear in the constant terms of Eisenstein series. His results were subsequently completed and refined by Shahidi [Sha10]. This method of proving the analytic continuation of L -functions is known as **the Langlands-Shahidi method**. Combining these results with the converse theorem (Theorem 11.9.1), one can obtain cases of Langlands functoriality that, at present, have not been obtained by any other method. The book [Sha10] and the surveys [CKM04] are useful references for this theory.

We record some important results obtained via the Langlands-Shahidi method. We assume throughout this section that F is a number field. The first case we discuss is the symmetric powers of cuspidal automorphic representations of GL_2 . More precisely, let $\pi = \otimes'_v \pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ and let

$$\mathrm{Sym}^k : {}^L\mathrm{GL}_2 \longrightarrow {}^L\mathrm{GL}_{k+1} \quad (13.10)$$

be the representation that is the identity on the Galois factors and the symmetric k th power representation on the neutral components. By the local Langlands correspondence explained in §12.3 and §12.4, $\mathrm{Sym}^k(\pi_v)$ is a well-defined irreducible admissible representation of $\mathrm{GL}_{k+1}(F_v)$ for all places v of F . Hence

$$\mathrm{Sym}^k(\pi) = \otimes'_v \mathrm{Sym}^k(\pi_v)$$

is an irreducible admissible representation of $\mathrm{GL}_{k+1}(\mathbb{A}_F)$.

Theorem 13.6.1 *If π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ then the irreducible admissible representation $\mathrm{Sym}^k(\pi)$ of $\mathrm{GL}_{k+1}(\mathbb{A}_F)$ is automorphic for $k \leq 4$. $\square$*

The proof of Theorem 13.6.1 was given by Gelbart-Jacquet [GJ78] for $k = 2$, by Kim-Shahidi [KS02] for $k = 3$, and by Kim [Kim03] for $k = 4$. As of this writing, the corresponding statement for $k \geq 5$ remains unproven. Gelbart and Jacquet's work occurred before the local Langlands correspondence was proven. In [Kim03] Kim gives an alternate proof that includes compatibility with the local Langlands correspondence.

If F is totally real and π_v is an essentially square integrable representation for all $v|\infty$ then Newton and Thorne [NT22] have shown that $\mathrm{Sym}^k(\pi)$ is automorphic for all k . Their methods are essentially disjoint from the Langlands-Shahidi method and build on the work of many. The proof relies heavily on the assumption on π_∞ . It appears unlikely that it can be adapted to treat the general case.

The second case we discuss is Langlands functoriality for Rankin-Selberg products of cuspidal automorphic representations of GL_n . For any $m, n \in$

$\mathbb{Z}_{>0}$, let

$$r_{\otimes} : {}^L(\mathrm{GL}_m \times \mathrm{GL}_n) \longrightarrow {}^L\mathrm{GL}_{mn} \quad (13.11)$$

be the representation that is the tensor product on the neutral components and the identity map on the Galois factors. This family of L -maps is extremely important; taking tensor products is a basic procedure for constructing new representations out of old representations.

For every place v of F and every admissible representation $\pi_{1v} \otimes \pi_{2v}$ of $\mathrm{GL}_m(F_v) \times \mathrm{GL}_n(F_v)$, we have

$$L(s, \pi_{1v} \otimes \pi_{2v}, r_{\otimes}) = L(s, \pi_{1v} \times \pi_{2v})$$

where the L -function on the right is the Rankin-Selberg L -function appearing in §11.5. This identity is built into the local Langlands correspondence. It is customary to write

$$\pi_{1v} \boxtimes \pi_{2v} := r_{\otimes}(\pi_{1v} \otimes \pi_{2v}).$$

Thus for automorphic representations π_1 and π_2 of $\mathrm{GL}_m(\mathbb{A}_F)$ and $\mathrm{GL}_n(\mathbb{A}_F)$, respectively, we have an admissible representation

$$\pi_1 \boxtimes \pi_2 := \prod_v \pi_{1v} \boxtimes \pi_{2v}$$

of $\mathrm{GL}_{mn}(\mathbb{A}_F)$. By Rankin-Selberg theory, specifically Theorem 11.7.1, the standard L -function of this representation behaves as predicted by Conjecture 12.7.1. In general this is not enough to deduce that $\pi_1 \boxtimes \pi_2$ is automorphic using the converse theorem (Theorem 11.9.1). However for small m and n a combination of the converse theorem and the Langlands-Shahidi method yields the following theorem:

Theorem 13.6.2 *For $1 \leq k \leq 3$, let π_1 and π_2 be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ and $\mathrm{GL}_k(\mathbb{A}_F)$, respectively. Then $\pi_1 \boxtimes \pi_2$ is automorphic.* $\square$

For the $\mathrm{GL}_2 \times \mathrm{GL}_2$ case, this was proven in [Ram00a]. For the $\mathrm{GL}_2 \times \mathrm{GL}_3$ case, this is [KS02, Theorem 5.1]. The paper [Ram00a] was written before the proof of the local Langlands correspondence. Compatibility of the transfer with the local Langlands correspondence was checked in [Kim03].

13.7 Functoriality for the classical groups

We continue to assume that F is a number field. Let

$$J_n := \begin{pmatrix} & & 1 \\ & \ddots & \\ & & \\ 1 & & \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Z}) \quad (13.12)$$

be the antidiagonal matrix and let

$$J'_{2n} := \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}. \quad (13.13)$$

We take our “standard” orthogonal and symplectic groups to be the groups over F whose points in an F -algebra R are given by

$$\begin{aligned} \mathrm{O}_n(R) &:= \{g \in \mathrm{GL}_n(R) : gJ_n g^t = J_n\}, \\ \mathrm{Sp}_{2n}(R) &:= \{g \in \mathrm{GL}_{2n}(R) : gJ'_{2n} g^t = J'_{2n}\}. \end{aligned} \quad (13.14)$$

Let E/F be a quadratic extension. We then define

$$\mathrm{U}_n(R) := \{g \in \mathrm{GL}_n(E \otimes_F R) : gJ_n \bar{g}^t = J_n\},$$

where the bar denotes the action of the nontrivial element of $\mathrm{Gal}(E/F)$. This is the quasi-split unitary group attached to the extension E/F .

The extension E/F also defines a quasi-split orthogonal group as follows: View E as a vector space over F equipped with the quadratic form $N_{E/F}$. Choosing a basis, we obtain a symmetric matrix $\gamma_E \in \mathrm{GL}_2(F)$ corresponding to the quadratic form in the usual manner. Then, for an F -algebra R ,

$$\mathrm{O}_{2n}^*(R) := \left\{ g \in \mathrm{GL}_n(R) : g \begin{pmatrix} & J_{n-1} \\ & \gamma_E \\ J_{n-1} & \end{pmatrix} g^t = \begin{pmatrix} & J_{n-1} \\ & \gamma_E \\ J_{n-1} & \end{pmatrix} \right\}.$$

We define $\mathrm{SO}_n < \mathrm{O}_n$ and $\mathrm{SO}_n^* < \mathrm{O}_n^*$ to be the subgroups of determinant 1 as usual. We refer to the groups above as the **quasi-split classical groups** of rank n defined over F and denote them by G_n . To avoid degenerate special cases,

when $G_n = \mathrm{SO}_{2n}$ or $G_n = \mathrm{SO}_{2n}^*$ we always assume $n \geq 2$.

Following [CPSS11] we have the following table of groups and L -maps:

G_n	$r : {}^L G_n \rightarrow {}^L H_N$	H_N
SO_{2n+1}	$\mathrm{Sp}_{2n}(\mathbb{C}) \times \mathrm{Gal}_F \rightarrow \mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{Gal}_F$	GL_{2n}
SO_{2n}	$\mathrm{SO}_{2n}(\mathbb{C}) \times \mathrm{Gal}_F \rightarrow \mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{Gal}_F$	GL_{2n}
SO_{2n}^*	$\mathrm{SO}_{2n}(\mathbb{C}) \rtimes \mathrm{Gal}_F \rightarrow \mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{Gal}_F$	GL_{2n}
Sp_{2n}	$\mathrm{SO}_{2n+1}(\mathbb{C}) \times \mathrm{Gal}_F \rightarrow \mathrm{GL}_{2n+1}(\mathbb{C}) \times \mathrm{Gal}_F$	GL_{2n+1}
U_n	$\mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}_F \rightarrow \mathrm{GL}_n(\mathbb{C})^2 \rtimes \mathrm{Gal}_F$	$\mathrm{Res}_{E/F} \mathrm{GL}_n$

The L -maps in the first, second, and fourth rows are given by the natural inclusions. For the third, let

$$\widehat{w} := \begin{pmatrix} I_{n-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{n-1} \end{pmatrix}.$$

Then Gal_F acts on $\mathrm{SO}_{2n}(\mathbb{C})$ via its quotient $\mathrm{Gal}(E/F)$, with the nontrivial element σ of $\mathrm{Gal}(E/F)$ acting by conjugation by $\widehat{w}$. The map

$$\mathrm{SO}_{2n}(\mathbb{C}) \rtimes \mathrm{Gal}_F \longrightarrow \mathrm{GL}_{2n}(\mathbb{C}) \times \mathrm{Gal}_F$$

is given by the natural embedding on the neutral component and sends $1 \rtimes \sigma$ to $\widehat{w} \times \sigma$. Finally when $G_n = \mathrm{U}_n$ and $H_N = \mathrm{Res}_{E/F} \mathrm{GL}_n$ the absolute Galois group of F again acts on $\mathrm{GL}_n(\mathbb{C})$ via its quotient $\mathrm{Gal}(E/F)$, with the nontrivial element σ acting via

$$\begin{aligned} {}^L \mathrm{U}_n^\circ &= \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}_n(\mathbb{C}) \\ g &\longmapsto J_n^{-1} g^{-t} J_n, \\ {}^L H_N^\circ &= \mathrm{GL}_n^2(\mathbb{C}) \longrightarrow \mathrm{GL}_n^2(\mathbb{C}) \\ (h_1, h_2) &\longmapsto (h_2, h_1) \end{aligned} \tag{13.15}$$

and the embedding r is given by

$$r(g \rtimes \sigma) = (g, J_n^{-1} g^{-t} J_n) \rtimes \sigma. \tag{13.16}$$

There is a great deal known about Langlands functoriality for this collection of L -maps. Definitive results were obtained for generic representations in [CKPSS04, CPSS11]. We focus on this case in the current section. Later Arthur [Art13] established the same result for arbitrary representations (in the orthogonal and symplectic cases) under suitable assumptions (see §13.8

for details). The unitary case was completed by Mok [Mok15] under the same assumptions.

The following theorem is [CPSS11, Theorem 1.1]. It uses the notation of the table above. It represents the culmination of an impressive body of work by Cogdell, Kim, Krishnamurthy, Piatetski-Shapiro, and Shahidi.

Theorem 13.7.1 *Let π be an irreducible globally generic cuspidal automorphic representation of $G_n(\mathbb{A}_F)$. Then there exists a functorial transfer of π to $H_N(\mathbb{A}_F)$ with respect to the L -map r . $\square$*

The main tools used in the proof of this theorem are the Langlands-Shahidi method and an analogue of the converse theorem stated in Theorem 11.9.1. Let π be as in the statement of Theorem 13.7.1. For every integer $m \geq 1$ and every cuspidal automorphic representation τ of $H_m(\mathbb{A}_F)$, the Langlands-Shahidi method provides an L -function $L(s, \pi \times \tau)$ that admits a meromorphic continuation to the plane and a functional equation. One constructs an admissible representation $r(\pi)$ of $H_N(\mathbb{A}_F)$ such that

$$L(s, \pi_v \times \tau_v) = L(s, r(\pi)_v \times \tau_v) \quad (13.17)$$

for all Archimedean places v and non-Archimedean places v where F is unramified over its prime field and π_v, τ_v are unramified. It is harder to obtain information at the ramified places. For this one studies γ -factors $\gamma(s, \pi_v \times \tau_v, \psi_v)$ defined for the Langlands-Shahidi L -function $L(s, \pi_v \times \tau_v)$. They are the analogues for Langlands-Shahidi L -functions of the γ -factors defined in Theorem 11.5.4 in the context of Rankin-Selberg theory. When $m = 1$, so τ_v is a quasi-character, it can be shown that $\gamma(s, \pi_v \times \tau_v, \psi_v)$ depends only on the central character of π_v if τ_v is sufficiently ramified. This result is known as **stability of γ -factors**. This allows one to prove (13.17) for all places v and a range of m after replacing τ_v by a ramified twist. Applying a suitable converse theorem one deduces the automorphy of $r(\pi)$.

Assuming Theorem 13.7.1, Ginzburg, Rallis, and Soudry [GRS01] had previously used their descent technique to characterize the image of the functorial transfer attached to r . In order to state their result, we digress to discuss a family of L -functions. As a first step we record a table of representations r' and characters $\chi_{G_n} : [\mathbb{G}_m]^\times \rightarrow \mathbb{C}^\times$. The representations r' listed below are representations of H_N , where H_N is the group attached to G_n in the table above:

G_n	r'	χ_{G_n}
SO_{2n+1}	$\wedge^2$	1
SO_{2n}	Sym^2	1
SO_{2n}^*	Sym^2	$\eta_{E/F}$
Sp_{2n}	Sym^2	1
U_{2n}	$\mathrm{As}_{E/F} \otimes \eta_{E/F}$	1
U_{2n+1}	$\mathrm{As}_{E/F}$	1

Here 1 denotes the trivial character, and $\eta_{E/F}$ is the character attached to E/F by class field theory. The only representation that is not self-explanatory is $\mathrm{As}_{E/F}$. For any field extension E/F of finite degree the **Asai representation**

$$\mathrm{As}_{E/F} : {}^L\mathrm{Res}_{E/F}\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})^{\mathrm{Hom}_F(E, \overline{F})} \rtimes \mathrm{Gal}_F \rightarrow \mathrm{GL}_{n[E:F]}(\mathbb{C}) \quad (13.18)$$

is defined as follows. We have a homomorphism

$$\mathrm{As}_{E/F} : \mathrm{GL}_n(\mathbb{C})^{\mathrm{Hom}_F(E, \overline{F})} \rtimes \mathrm{Gal}_F \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\otimes_{\xi \in \mathrm{Hom}_F(E, \overline{F})} \mathbb{C}^n)$$

given by stipulating that for $(g_\xi) \rtimes \sigma \in \mathrm{GL}_n(\mathbb{C})^{\mathrm{Hom}_F(E, \overline{F})} \rtimes \mathrm{Gal}_F$

$$\mathrm{As}_{E/F}((g_\xi) \rtimes 1)(\otimes_\xi v_\xi) = \otimes_\xi g_\xi v_\xi \quad \text{and} \quad \mathrm{As}_{E/F}(1 \rtimes \sigma)(\otimes_\xi v_\xi) = \otimes_\xi v_{\sigma^{-1} \circ \xi}.$$

The inverse is here because σ maps the ξ factor of a pure tensor to the $\sigma \circ \xi$ factor. Upon choosing an isomorphism

$$\mathrm{Aut}_{\mathbb{C}}(\otimes_{\xi \in \mathrm{Hom}_F(E, \overline{F})} \mathbb{C}^n) \cong \mathrm{GL}_{n[E:F]}(\mathbb{C})$$

and we obtain the representation (13.18). This representation plays an important role in §14.5 below.

Conjecture 12.7.1 on the analytic properties of Langlands L -functions is more or less known in the cases above. In more detail, let $\psi : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times$ be a nontrivial character. Langlands-Shahidi theory produces L -functions and ε -factors

$$L_{\mathrm{LS}}(s, \pi_v, r') \quad \text{and} \quad \varepsilon_{\mathrm{LS}}(s, \pi_v, r', \psi_v) \quad (13.19)$$

for every place v of F that are equal to the L -function and ε -factor

$$L(s, \pi_v, r') \quad \text{and} \quad \varepsilon(s, \pi_v, r', \psi_v), \quad (13.20)$$

respectively, when v is Archimedean or v lies outside of some finite set S of non-Archimedean places. Let

$$\begin{aligned} L_{\text{LS}}(s, \pi, r') &:= \prod_v L_{\text{LS}}(s, \pi_v, r'), \\ \varepsilon_{\text{LS}}(s, \pi, r') &:= \prod_v \varepsilon_{\text{LS}}(s, \pi_v, r', \psi_v). \end{aligned} \tag{13.21}$$

The global ε -factor is again independent of ψ . These factors are originally defined for $\text{Re}(s)$ large. One has the following theorem, due to Shahidi following work of Langlands [Sha90, Theorem 7.7]:

Theorem 13.7.2 *The L -function $L_{\text{LS}}(s, \pi, r')$ admits a meromorphic continuation to the plane. It satisfies the functional equation*

$$L_{\text{LS}}(s, \pi, r') = \varepsilon_{\text{LS}}(s, \pi, r') L_{\text{LS}}(1 - s, \pi^\vee, r').$$

□

The L -functions $L_{\text{LS}}(s, \pi, \wedge^2)$ and $L_{\text{LS}}(s, \pi, \text{Sym}^2)$ are moreover bounded in vertical strips [GS01].

Generalizations of the Rankin-Selberg theory discussed in Chapter 11 provide yet another definition of the L -functions and ε -factors above and one can prove a theorem analogous to Theorem 13.7.2 using these L -functions in most cases (see [Bum05] for a survey). The definition of L -functions in terms of Rankin-Selberg integrals is important because in practice it provides information (e.g. about poles) that is complementary to that obtained via the Langlands-Shahidi method.

Of course, one wants to know that all of these L -functions and ε -factors agree. This has been proven in important cases. For the agreement of Langlands-Shahidi and Rankin-Selberg L -factors in many cases, we refer to [Kap15]. We also have the following result [CST17]:

Theorem 13.7.3 *If r' is $\wedge^2$ or Sym^2 , then one has that*

$$L_{\text{LS}}(s, \pi_v, r') = L(s, \pi_v, r') \quad \text{and} \quad \varepsilon_{\text{LS}}(s, \pi_v, r') = \varepsilon(s, \pi_v, r').$$

□

After this digression on L -functions, we turn back to the characterization of the image of the functorial transfer. As above let E/F be a field extension of degree 2. The group $\text{Res}_{E/F} \text{GL}_n$ admits an outer automorphism σ induced by the action of the nontrivial element of $\text{Gal}(E/F)$. For irreducible admissible representations π' of $\text{Res}_{E/F} \text{GL}_n(\mathbb{A}_F)$, let

$$\pi'^\sigma(g) := \pi'(\sigma(g))$$

and let

$$\pi'^* := (\pi'^\sigma)^\vee.$$

This is sometimes known as the **conjugate dual representation**. We say that π' is **conjugate self-dual** if $\pi' \cong \pi'^*$.

For the purposes of stating the following theorem, it is convenient to let $\pi'^* := \pi'^\vee$ when $H_N = \mathrm{GL}_N$ and let π'^* be defined as above otherwise. Let $\omega_{\pi'}$ be the central quasi-character of π' . For the proof of the following theorem we refer to [GRS01, Sou05]:

Theorem 13.7.4 *Let π be a globally generic cuspidal automorphic representation of $G_n(\mathbb{A}_F)$. The functorial lift $r(\pi)$ satisfies $\omega_{r(\pi)}|_{\mathbb{A}_F^\times} = \chi_{G_n}$ and is of the form*

$$\pi' = \pi'_1 \boxplus \cdots \boxplus \pi'_d, \quad (13.22)$$

where each π'_i is a unitary cuspidal automorphic representation of $H_{N_i}(\mathbb{A}_F)$ with $\sum_i N_i = N$ satisfying

- (a) $\pi'^*_i \cong \pi'_i$,
- (b) $\pi'_i \cong \pi'_j$ if and only if $i = j$,
- (c) $L(s, \pi'_i, r')$ has a pole at $s = 1$.

Moreover, any π' as in (13.22) satisfying $\omega_{\pi'}|_{\mathbb{A}_F^\times} = \chi_{G_n}$ and (a)-(c) is of the form $r(\pi)$ for some globally generic cuspidal automorphic representation π of $G_n(\mathbb{A}_F)$. $\square$

Here $\omega_{\pi'}|_{\mathbb{A}_F^\times}$ is just $\omega_{\pi'}$ unless $G_n = \mathrm{U}_n$, in which case it is the restriction of $\omega_{\pi'}$ to the proper subgroup

$$\mathbb{A}_F^\times \hookrightarrow \mathbb{A}_E^\times \xrightarrow{\sim} Z_{\mathrm{Res}_{E/F} \mathrm{GL}_n}(\mathbb{A}_F).$$

Analogues of Theorem 13.7.1 and Theorem 13.7.4 are known for general spin groups by work of Asgari and Shahidi [AS06, AS14].

In this section we have concentrated on generic representations. This is because originally the approach explained above relied crucially on this assumption. Recently the genericity assumption has been removed in work of Cai, Friedberg, Ginzburg, and Kaplan. We refer to [CFK18] (which is based on [CFGK19]) for a precise statement. This provides a new, unconditional, and independent proof of a portion of the endoscopic classification of representations on classical groups that we discuss in the next section.

13.8 Endoscopic classification of representations

In this section we continue to use the notational conventions of the previous section. In particular F is a number field. In [Art13], Arthur proves the existence of functorial transfer with respect to the L -maps r of the previous

section (at least when $G_n \neq U_n$). In particular there is no genericity assumption in his work. Moreover, he gave a precise enough description of the fibers of the functorial transfer that he could classify the discrete spectrum of $L^2([G_n])$ in terms of automorphic representations on H_N . We explain these results in the current section. The main tool used to establish Arthur's result is the theory of **twisted endoscopy** (see §19.5), so one refers to Arthur's work and subsequent refinements as the **endoscopic classification** of representations.

In his book [Art13] Arthur gives a careful account of how to replace objects attached to the conjectural global Langlands group $\mathcal{L}_F$ discussed in §12.6 by well-defined, un conjectural objects. For the sake of clarity and brevity, we will not reproduce this discussion apart from some brief comments at the end of this section. Instead, we will state Arthur's main results as directly as possible.

There are two points the reader should bear in mind. First, we have not stated all of Arthur's results, and we have not included Mok's results in the $G_n = U_n$ case since this would require more notation. We refer to [Mok15] instead. Second, all of the work in this section (and in Mok's work) is conditional on Arthur's local claims discussed in §12.5 above and on the stabilization of the twisted weighted trace formula. Mœglin and Waldspurger have proven this under the assumption of the twisted weighted fundamental lemma [MW16a, MW16b]. The original fundamental lemma of Shelstad and Langlands was proved in the breakthrough work of Ngô in [Ngô10b] for which he received the Fields medal. Ngô's work is based on the work of many over decades and his survey [Ngô10c] is a good place to obtain some historical perspective. Other versions of the fundamental lemma have been proven by Laumon and Chaudouard using nontrivial generalizations of Ngô's techniques [CL10, CL12]. In principle a proof of the twisted weighted fundamental lemma should be within reach but as of this writing it is not complete.

For the remainder of this section, we assume $G_n \neq U_n$. The first main theorem is the following:

Theorem 13.8.1 *Every irreducible subrepresentation of $L^2([G_n])$ admits a functorial transfer to $H_N(\mathbb{A}_F)$ with respect to r . $\square$*

This result is implicit in the main theorems stated in [Art13]. Since irreducible subrepresentations of $L^2([G_n])$ need not be tempered, to make precise what one means by a functorial transfer, one needs more than the theory outlined in §12.5. We will not make this precise and instead refer the reader to [Art13].

There is a wrinkle which will continue to play a role below. When G_n is SO_{2n} or SO_{2n}^* let $\theta : G_n \rightarrow G_n$ be the automorphism induced by conjugation by an element of $\mathrm{O}_{2n}(F) - \mathrm{SO}_{2n}(F)$ or $\mathrm{O}_{2n}^*(F) - \mathrm{SO}_{2n}^*(F)$, respectively. The automorphism θ is unique up to composing with an inner automorphism of G_n . The functorial transfer given in Theorem 13.8.1 is insensitive to replacing a representation π by π^θ , where $\pi^\theta(g) := \pi(\theta(g))$. With this in mind let

$$\tilde{C}_c^\infty(G_n(\mathbb{A}_F)) \quad (13.23)$$

be $C_c^\infty(G_n(\mathbb{A}_F))$ except in the special case where G_n is SO_{2n} or SO_{2n}^* , in which case it is the subalgebra of $C_c^\infty(G_n(\mathbb{A}_F))$ invariant under θ .

Let

$$L_{\mathrm{disc}}^2([G_n]) < L^2([G_n])$$

be the largest closed subspace that decomposes discretely under $G_n(\mathbb{A}_F)$. In view of Theorem 13.8.1, it is natural to partition $L_{\mathrm{disc}}^2([G_n])$ into the fibers of the functorial transfer to $H_N(\mathbb{A}_F)$ and then try to describe the fibers, preferably in terms of local data. This is precisely what Arthur accomplished.

We state the main classification result first. The remainder of the section is devoted to defining the notation that appears.

Theorem 13.8.2 (Arthur) *There is an $\tilde{C}_c^\infty(G_n(\mathbb{A}_F))$ -module isomorphism*

$$L_{\mathrm{disc}}^2([G_n]) \cong \bigoplus_{\psi \in \tilde{\Psi}_2(G_n)} \bigoplus_{\pi \in \tilde{\Pi}_\psi(\varepsilon_\psi)} \pi^{\oplus m_\psi},$$

where $m_\psi = 1$ or 2 . □

This is [Art13, Theorem 1.5.2]. For a cuspidal automorphic representation τ of $H_N(\mathbb{A}_F)$ and $m \in \mathbb{Z}_{\geq 1}$, let (τ, m) be the Speh representation of §10.8. One says that τ is of **orthogonal type** if $L(s, \tau, \mathrm{Sym}^2)$ has a pole at $s = 1$ and of **symplectic type** if $L(s, \tau, \wedge^2)$ has a pole at $s = 1$. Since

$$L(s, \tau, \mathrm{Sym}^2)L(s, \tau, \wedge^2) = L(s, \tau \times \tau)$$

it follows from Theorem 11.7.1 that τ cannot be both orthogonal and symplectic, and it is orthogonal or symplectic if and only if $\tau \cong \tau^\vee$. We define the **type** of the representation (τ, m) as follows:

τ	m	(τ, m)
orthogonal	even	symplectic
symplectic	even	orthogonal
orthogonal	odd	orthogonal
symplectic	odd	symplectic

The set $\tilde{\Psi}_2(G_n)$ in Theorem 13.8.2 is the set of automorphic representations of $H_N(\mathbb{A}_F)$ of the form

$$\boxplus_{i=1}^d (\tau_i, m_i)$$

where

- (1) τ_i is a cuspidal automorphic representation of $H_{N_i}(\mathbb{A}_F)$,

- (2) $\sum_{i=1}^d N_i m_i = N$,
- (3) $\tau_i^\vee \cong \tau_i$ for all i ,
- (4) $\tau_i \cong \tau_j$ if and only if $i = j$,
- (5) If ${}^L G_n^\circ$ is orthogonal (resp. symplectic) then (τ_i, m_i) is orthogonal (resp. symplectic).

We note that condition (5) actually implies condition (3). The set $\tilde{\Psi}_2(G_n)$ is known as the set of **discrete global A -parameters** of G_n . The A in A -packets is in honor of Arthur, who first introduced these objects. The multiplicity m_ψ in Theorem 13.8.2 is 1 unless N is even, ${}^L G_n^\circ = \mathrm{SO}_N(\mathbb{C})$, and $N_i m_i$ is even for all i , in which case it is 2. The discrete global A -parameter is said to be **generic** if $m_i = 1$ for all i . In this case we also refer to the parameter as a **discrete generic global L -parameter**.

For every $\psi \in \tilde{\Psi}_2(G_n)$, Arthur defines a finite 2-group $\mathcal{S}_\psi$ and a character

$$\varepsilon_\psi : \mathcal{S}_\psi \longrightarrow \{\pm 1\}.$$

We omit the definition. For every place v of F and every $\psi \in \tilde{\Psi}_2(G_n)$, we define a representation

$$\psi_v : W'_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L H_N$$

by

$$\psi_v := \bigoplus_{i=1}^d \mathrm{rec}(\tau_{iv}) \otimes \mathrm{Sym}^{m_i}$$

where rec is the local Langlands reciprocity map of (12.17) in §12.4. We note that ψ_v is not an L -parameter, but one can obtain an L -parameter from it as follows:

$$\rho(\psi_v)(g) := \psi_v \left(g, \begin{pmatrix} |g|_v^{1/2} & \\ & |g|_v^{-1/2} \end{pmatrix} \right).$$

Here $|\cdot|_v$ is the composite

$$W'_{F_v} \longrightarrow W_{F_v}^{\mathrm{ab}} \xrightarrow{\mathrm{Art}_F^{-1}} F_v^\times \xrightarrow{|\cdot|_v} \mathbb{R}_{>0}$$

where the first arrow is the canonical quotient map.

The extra $\mathrm{SL}_2(\mathbb{C})$ factor occurring in the domain of ψ_v is colloquially known as the **Arthur- SL_2** and plays a role similar to the representation of SL_2 that appears in Hodge theory. One proves that an $\widehat{H}_N(\mathbb{C})$ -conjugate of ψ_v factors through the L -map r and hence ψ_v defines a homomorphism

$$\psi_v : W'_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^L G_n.$$

These are examples of **local A -parameters**, although we will not formally define this concept. One shows in addition the existence of maps

$$\mathrm{loc}_v : \mathcal{S}_\psi \longrightarrow \pi_0(\overline{S}_{\psi_v})$$

where

$$\overline{S}_{\psi_v} := C_{\widehat{G}_n(\mathbb{C})}(\mathrm{im}(\psi_v)) / Z_{\widehat{G}_n(\mathbb{C})}^{\mathrm{Gal}_{F_v}}$$

as in (12.24). We warn the reader that the component group $\pi_0(\overline{S}_{\psi_v})$ is denoted by $\mathcal{S}_{\psi_v}$ in [Art13].

For each A -parameter ψ_v Arthur defines a set $\widetilde{\Pi}(\psi_v)$ of irreducible admissible representations of $G_n(F_v)$ satisfying certain desiderata. This is already a substantial theorem as it amounts to a proof of a refined local Langlands correspondence for $G_n(F_v)$. The set $\widetilde{\Pi}(\psi_v)$ is the **local A -packet** attached to ψ_v . The use of $\widetilde{\Pi}(\psi_v)$ instead of the notation $\Pi(\rho_v)$ of §12.5 is intentional. First, the tilde is a reminder that one is only classifying representations up to the action of the outer automorphism θ when G_n is SO_{2n} or SO_{2n}^* . Second, the packets $\widetilde{\Pi}(\psi_v)$ are not L -packets. They are an enlargement of L -packets required due to the fact that the representations in question are often nontempered. In the special case where ψ_v is trivial on the Arthur- SL_2 , the packet $\widetilde{\Pi}(\psi_v)$ is an (almost tempered) L -packet in the sense of §12.5. We point out that ψ_v is trivial on the Arthur- SL_2 for all v for any discrete generic global L -parameter ψ .

As in §12.5, any $\pi_v \in \widetilde{\Pi}(\psi_v)$ comes equipped with a character

$$\langle \pi_v, \cdot \rangle : \pi_0(\overline{S}_{\psi_v}) \longrightarrow \mathbb{C}^\times$$

and thus for all $\pi \in \widetilde{\Pi}(\psi)$ obtain a character

$$\langle \pi, \cdot \rangle := \prod_v \langle \pi_v, \cdot \rangle.$$

This allows us to form the global adelic A -packet

$$\widetilde{\Pi}(\psi) := \left\{ \otimes'_v \pi_v : \pi_v \in \widetilde{\Pi}(\psi_v) \text{ and } \langle \pi_v, \cdot \rangle = 1 \text{ for almost all } v \right\}. \quad (13.24)$$

It consists of a set of admissible representations of $G_n(\mathbb{A}_F)$. Not all of them are automorphic, let alone occur in $L_{\mathrm{disc}}^2([G_n])$. The last piece of the classification theorem is a device for selecting which occur in $L^2([G_n])$. This is provided by ε_ψ . One defines

$$\widetilde{\Pi}_\psi(\varepsilon_\psi) := \{ \pi \in \widetilde{\Pi}(\psi) : \langle \pi, \cdot \rangle = \varepsilon_\psi \}. \quad (13.25)$$

This completes our discussion of the objects in Theorem 13.8.2.

There is one piece of terminology above that may seem puzzling. In this section discrete generic global L -parameters were defined in terms of cuspidal automorphic representations of general linear groups, whereas in §12.6 these objects were defined in terms of representations of the conjectural group $\mathcal{L}_F$. Since $\mathcal{L}_F$ is conjectural the earlier description is problematic, so Arthur

modified it as follows. Local L -packets for H_{NF_v} are singletons for all places v of F , at least in the almost tempered case. Hence global L -parameters into ${}^L H_N$ that are direct sums of discrete generic global L -parameters should correspond bijectively to isobaric sums

$$\boxplus_{i=1}^k \pi_i \quad (13.26)$$

of cuspidal automorphic representations of $A_{\mathrm{GL}_{n_i}} \backslash \mathrm{GL}_{n_i}(\mathbb{A}_F)$ satisfying

$$\sum_{i=1}^k n_i = N. \quad (13.27)$$

The bijection should be compatible with the local Langlands correspondence. On the other hand a discrete generic global parameter into ${}^L G_n$ is a particular type of homomorphism $\mathcal{L}_F \rightarrow {}^L H_N$ that factors through the L -map $r : {}^L G_n \rightarrow {}^L H_N$. Thus Arthur identifies the set of discrete generic global parameters $\mathcal{L}_F \rightarrow {}^L H_N$ with the set of isomorphism classes of cuspidal representations of $A_{\mathrm{GL}_N} \backslash \mathrm{GL}_N(\mathbb{A}_F)$. He then isolates exactly which isobaric sums (13.26) would have global L -parameters into ${}^L H_N$ that factor through $r : {}^L G_n \rightarrow {}^L H_N$ if we knew that $\mathcal{L}_F$ existed. This is how he arrives at the definition of discrete generic global L -parameters of G_n . Similar considerations lead to the definition of discrete global A -parameters.

Let us be more explicit about isolating discrete generic global L -parameters that factor through an L -map in a special case. Assume

$$G_n = \mathrm{Sp}_{2n}, \quad {}^L G_n = \mathrm{SO}_{2n+1}(\mathbb{C}) \times \mathrm{Gal}_F.$$

If we knew the existence of the global Langlands group $\mathcal{L}_F$, then the discrete generic global L -parameters $\rho : \mathcal{L}_F \rightarrow {}^L H_{2n+1}$ that factor through

$$r : {}^L \mathrm{Sp}_{2n} \longrightarrow {}^L \mathrm{GL}_{2n+1}$$

would be precisely those ρ whose image under the projection

$${}^L \mathrm{GL}_{2n+1} \longrightarrow \mathrm{GL}_{2n+1}(\mathbb{C})$$

fixes a symmetric bilinear form on $\mathbb{C}^{2n+1}$. This condition on ρ is equivalent to the assertion that the trivial representation is a subrepresentation of

$$\mathrm{Sym}^2 \circ \rho.$$

By a suitable generalization of the Artin conjecture, the trivial representation occurs in $\mathrm{Sym}^2 \circ \rho$ if and only if $L(s, \mathrm{Sym}^2 \circ \rho)$ has a pole at $s = 1$. The translation of this discussion to the automorphic side of the conjectural global Langlands correspondence is the assertion that a cuspidal automorphic representation π of $\mathrm{GL}_{2n+1}(\mathbb{A}_F)$ is a functorial transfer from $\mathrm{Sp}_{2n}(\mathbb{A}_F)$

if and only if $L(s, \pi, \text{Sym}^2)$ has a pole at $s = 1$. Assume the notation of the previous section. The line of reasoning above leads to the expectation that the cuspidal automorphic representations π' of $A_{H_N} \backslash H_N(\mathbb{A}_F)$ that are functorial transfers from $G_n(\mathbb{A}_F)$ with respect to $r : {}^L G_n \rightarrow {}^L H_N$ are precisely those π' such that $L(s, \pi', r')$ has a pole at $s = 1$. Up to some additional conditions on the central character, this is implied by either Theorem 13.7.4 or Theorem 13.8.2 and its analogue in the unitary case [Mok15].

This sort of characterization of the image of functorial transfers in terms of L -functions is in principle not limited to the case of twisted endoscopy. This observation forms the basis of Langlands' beyond endoscopy idea. It is a proposal for establishing Langlands functoriality in general [Lan04] (see also [Alt20, Get18, Get20, Her11a, Her16, Rud90, Sak22, Sar, Ven04]). It is important to note that the problem of characterizing the image of functorial lifts in terms of L -functions in general is much more complicated than the discussion above may lead one to believe [GK15, Hah16a, Hah16b, Hah21].

13.9 The function field case

In the function field case much more is known. Let F be a function field of characteristic p . For GL_2 , the Langlands correspondence over F was proved by Drinfeld [Dri80, Dri87b, Dri87a, Dri88]. The work earned Drinfeld a Fields medal. For GL_n with n arbitrary the Langlands correspondence over F was proved by L. Lafforgue [Laf02]. The work earned him a Fields' medal as well. Lafforgue's argument follows Drinfeld's, although there were substantial technical obstacles to overcome to adapt the techniques from the GL_2 case to higher rank.

L. Lafforgue constructed a bijection between the set of isomorphism classes of continuous irreducible representations $\text{Gal}_F \rightarrow \text{GL}_n(\overline{\mathbb{Q}}_\ell)$ that are unramified almost everywhere and have finite order determinant and isomorphism classes of cuspidal automorphic representations of $\text{GL}_n(\mathbb{A}_F)$ with finite order central character. He moreover proved that the bijection is compatible with the local Langlands correspondence at all unramified places. This is enough to uniquely determine the correspondence by the Chebotarev density theorem on the Galois side and the strong multiplicity one assertion of Theorem 11.7.2 on the automorphic side. The proof involves comparing the Grothendieck-Lefschetz trace formula for the trace of Frobenius on a certain stack and the Arthur-Selberg trace formula. The Grothendieck-Lefschetz trace formula encodes Galois theoretic information ultimately coming from the étale site and the Arthur-Selberg trace formula encodes automorphic information.

More recently, V. Lafforgue gave a decomposition of the space of cusp forms on an arbitrary reductive group over F in terms of L -parameters that is compatible with the local Langlands correspondence at almost all places

[Laf18]. This can be viewed as a construction of the map (12.36). Interestingly his proof does not involve the Arthur-Selberg trace formula.

All of the work above relies on objects known as *shtukas* (Russian for “thing”). The very definition of a *shtuka* involves the scheme

$$\mathrm{Spec}(\mathcal{O}_F \otimes_{\mathbb{F}_p} \mathcal{O}_F). \quad (13.28)$$

This sort of self product also comes up in Deligne’s proof of the Riemann hypothesis in the function field case. The analogue of this scheme does not exist in the number field case, since the various residue fields of the ring of integers of the number field all have different characteristics. However, when one restricts to a particular prime of the number field and completes $\mathcal{O}_F$ at that prime then an analogue of (13.28) does exist [SW20, §1.2]. This analogue plays a key role in Fargues and Scholze’s work on the local Langlands correspondence [FS21].

There has been a great deal of thought about what can be used to replace (13.28) in the number field case. The idea is that instead of $\mathbb{F}_p$ one should use the field with one element. This remains a very interesting but mostly speculative prospect. The motivation comes not only from the link with Langlands functoriality but also from the link to the Riemann hypothesis.

Exercises

13.1. Suppose that G is a reductive group over a number field F . Let

$$r : {}^L G \longrightarrow {}^L \mathrm{Res}_{E/F} \mathrm{GL}_n$$

be an L -map and let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$. Assume that a weak transfer π' of π to $\mathrm{Res}_{E/F} \mathrm{GL}_n(\mathbb{A}_F) = \mathrm{GL}_n(\mathbb{A}_E)$ with respect to r exists and π' is isobaric. Prove that π' is the unique isobaric automorphic representation of $\mathrm{GL}_n(\mathbb{A}_E)$ that is a weak transfer of π with respect to r .

13.2. Prove (13.4).

13.3. Prove (13.6).

13.4. Let π be a cuspidal automorphic representation of $A_{\mathrm{GL}_2} \backslash \mathrm{GL}_2(\mathbb{A}_F)$. Let

$$r_k : {}^L \mathrm{GL}_2 \longrightarrow \mathrm{GL}_{(k+1)^2}(\mathbb{C})$$

be the representation given by $\mathrm{Sym}^k \otimes (\mathrm{Sym}^k)^\vee$. Assume that $L(s, \pi, r_k)$ converges absolutely for $\mathrm{Re}(s) > 1$ for all $k \geq 1$. Prove the unramified Ramanujan conjecture for π . In other words, for all places v of F such that π_v is unramified, say $\pi_v \cong J(\lambda)$, the Langlands class (or Satake parameter) $q^{-\lambda} \in \mathrm{GL}_2(\mathbb{C})$ has eigenvalues of complex norm equal to 1.

Chapter 14

Distinction and Period Integrals



Je serais reconnaissant a toute
personne ayant compris cette
demonstration de me l'expliquer.

Pierre Deligne

In this chapter we discuss the notion of a distinguished representation in local and global contexts. The global notion is defined in terms of period integrals and hence isolates a crucial link between automorphic representation theory and the geometry of locally symmetric spaces.

14.1 Introduction

The notion of distinction of automorphic representations with respect to subgroups of the ambient group was introduced in a global setting by Harder, Langlands, and Rapoport in their work on the Tate conjecture for Hilbert modular surfaces [HLR86]. Jacquet later developed a tool for studying distinction, namely, the **relative trace formula**, which includes the usual trace formula as a special case. This will be treated in Chapter 18. In retrospect, the notion of distinction has been an integral part of representation theory for some time, as it often isolates models of representations of considerable interest. For example, the whole theory of generic representations can be thought of as a special case of the study of distinguished representations.

In this chapter we discuss the notion of distinction, starting from the local setting and moving to the global. The special case of spherical subgroups is discussed in §14.4 and important conjectures of Sakellaridis and Venkatesh are described in vague terms. We give a fairly thorough treatment of symmetric subgroups of the general linear group in §14.5 and then discuss the exciting prospect of generalizing this to classical groups in §14.6. A brief

introduction to the Gan-Gross-Prasad conjecture follows in §14.7. Finally we complete the chapter with some negative results in §14.8.

14.2 Distinction in the local setting

Let G be a reductive group over a local field F and let $H \leq G$ be a subgroup. Assume we are given a quasi-character $\chi : H(F) \rightarrow \mathbb{C}^\times$. Let V_χ be the space of χ . In other words, it is the 1-dimensional complex vector space on which $H(F)$ acts via χ .

Definition 14.2.1. A smooth representation (π, V) of $G(F)$ is said to be (H, χ) -**distinguished** if

$$\mathrm{Hom}_{H(F)}(V, V_\chi) \neq 0.$$

If H and/or χ are understood then we often speak of χ -distinguished, H -distinguished or simply distinguished representations. An element of

$$\mathrm{Hom}_{H(F)}(V, V_\chi)$$

is an (H, χ) -**functional**.

Here $\mathrm{Hom}_{H(F)}(V, V_\chi)$ is the space of $H(F)$ -equivariant continuous $\mathbb{C}$ -linear maps from V to V_χ where $V_\chi = \mathbb{C}$ is given its usual topology. In the non-Archimedean case, V is endowed with the discrete topology, and hence any $\mathbb{C}$ -linear map from V to V_χ is continuous.

For all $h \in H(F)$ and $g \in G(F)$, let

$$C^\infty(H(F) \backslash G(F), \chi) := \{f \in C^\infty(G(F)) : f(hg) = \chi(h)f(g)\}.$$

In the Archimedean case $C^\infty(G(F))$ admits a natural Fréchet topology. The subspace $C^\infty(H(F) \backslash G(F), \chi)$ is closed and we give it the subspace topology.

Lemma 14.2.2 *There is a $\mathbb{C}$ -linear isomorphism*

$$\mathrm{Hom}_{H(F)}(V, V_\chi) \xrightarrow{\sim} \mathrm{Hom}_{G(F)}(V, C^\infty(H(F) \backslash G(F), \chi)). \quad (14.1)$$

Here elements of $\mathrm{Hom}_{G(F)}(V, C^\infty(H(F) \backslash G(F), \chi))$ are required to be continuous (which is automatic in the non-Archimedean case).

Proof. To an (H, χ) -functional λ and $\varphi \in V$, we associate the function

$$I_\lambda(\varphi)(g) := \lambda(\pi(g)\varphi).$$

In the non-Archimedean case, it is smooth because φ is smooth. In the Archimedean case, the functional lies in $C^\infty(G(F))$ by theorems 4.2.3 and 4.2.4.

To an intertwining map $I : V \rightarrow C^\infty(H(F)\backslash G(F), \chi)$, we associate the (H, χ) -functional

$$\lambda_I(\varphi) := I(\varphi)(1)$$

where 1 is the identity of $G(F)$. One checks that the two maps we have defined are inverses of each other. $\square$

Corollary 14.2.3 *A smooth representation (π, V) of $G(F)$ is $(H(F), \chi)$ -distinguished if and only if there is a nonzero continuous intertwining map*

$$V \longrightarrow C^\infty(H(F)\backslash G(F), \chi).$$

$\square$

Again, the continuity requirement is vacuous in the non-Archimedean case.

Lemma 14.2.2 can be refined in the non-Archimedean case. A function $f \in C^\infty(H(F)\backslash G(F), \chi)$ is **uniformly smooth** if there is a compact open subgroup $K \leq G(F)$ such that $f(gk) = f(g)$ for all $(g, k) \in G(F) \times K$. When F is non-Archimedean we let

$$C_u^\infty(H(F)\backslash G(F), \chi) \tag{14.2}$$

be the subspace of $C^\infty(H(F)\backslash G(F), \chi)$ consisting of uniformly smooth functions. When $\chi = 1$ these spaces are used in [SV17], for example, where they are simply denoted $C^\infty(H(F)\backslash G(F))$. The fact that V is smooth implies that

$$\mathrm{Hom}_{G(F)}(V, C^\infty(H(F)\backslash G(F), \chi)) = \mathrm{Hom}_{G(F)}(V, C_u^\infty(H(F)\backslash G(F), \chi)).$$

Hence the isomorphism in Lemma 14.2.2 is in fact a $\mathbb{C}$ -linear isomorphism

$$\mathrm{Hom}_{H(F)}(V, V_\chi) \xrightarrow{\sim} \mathrm{Hom}_{G(F)}(V, C_u^\infty(H(F)\backslash G(F), \chi)). \tag{14.3}$$

It is instructive to observe that one can view the theory of generic representations discussed in §11.3 as a special case of the more general concept of distinguished representations. Assume for simplicity that F is non-Archimedean. If G is quasi-split reductive over F , $N \leq G$ is the unipotent radical of a Borel subgroup, and $\psi : N(F) \rightarrow \mathbb{C}^\times$ is a generic character then an irreducible admissible representation π of $G(F)$ is ψ -generic if and only if it is (N, ψ) -distinguished. The space $\mathrm{Hom}_{N(F)}(V, V_\psi)$ is the space of Whittaker functions, and $\mathrm{Hom}_{G(F)}(V, C^\infty(N(F)\backslash G(F), \psi))$ is the space of Whittaker models. Both are at most 1 dimensional. For many interesting cases, the phenomenon that $\mathrm{Hom}_{H(F)}(V, V_\chi)$ is at most 1 dimensional (or at least finite dimensional) persists. We discuss this in more detail in §14.4 below.

14.3 Global distinction and period integrals

The global version of distinction involves period integrals of cusp forms. Let G be a reductive group over a global field F and let H be a subgroup of G . Let

$$\chi : H(\mathbb{A}_F) \longrightarrow \mathbb{C}^\times$$

be a quasi-character trivial on $(A_G \cap H(\mathbb{A}_F))H(F)$. For the definition of the central subgroup A_G , see (2.20). Thus χ induces a function

$$A_G H(F) \backslash A_G H(\mathbb{A}_F) = (A_G \cap H(\mathbb{A}_F))H(F) \backslash H(\mathbb{A}_F) \longrightarrow \mathbb{C}^\times$$

which we continue to denote by χ . If $\varphi : [G] \rightarrow \mathbb{C}$ is a continuous function then we define the **period integral**

$$\mathcal{P}_\chi(\varphi) := \int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} \varphi(h) \chi^{-1}(h) dh \quad (14.4)$$

whenever this integral is absolutely convergent.

For a cuspidal automorphic representation π of $A_G \backslash G(\mathbb{A}_F)$, let $L^2_{\text{cusp}}(\pi)$ be the π -isotypic subspace of $L^2_{\text{cusp}}([G])$. We assume that $\mathcal{P}_\chi(\varphi)$ is absolutely convergent for all smooth $\varphi \in L^2_{\text{cusp}}([G])$.

Definition 14.3.1. A cuspidal automorphic representation π of $A_G \backslash G(\mathbb{A}_F)$ is (H, χ) -**distinguished** if $\mathcal{P}_\chi(\varphi)$ is nonzero for some smooth φ in $L^2_{\text{cusp}}(\pi)$.

The relationship between global distinction and local distinction is described precisely in Exercise 14.4 below. This exercise also explains why a χ^{-1} appears in the definition of $\mathcal{P}_\chi$ instead of just a χ .

If H and/or χ are understood then we often speak of H -distinguished, χ -distinguished or simply distinguished representations. If χ is trivial, then we often write $\mathcal{P}$ for $\mathcal{P}_\chi$.

Example 14.1. If G is quasi-split, $H = N$ is the unipotent radical of a Borel subgroup, and $\psi : N(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ is a generic character, then a cuspidal representation is ψ -generic if and only if it is (H, ψ) -distinguished. In this case the integral $\mathcal{P}_\chi(\varphi)$ is absolutely convergent for all smooth functions $\varphi : [G] \rightarrow \mathbb{C}$ since $[N]$ is compact.

The notion of distinction is also of interest for noncuspidal representations. However the definition is more complicated and usually involves truncation. See [JLR99, LR03, Off06a, Zyd19].

We now develop criteria for $\mathcal{P}_\chi(\varphi)$ to be convergent. First we state a preparatory result:

Proposition 14.3.2 *For any affine algebraic group Q over F , the topological group $Q^\circ(\mathbb{A}_F) \backslash Q(\mathbb{A}_F)$ is compact. In fact, it is profinite.*

Proof. This is a result of Borel [Con12a, Proposition 3.2.1]. $\square$

Assume that the neutral component H° of H is of the form

$$H^\circ = H_r \times H_u \quad (14.5)$$

where H_r is reductive and H_u is unipotent. A version of the following proposition is proven in [AGR93]:

Proposition 14.3.3 *Assume (14.5) and that $\Omega_G \subset G(\mathbb{A}_F)$ is a compact set. Let $\varphi : [G] \rightarrow \mathbb{C}$ be a continuous function. If F is a number field we assume φ is rapidly decreasing. If F is a function field we assume that φ is compactly supported. We have*

$$\int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} |\varphi(hg)\chi^{-1}(h)| dh \ll_{\chi, \varphi, \Omega_G} 1$$

for $g \in \Omega_G$.

Proof. We first reduce the proposition to the case where H is connected. Note that

$$A_G H^\circ(F) \backslash A_G H^\circ(\mathbb{A}_F) = A_G H(F) \backslash A_G H(F) H^\circ(\mathbb{A}_F).$$

Thus

$$\begin{aligned} & \int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} |\varphi(hg)\chi^{-1}(h)| dh \\ &= \int_{A_G H(F) H^\circ(\mathbb{A}_F) \backslash A_G H(\mathbb{A}_F)} \int_{A_G H^\circ(F) \backslash A_G H^\circ(\mathbb{A}_F)} |\varphi(h^\circ hg)\chi^{-1}(h^\circ h)| dh^\circ \frac{dh}{dh^\circ} \end{aligned}$$

where dh° is induced by a choice of Haar measure on $H^\circ(\mathbb{A}_F)$. The group $H^\circ(\mathbb{A}_F) \backslash H(\mathbb{A}_F)$ is compact by Proposition 14.3.2. Thus applying the Fubini-Tonelli theorem, we are reduced to the case where H is connected.

We now assume H is connected, so $H = H_r \times H_u$. We then have

$$\begin{aligned} & \int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} |\varphi(hg)\chi^{-1}(h)| dh \\ &= \int_{[H_u]} \int_{A_G H_r(F) \backslash A_G H_r(\mathbb{A}_F)} |\varphi(h_r h_u g)\chi^{-1}(h_r h_u)| dh_r dh_u. \end{aligned}$$

The set $[H_u]$ is compact by Theorem 2.6.3. Thus again applying the Fubini-Tonelli theorem, we see that it suffices to establish the proposition in the special case where H is reductive.

Finally, assuming now that H is reductive, we have that $A_H H(\mathbb{A}_F)^1$ is of finite index in $H(\mathbb{A}_F)$, and in fact is equal to $H(\mathbb{A}_F)$ if F is a number field by Lemma 2.6.2. It therefore suffices to show that

$$\begin{aligned}
& \int_{A_G H(F) \backslash A_G A_H H(\mathbb{A}_F)^1} |\varphi(hg)\chi^{-1}(h)| dh \\
&= \int_{A_G \cap A_H H(F) \backslash A_H H(\mathbb{A}_F)^1} |\varphi(hg)\chi^{-1}(h)| dh
\end{aligned}$$

is bounded by a constant depending on χ , φ , and Ω_G for $g \in \Omega_G$.

Let $T_{H_0} \leq H^{\text{der}}$ and $T_0 \leq G^{\text{der}}$ be maximal F -split tori in H^{der} and G^{der} , respectively. We can assume $T_0 \cap H^{\text{der}} = T_{H_0}$. Let P_H be a minimal parabolic subgroup of H containing T_{H_0} . It admits a Levi decomposition $M_H N_H = P_H$, where M_H is the centralizer of T_{H_0} in H (it is a Levi subgroup of P_H by Lemma 1.9.5) and $N_H < P_H$ is the unipotent radical of P_H . By Theorem 2.7.2, there is an $t \in \mathbb{R}_{>0}$ such that

$$H(\mathbb{A}_F)^1 = H(F) A_{T_{H_0}}(t) \Omega K_H$$

where Ω is a compact subset of $N_H(\mathbb{A}_F) M_H(\mathbb{A}_F)^1$, $K_H \leq H(\mathbb{A}_F)$ is a maximal compact subgroup such that $H(\mathbb{A}_F) = P_H(\mathbb{A}_F) K_H$, and $A_{T_{H_0}}(t)$ is defined with respect to the set of simple roots Δ_H of T_{H_0} in H attached to the parabolic subgroup P_H . We refer to §2.7 for notation.

The Haar measure on $A_H H(\mathbb{A}_F)^1$ decomposes as

$$d(anmk) = dadndmdk$$

for $(a, n, m, k) \in A_H A_{T_{H_0}}(t) \times N_H(\mathbb{A}_F) \times M_H(\mathbb{A}_F)^1 \times K_H$ by Proposition 3.2.2. Thus

$$\begin{aligned}
& \int_{A_G \cap A_H H(F) \backslash A_H H(\mathbb{A}_F)^1} |\varphi(hg)\chi^{-1}(h)| dh \\
& \leq \int |\varphi(anmkg)\chi^{-1}(anmk)| dadndmdk,
\end{aligned} \tag{14.6}$$

where the second integral is over

$$(A_G \cap A_H) \backslash A_H A_{T_{H_0}}(t) \times \Omega \times K_H.$$

Assume F is a function field. Then (14.6) has support in a compact subset depending only on φ , Ω , K_H , and Ω_G and we deduce the proposition.

Assume now that F is a number field. Let Z_0 be the maximal split subtorus of Z_G , let $T = Z_0 T_0$, and let $\Delta \subset \Phi(G, T)$ be a base. We form the corresponding set

$$A_{T_0}(t_0)$$

for $t_0 \in \mathbb{R}_{>0}$. There is a Weyl chamber $C \subset \text{Lie } A_{T_0}$ such that the closure of the image of C under the exponential map is $A_{T_0}(1)$. Weyl chambers in $\text{Lie } A_{T_0}$ are permuted simply transitively by $W(G, T)(F)$ [Bor91, Theorems 21.2, and 21.6] [Bou02, §VI.1.5, Theorem 2]. It follows that

$$A_{T_0} = \bigcup_{w \in W(G, T)(F)} w A_{T_0}(1) w^{-1}$$

and the intersection of any two $W(G, T)(F)$ -conjugates of $A_{T_0}(1)$ is a set of measure zero with respect to any Haar measure on A_{T_0} . Hence (14.6) is equal to

$$\sum_{w \in W(G, T)(F)} \int |\varphi(anmkg) \chi^{-1}(anmk)| \, dadndmdk,$$

where the integral is over

$$(A_G \cap A_H) \backslash A_H A_{T_{H_0}}(t) \cap w A_{T_0}(1) w^{-1} \times \Omega \times K_H.$$

Changing variables and using the left invariance of φ under w , we see that this is

$$\sum_{w \in W(G, T)(F)} \int |\varphi(aw^{-1}nmkg) \chi^{-1}(waw^{-1}nmk)| \, dadndmdk, \quad (14.7)$$

where the integral is now over

$$(A_G \cap w^{-1} A_H w) \backslash w^{-1} A_H A_{T_{H_0}}(t) w \cap A_{T_0}(1) \times \Omega \times K_H.$$

Using Lemma 9.6.3, for any $r \in \mathbb{R}_{>0}$ one has that

$$|\varphi(aw^{-1}nmkg)| \ll_{\varphi, r, \Omega, K_H, \Omega_G} \min_{\alpha \in \Delta} \alpha(a)^{-r}.$$

Combining this estimate with (14.7) we deduce the proposition when F is a number field. $\square$

Corollary 14.3.4 *Assume (14.5). If $\varphi \in L^2_{\text{cusp}}([G])$ is smooth then the integral defining $\mathcal{P}_\chi(\varphi)$ is absolutely convergent and the functions*

$$g \mapsto \mathcal{P}_\chi(\pi(g)\varphi) \quad \text{and} \quad g \mapsto \int_{A_G H(F) \backslash A_G H(\mathbb{A}_F)} |\varphi(hg) \chi^{-1}(h)| \, dh$$

are continuous on $G(\mathbb{A}_F)$.

Proof. The function φ is rapidly decreasing when F is a number field by Corollary 9.6.5 and is compactly supported when F is a function field by Theorem 9.5.1. Therefore the corollary follows from the proof of Proposition 14.3.3. $\square$

One might ask whether our assumption (14.5) is actually necessary for the validity of Proposition 14.3.3 and Corollary 14.3.4. Certainly these results do not hold for arbitrary H (see Exercise 14.5). However under suitable assumptions on φ Proposition 14.3.3 is still valid if we replace assumption (14.5) with

the weaker assumption that G/H is quasi-affine (i.e. is an open subscheme of an affine scheme) [BP21, Proposition A.1.2]. This is a strict generalization: our assumption (14.5) implies that G/H is quasi-affine [Gro97, Theorem 2.1, Corollary 2.8, Corollary 2.10].

Assume F is a number field. Let $K_\infty \leq G(F_\infty)$ be a maximal compact subgroup. It is often useful to work with K_∞ -finite functions in $L^2_{\text{cusp}}(\pi)$ instead of merely smooth ones. For this purpose we prove the following lemma:

Lemma 14.3.5 *Assume (14.5). Let π be a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$. The representation π is (H, χ) -distinguished if and only if there is a K_∞ -finite smooth function $\varphi \in L^2_{\text{cusp}}(\pi)$ such that $\mathcal{P}_\chi(\varphi) \neq 0$.*

Proof. The “if” direction is obvious. We prove the “only if.” Let U be a neighborhood of 1 in $A_G \backslash G(F_\infty)$ and let $\varepsilon > 0$. By the proof of Proposition 4.4.7, we can choose a nonnegative K_∞ -finite $f_\infty \in C_c^\infty(A_G \backslash G(F_\infty))$ with support in $K_\infty U$ such that

$$\int_{A_G \backslash G(F_\infty)} f(g) dg = 1 \quad \text{and} \quad \int_{A_G \backslash G(F_\infty) - U} f(g) dg < \varepsilon.$$

Choose a smooth function φ in $L^2_{\text{cusp}}(\pi)$ such that $\mathcal{P}_\chi(\varphi) \neq 0$. It is fixed by some compact open subgroup $K^\infty \leq G(\mathbb{A}_F^\infty)$. Let

$$f := \text{meas}_{dg^\infty}(K^\infty)^{-1} f_\infty \mathbb{1}_{K^\infty},$$

where $\mathbb{1}_{K^\infty}$ is the characteristic function. Then $\pi(f)\varphi$ is K_∞ -finite. We have

$$|\mathcal{P}_\chi(\pi(f)\varphi) - \mathcal{P}_\chi(\varphi)| = \left| \int_{A_G \backslash G(\mathbb{A}_F)} f(g) (\mathcal{P}_\chi(\pi(g)\varphi) - \mathcal{P}_\chi(\varphi)) dg \right|.$$

Here we have used Corollary 14.3.4 to justify switching the order of integration. The above is bounded by

$$\int_{A_G \backslash G(\mathbb{A}_F)} f(g) |\mathcal{P}_\chi(\pi(g)\varphi) - \mathcal{P}_\chi(\varphi)| dg. \quad (14.8)$$

By Corollary 14.3.4 $\mathcal{P}_\chi(\pi(g)\varphi)$ is continuous as a function of $g \in G(\mathbb{A}_F)$. Thus (14.8) can be made as small as we wish by adjusting U and ε . $\square$

14.4 Spherical varieties

Let G be a reductive group over a field F and let $H \leq G$ be a subgroup. We assume for this section that the characteristic of F is zero because the majority of references restrict to this case.

There is an important class of subgroups H for which one can conjecturally characterize H -distinguished representations. Define the quotient scheme

$$X := G/H \quad (14.9)$$

as in §1.10. It may not be affine. Let $B \leq G_{\overline{F}}$ be a Borel subgroup.

Definition 14.4.1. A subgroup $H \leq G$ is said to be **spherical** if B has an open orbit on $G_{\overline{F}}/H_{\overline{F}}$.

More generally, let X be a separated normal (not necessarily affine) scheme of finite type over F equipped with an action of G .

Definition 14.4.2. The scheme X is a **spherical variety** if B has an open orbit on $X_{\overline{F}}$.

Thus if $H \leq G$ is a spherical subgroup then G/H is a spherical variety. A spherical variety X is of the form G/H for some spherical subgroup H if and only if it is homogeneous, that is, consists of a single G -orbit (see Proposition 17.2.2).

For any spherical variety X , Gaitsgory and Nadler [GN10] defined a dual group $\widehat{G}_X$. In [SV17] Sakellaridis and Venkatesh sketched an argument implying the existence of an embedding

$$\widehat{G}_X \times \mathrm{SL}_2 \longrightarrow \widehat{G} \quad (14.10)$$

of reductive groups over $\mathbb{C}$ satisfying certain desiderata. The existence of the embedding was later proved by Knop and Schalke [KS17]. When $X = G/H$ and H is a reductive symmetric subgroup in the sense of §14.5 there are antecedents of the definition of $\widehat{G}_X$ in [Ric82, JLR93]. These antecedents do not include the SL_2 -factor. This factor is crucial for understanding nontempered distinguished representations. We give an example at the end of §14.5 below.

Assume that $X = G/H$ is a spherical variety where G is split. Assume moreover that F is a local field. Let G_X be the split group over F with ${}^L G_X^\circ = \widehat{G}_X$. To avoid discussing A -parameters we assume that the SL_2 -factor of (14.10) is trivial. Thus (14.10) is an embedding

$${}^L G_X \longrightarrow {}^L G. \quad (14.11)$$

Assume that the local Langlands correspondence is known for tempered representations of $G(F)$. In this case the basic local principle of Sakellaridis and Venkatesh is the following:

Principle 14.4.3 *A tempered L -parameter $\rho : W_F' \rightarrow {}^L G$ factors through (14.11) if and only if some $\pi \in \Pi(\rho)$ is H -distinguished.*

Due to work of Sakellaridis [Sak08, Sak13] we have a robust understanding of Principle 14.4.3 when the L -parameter is unramified. Generalizing Sakellaridis' work to arbitrary unramified groups G is an important (and probably accessible) task.

We have used “principle” instead of “conjecture” because it might be false as stated. One issue is that it might be necessary to enlarge the L -packet to a so-called **Vogan L -packet** that includes representations of pure inner forms of G . See §14.7 for the definition of pure inner forms. More serious issues can also occur. A case in point is when H is an orthogonal group O_n and G is the general linear group GL_n (see §14.5 below). We refer to [SV17, §16] for explicit conjectures.

Now assume that F is a global field and G is a reductive group over F . Let $H \leq G$ be a spherical subgroup. By Exercise 14.4, if $\pi = \otimes'_v \pi_v$ is a cuspidal automorphic representation of $G(\mathbb{A}_F)$ that is H -distinguished then π_v is H -distinguished for all places v of F . In general, the converse statement is false. There is usually an additional global constraint that can be phrased in terms of L -functions. We will illustrate this with examples in the next few sections.

Under suitable assumptions, in [SV17] Sakellaridis and Venkatesh gave a systematic conjectural description of this additional global constraint. Assume that G is split, let G_X be the split reductive group over F with ${}^L G_X^\circ = \widehat{G}_X$, and assume that the SL_2 -factor in (14.10) is trivial. Thus we again have the embedding (14.11) in this global setting.

Principle 14.4.4 *A tempered cuspidal automorphic representation π of $G(\mathbb{A}_F)$ is H -distinguished if and only if*

- (a) *it is a functorial transfer from $G_X(\mathbb{A}_F)$ with respect to (14.11) and*
- (b) *a certain value or residue of a particular quotient of L -functions is nonzero.*

Assuming Principle 14.4.3, condition (a) is essentially equivalent to π_v being H -distinguished for all v . This is only a principle and not a conjecture for the same reasons mentioned in the local setting. We refer to [SV17, §17] for a precise conjecture.

There are antecedents of this principle in the special case of symmetric varieties in the work of Jacquet and his collaborators [JLR93, Jac05a]. These theorems together with the Gan-Gross-Prasad conjecture discussed in §14.7 were among the motivations for Principle 14.4.4.

In [SV17, §17] there is even a conjectural formula for the period integral $\mathcal{P}(\varphi)$ for φ in the space of π in terms of L -functions. It is known as the **Ichino-Ikeda conjecture**. It was first posed in a special case in [II10]. The power and beauty of the conjectures of Sakellaridis and Venkatesh lies in the fact that they provide a uniform perspective on many proved and unproved conjectures in local representation theory and period integrals. At the same time they suggest many new avenues of research. They are currently the most precise enunciation of what has become known as the **relative Langlands program**.

Before moving to examples of spherical subgroups in the next few sections, let us consider a general separated integral scheme X of finite type over F (not necessarily affine) equipped with an action of G . Knop and Schalke

[KS17] have defined a dual group $\widehat{G}_X$ equipped with a morphism $\widehat{G}_X \rightarrow \widehat{G}$ in this generality. It would be interesting to investigate whether or not this generalized dual group can be used to generalize Principle 14.4.3.

14.5 Symmetric subgroups

Assume that F is a characteristic zero field. Let $\sigma : G \rightarrow G$ be an automorphism of order 2. For F -algebras R , let

$$G^\sigma(R) := \{g \in G(R) : g^\sigma = g\} \quad (14.12)$$

be the subgroup fixed by σ and denote by $H = (G^\sigma)^\circ$ its neutral component.

In this case we refer to $X := G/H$ as a **symmetric variety** and H as a **symmetric subgroup**. We have the following theorem of Vust [Vus74, §1.3]:

Theorem 14.5.1 *A symmetric subgroup is a spherical subgroup.* □

When F is the real numbers and $H(F)$ is compact, the quotient $G(F)/H(F)$ is a finite union of Riemannian symmetric spaces. These were classified by Élie Cartan. A standard reference for this result is [Hel01]. Over arbitrary fields symmetric subgroups have been classified by Helminck [Hel00]. The classification is somewhat complicated. To get a feel for what is going on, it is illuminating to consider the case of classical groups over an algebraically closed field. When the field is $\mathbb{C}$, there is a nice discussion in [GW09, §11.3.4]. In particular, SO_n and $\mathrm{GL}_m \times \mathrm{GL}_{n-m}$ for $1 \leq m < n$ are symmetric subgroups of GL_n and Sp_n is a symmetric subgroup of GL_n when n is even. Every symmetric subgroup of GL_n is $\mathrm{GL}_n(\overline{F})$ -conjugate to one of these groups.

We now describe some results on distinction by symmetric subgroups of GL_n or $\mathrm{Res}_{E/F}\mathrm{GL}_n$ when E/F is a quadratic extension of number fields. There is a great deal known in this setting.

One of the motivations for Jacquet's development of the relative trace formula was to prove the following theorem [Jac10]:

Theorem 14.5.2 *Let E/F be a quadratic extension of number fields, let $G = \mathrm{Res}_{E/F}\mathrm{GL}_n$ and let H be a quasi-split unitary group attached to the extension E/F . Then a cuspidal automorphic representation π of $G(\mathbb{A}_F)$ is H -distinguished if and only if $\pi \cong \pi^\sigma$, where $\langle \sigma \rangle = \mathrm{Gal}(E/F)$.* □

See also [FLO12] for refinements of this theorem. This is sometimes known as the Jacquet-Ye case since its proof was more or less reduced to a local statement known as the Jacquet-Ye fundamental lemma in [JY96] and later proved in two different manners by Ngô [Ngô99a, Ngô99b] and Jacquet [Jac04, Jac05b] (see §19.6 for more details). By Theorem 13.4.1 $\pi \cong \pi^\sigma$ if and only if π is a base change from $\mathrm{GL}_n(\mathbb{A}_F)$.

We point out that Theorem 14.5.2 appeared in the literature prior to the conjectures of Sakellaridis and Venkatesh discussed in the previous section. The same is true of all of the theorems in this section and the principal theorems and conjectures in §14.6 and §14.7. Thus it would be misleading to say that these results confirm the predictions of the conjectures of Sakellaridis and Venkatesh because they predate the formulation of the conjectures. In fact, whenever the group G is nonsplit principles 14.4.3 and 14.4.4 have never been made precise. Moreover there are additional settings in which the conjectures require refinement. One example, the case of distinction of automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ by O_n , is discussed below. However, even in cases that technically are not addressed by the conjectures of Sakellaridis and Venkatesh known results are consonant with the essence of these conjectures. Moreover the conjectures of Sakellaridis and Venkatesh provide an influential organizing principle and we will indicate the relationship of it to the results below.

Returning to the case of $G = \mathrm{Res}_{E/F}\mathrm{GL}_n$, we could also take the H to be GL_n , the split outer form of the group in Theorem 14.5.2. Using the Rankin-Selberg method Flicker and Zinoviev proved the following theorem (see [Fli88, FZ95], Theorem 11.5.2 and Exercise 14.7):

Theorem 14.5.3 *Let E/F be a quadratic extension of number fields and let $G = \mathrm{Res}_{E/F}\mathrm{GL}_n$. A cuspidal automorphic representation π of $A_G \backslash G(\mathbb{A}_F)$ is GL_n -distinguished if and only if the Asai L -function $L(s, \pi, \mathrm{As}_{E/F})$ has a pole at $s = 1$. $\square$*

Here the Asai L -function is the Langlands L -function attached to the Asai representation defined in (13.18).

A representation

$$\rho : \mathrm{Gal}_E \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

extends uniquely to a homomorphism

$$\rho : \mathrm{Gal}_F \longrightarrow {}^L\mathrm{Res}_{E/F}\mathrm{GL}_n$$

commuting with the projections to Gal_F on the L -group side. Thus to each such ρ we can associate the representation

$$\mathrm{As}_{E/F}(\rho) := \mathrm{As}_{E/F} \circ \rho : \mathrm{Gal}_F \longrightarrow \mathrm{GL}_{n[E:F]}(\mathbb{C}).$$

Assume that E/F is Galois. Then

$$\mathrm{As}_{E/F}(\rho)|_{\mathrm{Gal}_E} \cong \otimes_{\tau \in \mathrm{Gal}(E/F)} \rho^\tau.$$

Thus the Asai representation $\mathrm{As}_{E/F}(\rho)$ is a canonical extension of

$$\otimes_{\tau \in \mathrm{Gal}(E/F)} \rho^\tau$$

to Gal_F . Suppose now that E/F is quadratic and that τ is the generator of $\text{Gal}(E/F)$. Let π be a cuspidal automorphic representation of $A_{\text{Res}_{E/F}\text{GL}_n} \backslash \text{Res}_{E/F}\text{GL}_n(\mathbb{A}_F)$. In this case the analytic properties of the Asai L -function are understood by Theorem 13.7.2. The discussion above (with global Galois groups replaced by local Weil-Deligne groups) implies that

$$L(s, \pi \times \pi^\tau) = L(s, \pi, \text{As}_{E/F}) L(s, \pi, \text{As}_{E/F} \otimes \eta_{E/F}) \quad (14.13)$$

where $\eta_{E/F}$ is the character attached to $\text{Gal}(E/F)$ by class field theory. In particular, by Theorem 11.7.1 if $L(s, \pi, \text{As}_{E/F} \otimes \eta_{E/F}^i)$ has a pole for some $i \in \{0, 1\}$ then $\pi \cong \pi^{\vee\tau}$. Moreover, if $L(s, \pi, \text{As}_{E/F} \otimes \eta_{E/F}^i)$ has a pole at $s = 1$ then π is a functorial transfer from a quasi-split unitary group with respect to an L -map depending on the parity of n and i [Mok15, Theorem 2.5.4(a), Remark 2.5.5].

If $n = 2$, $F = \mathbb{Q}$ and E is a real quadratic extension then Theorem 14.5.3 was proven in [HLR86]. It was then used to prove many cases of the Tate conjecture for Hilbert modular surfaces. As mentioned in the introduction, this paper is significant because it was where the notion of a distinguished representation was first defined. Extensions of Harder, Langlands, and Rapoport's results are contained in [Ram04, GH14].

We now move on to symmetric subgroups of GL_n (not $\text{Res}_{E/F}\text{GL}_n$). Let

$$\sigma_{m,n} : \text{GL}_{m+n} \longrightarrow \text{GL}_{m+n}$$

be conjugation by $\begin{pmatrix} I_m & \\ & -I_n \end{pmatrix}$ and let

$$H := \text{GL}_{m+n}^{\sigma_{m,n}} = \text{GL}_m \times \text{GL}_n.$$

The following is [FJ93, Proposition 2.1]:

Proposition 14.5.4 *Suppose that $m > n$. Then no cuspidal automorphic representation of $\text{GL}_{m+n}(\mathbb{A}_F)$ is $(\text{GL}_m, \chi \circ \det)$ -distinguished for any quasi-character $\chi : F^\times \backslash \mathbb{A}_F^\times \rightarrow \mathbb{C}^\times$. $\square$*

Here we view GL_m as the subgroup $\text{GL}_m \times 1$ of $\text{GL}_m \times \text{GL}_n = H$. In particular, if $m > n$ no cuspidal automorphic representation of $\text{GL}_{m+n}(\mathbb{A}_F)$ is (H, χ) -distinguished for any χ .

Thus from the point of view of cuspidal automorphic representations, the only interesting case is when $m = n$. In this case, consider the quasi-character

$$\begin{aligned} \mu_s : H(\mathbb{A}_F) &\longrightarrow \mathbb{C}^\times \\ \begin{pmatrix} a & \\ & b \end{pmatrix} &\longmapsto \left| \frac{\det b}{\det a} \right|^{s-1/2} \chi \left(\frac{\det b}{\det a} \right) \eta^{-1}(\det b), \end{aligned}$$

where χ and η are characters of $[\mathbb{G}_m]$. In particular they are trivial on $A_{\mathbb{G}_m}$. In this setting, Jacquet and Friedberg [FJ93, Theorem 4.1] prove the following:

Theorem 14.5.5 *Let π be a cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ with central character ω_π . Assume that $\eta^n \omega_\pi = 1$. The representation π is (H, μ_{s_0}) -distinguished if and only if $L(s, \pi, \wedge^2 \otimes \eta)$ has a pole at $s = 1$ and $L(s_0, \pi \otimes \chi) \neq 0$. $\square$*

In fact Friedberg and Jacquet prove that, for φ in the space of π , the period integral $\mathcal{P}_{\mu_{s_0}}(\varphi)$ is a holomorphic multiple of $L(s_0, \pi \otimes \chi)$ (under the assumption that $L(s, \pi, \wedge^2 \otimes \eta)$ has a pole at $s = 1$). Ash and Ginzburg [AG94] use Theorem 14.5.5 to construct p -adic L -functions under certain technical hypotheses, one of which was removed in [Sun19].

Consider the L -map

$$r : \widehat{\mathrm{SO}}_{2n+1}(\mathbb{C}) \times \mathrm{Gal}_F = \mathrm{Sp}_{2n}(\mathbb{C}) \times \mathrm{Gal}_F \longrightarrow \widehat{\mathrm{GL}}_{2n}(\mathbb{C}) \times \mathrm{Gal}_F.$$

If $\eta = 1$ and π has trivial central character, the condition that $L(s, \pi, \wedge^2)$ has a pole at $s = 1$ is equivalent to the statement that π is a transfer of a cuspidal automorphic representation of $\mathrm{SO}_{2n+1}(\mathbb{A}_F)$ with respect to r by Theorem 13.7.4. On the other hand the map

$$\widehat{G}_{\mathrm{GL}_{2n}/\mathrm{GL}_n \times \mathrm{GL}_n} \times \mathrm{SL}_2 \longrightarrow \widehat{\mathrm{GL}}_{2n}$$

of (14.10) is trivial on the SL_2 -factor and on the

$$\widehat{G}_{\mathrm{GL}_{2n}/\mathrm{GL}_n \times \mathrm{GL}_n} = \mathrm{Sp}_{2n}$$

factor it is the natural inclusion $\mathrm{Sp}_{2n} \rightarrow \widehat{\mathrm{GL}}_{2n} = \mathrm{GL}_{2n}$, at least after passing to Lie algebras [KS17, Table 3]. Thus Theorem 14.5.5 is consistent with Principle 14.4.4.

If we take $G = \mathrm{GL}_n$ and take $\sigma(g) = J^{-1}g^{-t}J$ for a symmetric matrix $J \in \mathrm{GL}_n(F)$ then G^σ is an orthogonal group O_n . In the case $n = 2$, one has a complete characterization of representations of GL_2 that are O_2 -distinguished via work of Waldspurger reinterpreted and reproved in [Jac86]. In general, the representations of $\mathrm{GL}_n(\mathbb{A}_F)$ that are O_n -distinguished should be functorial lifts from the topological double cover of $\mathrm{GL}_n(\mathbb{A}_F)$. This expectation is stated as a precise conjecture in [Mao98]. In the same paper, it is more or less reduced to a local conjecture known as the Jacquet-Mao fundamental lemma. The Jacquet-Mao fundamental lemma is now known [Do15, Do20] so it seems likely that the expected characterization of distinguished representations mentioned above will be proven in the near future. We point out that the topological double cover of $\mathrm{GL}_n(\mathbb{A}_F)$ is not the adelic points of an algebraic group. Thus Principle 14.4.4 must be modified to incorporate this setting.

If we take $G = \mathrm{GL}_{2n}$ and take $\sigma(g) := J^{-1}g^{-t}J$ for a skew-symmetric matrix $J \in \mathrm{GL}_{2n}(F)$ then G^σ is a symplectic group Sp_{2n} . It is not hard to check that all of these subgroups are conjugate under $\mathrm{GL}_{2n}(F)$, so the choice of J is irrelevant (see Exercise 14.1). It is known that there are no cuspidal

automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ that are Sp_{2n} -distinguished by [JR92, Proposition 1]. However there are discrete automorphic representations of GL_{2n} that are Sp_{2n} -distinguished; these are classified in [Off06b]. In this case the map (14.10) is the tensor product of the two standard representations

$$\mathrm{GL}_n \times \mathrm{SL}_2 \longrightarrow \mathrm{GL}_{2n}.$$

[SV17, Example 1.3.2]. We have omitted a discussion of Arthur's conjectures on nontempered automorphic representations in this book. However, a suitable combination of Arthur's conjectures and the conjectures of Sakellaridis and Venkatesh has the following implication. Whenever the SL_2 -factor of (14.10) is nontrivial, a distinguished cuspidal automorphic representation should have the property that its local components are not tempered. In fact the degree to which they fail to be tempered should be controlled by the SL_2 -factor. Thus Sakellaridis and Venkatesh's conjectures, when restricted to the case at hand, imply the known result that cuspidal automorphic representations of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ are never Sp_{2n} -distinguished. This puts the theory on a more satisfactory conceptual framework than the less precise antecedents to the conjectures mentioned after (14.10) above.

14.6 Relationship with the endoscopic classification

We continue to let F be a number field. When the locally symmetric spaces attached to a reductive F -group G are hermitian, one can exploit the link between algebraic and arithmetic geometry and distinction explained in §15.6 to produce many interesting results. As recalled in the introduction to this chapter, this was the original motivation for the notion of distinction. From this point of view, the results in §14.5 on distinction of cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$ and $\mathrm{Res}_{E/F}\mathrm{GL}_n(\mathbb{A}_F)$ with respect to symmetric subgroups suffer a serious drawback: the locally symmetric spaces attached to GL_n are never hermitian for $n > 2$.

In order to obtain results on distinction relevant to algebraic and arithmetic geometry, the theory of Shimura varieties exposed in §15.8 suggests investigating automorphic representations of classical groups. Of course this is also of interest independently of applications to algebraic or arithmetic geometry.

The first author and Wambach [GW14] provide one means of relating distinguished automorphic representations of general linear groups to distinguished automorphic representations of unitary groups (including those defining hermitian locally symmetric spaces). It is suggestive of broader phenomena that merit study.

We observe that for any quadratic extension of number fields M/F and any reductive group G over F we have a canonical L -map

$$r : {}^L G \longrightarrow {}^L \mathrm{Res}_{M/F} G. \quad (14.14)$$

It is written down explicitly in a special case in (13.16). Assume that E/F is a quadratic extension with E totally real. Let $\langle \sigma \rangle = \mathrm{Gal}(E/F)$, and let M/F be a quadratic extension with M totally imaginary. Let H be a quasi-split unitary group in n variables attached to the extension M/F and let

$$G := \mathrm{Res}_{E/F} H.$$

It is the restriction of scalars down to F of a quasi-split unitary group attached to the extension ME/E . In this case

$$\mathrm{Res}_{M/F} G = \mathrm{Res}_{ME/F} \mathrm{GL}_n.$$

The following is the main theorem of [GW14] under some simplifying assumptions:

Theorem 14.6.1 *Let π be a cuspidal automorphic representation of $G(\mathbb{A}_F)$. Suppose that π satisfies the following assumptions:*

- (a) *There is a finite dimensional representation V of $\mathrm{Res}_{F/\mathbb{Q}} G$ such that π is V -cohomological.*
- (b) *There are finite places $v_1 \neq v_2$ of F totally split in ME/F such that π_{v_1} and π_{v_2} are supercuspidal.*
- (d) *For all places v of F such that ME/F is ramified, one of the extensions M/F or E/F is split.*

Then the representation π admits a transfer $r(\pi)$ to

$$\mathrm{Res}_{M/E} G(\mathbb{A}_E) = \mathrm{GL}_n(\mathbb{A}_{ME}) = \mathrm{Res}_{ME/M} \mathrm{GL}_n(\mathbb{A}_M).$$

If the Asai L -function $L(s, r(\pi), \mathrm{As}_{ME/M})$ has a pole at $s = 1$ then some cuspidal automorphic representation π' of $G(\mathbb{A}_F)$ in the weak L -packet of π is H -distinguished. Moreover, we can take π' to be V -cohomological. $\square$

The notion of V -cohomological is explained in §15.5 below. To ease comparison with §15.5, in assumption (a) we are identifying π with a cuspidal automorphic representation of $\mathrm{Res}_{F/\mathbb{Q}} G(\mathbb{A}_{\mathbb{Q}})$.

The proof is based on a particular case of a general principle which we now explain. Let G be a reductive group over the global field F , let M/F be a quadratic extension, let $\langle \tau \rangle = \mathrm{Gal}(M/F)$, and let

$$\tilde{G} := \mathrm{Res}_{M/F} G.$$

Let σ be an automorphism of G of order 2 and let $H := G^\sigma \leq G$ and $\tilde{G}^\sigma \leq \tilde{G}$ be the subgroups fixed by σ . Let $\theta = \sigma \circ \tau$, viewed as an automorphism of $\tilde{G}$, and let $\tilde{G}^\theta \leq \tilde{G}$ be the subgroup fixed by θ . We have an L -map

$$r : {}^L G \longrightarrow {}^L \tilde{G} \quad (14.15)$$

as in (14.14). Let $\Pi \in \text{LP}_{\text{w,aut}}(H)$ be a weak automorphic L -packet. The machinery developed in [GW14] together with suitable fundamental lemmas should imply the following general principle:

Principle 14.6.2 *There is a cuspidal automorphic representation in $r(\Pi)$ that is $\tilde{G}^\sigma$ -distinguished and $\tilde{G}^\theta$ -distinguished if and only if there is a cuspidal automorphic representation in Π that is H -distinguished.*

We have not asserted this principle as a conjecture because it may be false as stated. In particular one might need to work with, for instance, (H, χ) -distinguished representations for characters χ or incorporate pure inner forms of the groups involved. Moreover there might be an additional constraint involving L -functions.

In Theorem 14.6.1 the group G is a unitary group and $\tilde{G}$ is the restriction of scalars of a general linear group. Even if we only consider G of this form there are more interesting cases to consider. For example, one can apply Principle 14.6.2 to study the unitary analogue of Theorem 14.5.5. This was first investigated in the PhD thesis of J. Polák [Pol15, Pol16]. Part of the proof of Principle 14.6.2 in this case was announced by W. Zhang, and another part has been completed by Leslie [Les19, Les22]. It is also natural to consider the unitary analogue of the setting of Jacquet and Mao mentioned above. For preliminary work in this case, see [Lee21].

It is reasonable to expect a generalization of Principle 14.6.2 to the setting where σ and τ are arbitrary commuting involutions of an arbitrary reductive group $\tilde{G}$. Since classical groups are, roughly, the neutral components of the fixed points of involutions on GL_n , this would allow one to classify the distinguished representations of classical groups in terms of their transfer to general linear groups. Thus one would have an analogue of the twisted endoscopic classification of the automorphic discrete spectrum of classical groups in terms of the general linear group (see §13.8 and [Art13, Mok15]). We remark that in many cases one direction of the principle can be obtained as a consequence of known results on Langlands functoriality and the so-called **residue method**. For a family of examples, we refer to [PWZ21].

14.7 Period integrals in the Gan-Gross-Prasad setting

The Gan-Gross-Prasad conjecture characterizes representations of suitable products of groups distinguished by a “diagonally embedded” subgroup. It can be viewed as a special case of the Sakellaridis-Venkatesh conjectures mentioned in §14.4, though the Gan-Gross-Prasad conjecture is older (the orthogonal case was formulated in [GP92]).

For simplicity, we will only state the conjectures of Gan, Gross and Prasad in a restricted setting, referring to [GGP12a, GGP12b] for the general setup. Let F be a local or global field and let E be an étale F -algebra of degree 1 or 2.

In other words $E = F$, $E = F \oplus F$, or E/F is a quadratic field extension. We let

$$\sigma : E \longrightarrow E$$

be the generator of $\text{Gal}(E/F)$. Let V be a finite rank free E -module. We let V_0 be the fixed points of σ acting on V . Thus V_0 is an F -vector space. An **E -Hermitian form** $\langle \cdot, \cdot \rangle$ on V is a pairing

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow E$$

that is E -linear in the first variable and satisfies

$$\langle v, u \rangle = \sigma(\langle u, v \rangle). \quad (14.16)$$

We assume that $\langle \cdot, \cdot \rangle$ is nondegenerate. We let $\tilde{G}_V$ be the group whose points in an F -algebra R are given by

$$\tilde{G}_V(R) := \{g \in \text{GL}_{V_0}(E \otimes_F R) : \langle gv, gw \rangle = \langle v, w \rangle \text{ for all } v, w \in V \otimes_F R\}.$$

Let G_V be the neutral component of $\tilde{G}_V$. Then G_V is a classical group. We list the possibilities in the following table:

E	G_V
$E = F$	SO_V
$E = F \oplus F$	GL_{V_0}
E/F quadratic	U_V

Here SO_V and U_V are the special orthogonal and unitary group of V , respectively.

In order to proceed we require the definition of a pure inner form of an algebraic group G . We will give two equivalent definitions, one using Galois cohomology, the other an ad-hoc definition using the specific features of our given situation. The reader unfamiliar with Galois cohomology could consult §17.4, but should feel free to skip to the ad-hoc definition below.

Let k be a field with separable closure k^{sep} . Two algebraic groups G and G' over k are said to be **forms** of each other if $G_{k^{\text{sep}}} \cong G'_{k^{\text{sep}}}$. We alert the reader that in positive characteristic this terminology is not universally accepted; sometimes one merely requires forms to become isomorphic upon base change to an algebraic closure $\bar{k}$ of k . The forms G and G' are said to be **equivalent** if $G \cong G'$ as algebraic groups over k . For an algebraic group H over a field k , we let

$$H^1(k, H) := H^1(\text{Gal}_k, H(k^{\text{sep}})) \quad (14.17)$$

denote the usual Galois cohomology set. The set of equivalence classes of forms of G is in bijection with $H^1(k, \text{Aut}(G))$. The so-called **inner forms** are those forms corresponding to the subgroup of inner automorphisms. This means that the corresponding cocycles are in the image of the map

$$H^1(k, G/Z_G) \longrightarrow H^1(k, \text{Aut}(G))$$

where Z_G is the center of G . The **pure inner forms** are those corresponding to cocycles in the image of the map

$$H^1(k, G) \longrightarrow H^1(k, G/Z_G) \longrightarrow H^1(k, \text{Aut}(G)).$$

Two pure inner forms are said to be **equivalent** if the corresponding cocycles in $H^1(k, G)$ are equivalent.

Now let us revert to the notation at the beginning of this section. Thus F is a global or local field and E is an étale F -algebra of degree 1 or 2. The pure inner forms of G_V are the groups of the form $G_{V'}$ for free E -modules V' equipped with nondegenerate E -Hermitian forms such that $\text{rank}_E V = \text{rank}_E V'$ and $\text{disc}(V) = \text{disc}(V')$ when $E = F$. When $E = F$, two pure inner forms G_V and $G_{V'}$ are equivalent if and only if V and V' are isomorphic as quadratic spaces. When $E = F \oplus F$ all pure inner forms are equivalent. Finally, when E/F is quadratic two pure inner forms G_V and $G_{V'}$ are equivalent if and only if V is isomorphic to V' as a hermitian space [GGP12b, §2].

We have the following:

Proposition 14.7.1 *If G and G' are reductive groups over a global field F and are inner forms of each other, then there exists a finite set S of places of F such that for $v \notin S$ one has an isomorphism $G_{F_v} \xrightarrow{\sim} G'_{F_v}$.*

Proof. There is a finite set S of places of F such that G_{F_v} and G'_{F_v} are quasi-split for $v \notin S$ [Spr79, Lemma 4.9]. On the other hand, two inner forms of the same quasi-split reductive group are necessarily isomorphic [Mil17, Corollary 23.53]. $\square$

Thus for S as in the proposition, we can unambiguously identify isomorphism classes of representations of $G(\mathbb{A}_F^S)$ and $G'(\mathbb{A}_F^S)$. We say that automorphic representations π of $G(\mathbb{A}_F)$ and π' of $G'(\mathbb{A}_F)$ are **nearly equivalent** if $\pi^S \cong \pi'^S$ for some finite set S of places as above. The notion of near equivalence is unchanged if we enlarge S .

Let $W < V$ be a subspace nondegenerate with respect to $\langle, \rangle$ such that $\text{rank}_E V/W = 1$. Then the restriction of $\langle, \rangle$ to W is naturally an E -Hermitian form and we can form the group G_W as above. Moreover one has a natural embedding $\iota: G_W \hookrightarrow G_V$ and hence an embedding of G_W into

$$G := G_V \times G_W$$

given on points in an F -algebra R by

$$\begin{aligned} G_W(R) &\longrightarrow G_V(R) \times G_{W'}(R) \\ g &\longmapsto (\iota(g), g). \end{aligned} \tag{14.18}$$

We let H be the image of G_W . Let $G_{W'}$ be a pure inner form of G_W with corresponding cocycle $c \in Z^1(F, G_W)$. The image of c under the map $Z^1(F, G_W) \rightarrow Z^1(F, G_V)$ induced by the inclusion yields a pure inner form $G_{V'}$ of G_V . We call the pair $(G_{W'}, G_{V'})$ a **pure inner form** of (G_W, G_V) , and say two such forms are **equivalent** if the corresponding cocycles in $Z^1(F, G_W)$ are equivalent. Thus the equivalence classes of pure inner forms of (G_W, G_V) are in bijection with $H^1(F, G_W)$. If $(G_{W'}, G_{V'})$ is a pure inner form of (G_W, G_V) then we let

$$G' := G_{W'} \times G_{V'}$$

and let

$$H' < G' \tag{14.19}$$

be the image of $G_{W'}$ under the obvious analogue of (14.18).

Assume that G_W and G_V are quasi-split. We then have the L -maps r_V from ${}^L G_V$ and r_W from ${}^L G_W$ given in the table of §13.7 and we can form the tensor product representation $r_V \otimes r_W$. In the special case where G_V and G_W are general linear groups, we take r_V and r_W to be the identity map.

Assume for the remainder of the section that F is a global field. Let π be a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ that lies in a global discrete generic L -packet in the sense of §13.8. In the special case where G_V and G_W are general linear groups, this simply means that π is a cuspidal automorphic representation.

Conjecture 14.7.2 (Gan-Gross-Prasad) *One has that*

$$L\left(\frac{1}{2}, \pi, r_V \otimes r_W\right) \neq 0$$

if and only if there is a pure inner form $(G_{W'}, G_{V'})$ of (G_W, G_V) and a cuspidal automorphic representation π' of $G'(\mathbb{A}_F)$ nearly equivalent to π that is H' -distinguished.

There is a local version of the conjecture which we omit. The import of the local version for the global Conjecture 14.7.2 is that it allows one to detect fairly explicitly which π' in the near equivalence class of π is distinguished. Stating this in a reasonable form seems to require one to collect representations on the adelic points of all the pure inner forms of G into a so-called **Vogan L -packet**. We prefer not to make this precise and instead refer the reader to the lucid explanation in [GGP12b]. Assuming certain expected properties of Vogan L -packets, the local Gan-Gross-Prasad conjecture has been proven for generic L -packets of representations on orthogonal groups over non-Archimedean local fields of characteristic zero by Mœglin and

Waldspurger [MW12]. For tempered L -packets of representations of unitary groups over characteristic zero local fields (either Archimedean or non-Archimedean), it has been proven by Beuzart-Plessis [BP16a].

We state now what is known about the global conjecture. When $E = F \oplus F$, the groups G_V and G_W are general linear groups. In this case, the conjecture is true (see Exercise 14.8). In fact, by Hilbert's theorem 90, the general linear group admits only one isomorphism class of pure inner form. Thus in the statement of the conjecture we may take $G = G'$ and $H = H'$ in this case.

Under suitable assumptions, Ginzburg, Jiang, and Rallis proved one implication of the Gan-Gross-Prasad conjecture. More precisely they proved that if some representation in the near equivalence class of π is distinguished, then the central L -value is nonzero [GJR04, GJR05, GJR09]. The proof is elegant in its simplicity. It is a clever application of the theory of automorphic descent which explicitly constructs the inverse of the functorial lift from classical groups to the general linear group. It is this theory that underlies one proof of Theorem 13.7.4.

Currently the most definitive result on the global Gan-Gross-Prasad conjecture is in the unitary case:

Theorem 14.7.3 *Assume that E/F is a quadratic extension of number fields so that G is a product of unitary groups. Then Conjecture 14.7.2 is true. $\square$*

The method of proof is quite beautiful and was suggested by Jacquet and Rallis [JR11]. One reduces Theorem 14.7.3, the unitary case of the Gan-Gross-Prasad conjecture, to the (known) general linear case using a comparison of relative trace formulae. The approach is an antecedent of Principle 14.6.2. It is interesting to note that allowing pure inner forms is necessary for Theorem 14.7.3 to be true. This was not necessary in the settings considered in §14.5 and in Theorem 14.6.1. It is expected that including pure inner forms will also be necessary to precisely formulate and prove Principle 14.6.2 in general.

Executing the strategy of Jacquet and Rallis required the work of many mathematicians. The first key step was the proof of a local conjecture known as the Jacquet-Rallis fundamental lemma by Z. Yun [Yun11]. W. Zhang established another important local result (the smooth transfer) in the non-Archimedean case, and then proved a slightly weaker version of Theorem 14.7.3 [Zha14]. Smooth transfer in the Archimedean case was proved by H. Xue [Xue17]. Other necessary local results were completed by Beuzart-Plessis [BP21]. Finally, as with most trace formula comparisons, to obtain a complete result requires delicate arguments to deal with the fact that the geometric and spectral sides of trace formulae are rarely convergent. In the current setting, this work was undertaken by Chaudouard and Zydor. The theorem in the form stated above is [BPCZ22, Theorem 1.1.5.1] (see also [BPLZZ21]). Since the proof of Theorem 14.7.3 does not proceed via the twisted trace formula, it is not contingent on the stabilization of the twisted trace formula (see §13.8 for a discussion).

The work above does not generalize in an obvious way to the orthogonal case of Conjecture 14.7.2. However, an approach to this case has recently been proposed [Kri19].

We close this section by mentioning an exciting analogue of Conjecture 14.7.2 that (in certain cases) gives a formula for $L'(\frac{1}{2}, \pi, r_V \otimes r_W)$ when $L(\frac{1}{2}, \pi, r_V \otimes r_W) = 0$. The derivative can in some cases be related to heights of algebraic cycles, and the relationship represents an important link between representation theory and arithmetic geometry. A formula of this type was conjectured in [GGP12b] and is known as the **arithmetic Gan-Gross-Prasad conjecture**. It generalizes the famous Gross-Zagier formula [GZ86], which is a key input into the best known results on the Birch and Swinnerton-Dyer conjecture in the number field case. In the unitary setting, the arithmetic Gan-Gross-Prasad conjecture was essentially reduced to a local conjecture known as the arithmetic fundamental lemma by W. Zhang [Zha12b]. Z. Yun and W. Zhang together used related ideas to prove the best known results towards the Birch and Swinnerton-Dyer conjecture in the function field case [YZ17, YZ19].

14.8 Necessary conditions for distinction

Assume for the moment that F is a global field. One should be aware that for general reductive groups G and connected subgroups $H \leq G$, there do not exist cuspidal automorphic representations of $G(\mathbb{A}_F)$ that are H -distinguished. A somewhat trivial example is given by taking H to be the unipotent radical of a parabolic subgroup. In this case there are no cuspidal automorphic representations of $G(\mathbb{A}_F)$ that are H -distinguished by the very definition of a cuspidal representation.

A more substantial result was obtained by Ash, Ginzburg, and Rallis in [AGR93]:

Theorem 14.8.1 *If (G, H) is one of the following pairs of split reductive groups over a number field F , then there are no cuspidal automorphic representations of $G(\mathbb{A}_F)$ that are H -distinguished:*

- $(\mathrm{GL}_{n+k}, \mathrm{SL}_n \times \mathrm{SL}_k)$ for $n \neq k$,
- $(\mathrm{GL}_{2n}, \mathrm{Sp}_{2n})$,
- $(\mathrm{GL}_{2n+1}, \mathrm{Sp}_{2n})$,
- $(\mathrm{SO}_{2n}, \mathrm{SL}_n)$ for n odd,
- $(\mathrm{Sp}_{2(n+k)}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2k})$,
- $(\mathrm{Sp}_{2n}, \mathrm{Sp}_{2(n-\ell)})$ with $4\ell < n$.

□

There is an additional case in [AGR93] which we omit since it involves cuspidal automorphic representations on (disconnected) orthogonal groups and

we have not defined cuspidal automorphic representations on such groups. We also point out that the theorem is stated in the case $F = \mathbb{Q}$ in [AGR93] but this is not used in the proofs. We refer to [AGR93] for the definition of the embeddings of H into G .

Let F be a non-Archimedean local field of characteristic not equal to 2 and let G be a quasi-split reductive group over F equipped with an automorphism

$$\sigma : G \longrightarrow G$$

of order 2. Let

$$H := ((G^\sigma)^\circ)^{\text{der}}.$$

In this setting, Prasad [Pra19, Theorem 1] gave a beautifully simple criterion for there to exist generic representations of $G(F)$ distinguished by H :

Theorem 14.8.2 *If there is an irreducible admissible generic H -distinguished representation of $G(F)$, then there is a Borel subgroup $B \leq G_{\overline{F}}$ such that $B \cap \sigma(B)$ is a maximal torus of $G_{\overline{F}}$. $\square$*

One also has the following partial converse [Pra19, Proposition 11]:

Proposition 14.8.3 *If there is a Borel subgroup $B \leq G$ such that $B \cap \sigma(B)$ is a maximal torus of G , then there exists an irreducible unitary admissible generic G^σ -distinguished representation of $G(F)$. $\square$*

In this proposition we are assuming B is a subgroup of G , not $G_{\overline{F}}$.

Exercises

14.1. Let G be a reductive group over a local or global field F and let $H, H' \leq G$ be subgroups that are $G(F)$ -conjugate. If F is local, prove that an irreducible admissible representation of $G(F)$ is H -distinguished if and only if it is H' -distinguished. If F is global and H° is the direct product of a reductive group and a unipotent group, prove that a cuspidal automorphic representation of $G(\mathbb{A}_F)$ is H -distinguished if and only if it is H' -distinguished.

14.2. Let H be a reductive group over the non-Archimedean local field F , viewed as a subgroup of $G := H \times H$ via the diagonal embedding. Show that an irreducible admissible representation $\pi_1 \times \pi_2$ of $G(F)$ is H -distinguished if and only if $\pi_1 \cong \pi_2^\vee$.

14.3. Let H be a reductive group over the global field F viewed as a subgroup of $G := H \times H$ via the diagonal embedding. Let π_1 and π_2 be cuspidal automorphic representations of $A_H \backslash H(\mathbb{A}_F)$ and let $\pi_1 \otimes \pi_2$ be their exterior tensor product, viewed as a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$. Show that $\pi_1 \otimes \pi_2$ is H -distinguished if and only if $\pi_1 \cong \pi_2^\vee$.

14.4. Let G be a reductive group over a global field F , let $H \leq G$ be a subgroup, and let $\chi : H(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ be a quasi-character trivial on $H(F)$. Let $\pi = \otimes'_v \pi_v$ be a cuspidal automorphic representation of $G(\mathbb{A}_F)$ that is (H, χ) -distinguished. Prove that for all non-Archimedean places v , the representation π_v is (H, χ_v) -distinguished.

14.5. Give an example of a subgroup $H \leq \mathrm{GL}_2 \times \mathrm{GL}_2$ over a global field F , a quasi-character $\chi : A_{\mathrm{GL}_2^2} \backslash A_{\mathrm{GL}_2^2} H(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ trivial on $H(F)$ and a smooth function φ in a cuspidal automorphic representation of $(A_{\mathrm{GL}_2} \backslash \mathrm{GL}_2(\mathbb{A}_F))^2$ such that the integral

$$\int_{A_{\mathrm{GL}_2^2} H(F) \backslash A_{\mathrm{GL}_2^2} H(\mathbb{A}_F)} |\varphi(h) \chi^{-1}(h)| d_r h$$

diverges.

14.6. Prove (14.13).

14.7. Let E/F be a quadratic extension of local fields. Prove that if π is a unitary generic admissible representation of $\mathrm{Res}_{E/F} \mathrm{GL}_n(F)$ then $L(s, \pi, \mathrm{As}_{E/F})$ is holomorphic and nonzero for $\mathrm{Re}(s) \geq 1$.

14.8. Prove the Gan-Gross-Prasad conjecture when $E = F \oplus F$ (and hence G_V and G_W are general linear groups) using the Rankin-Selberg method.

Chapter 15

The Cohomology of Locally Symmetric Spaces



We are in a forest whose trees
will not fall with a few timid
hatchet blows. We have to take
up the double-bitted axe and the
cross-cut saw, and hope that our
muscles are equal to them.

R. P. Langlands

We discuss how automorphic representation theory controls the cohomology of arithmetic locally symmetric spaces.

15.1 Introduction

A fruitful application of the theory of automorphic representations is to give the decomposition of the cohomology of locally symmetric spaces as a module under certain algebras of correspondences. We survey this application in this chapter. The primary technical tool one uses to make the connection is $(\mathfrak{g}, K_\infty)$ -cohomology. The definition is given in §15.4.

One case that is particularly important is when the locally symmetric space is the $\mathbb{C}$ -points of a quasi-projective scheme over a number field. Shimura was the first to study this case systematically, and in his honor, such varieties are called **Shimura varieties**. We state Deligne's reformulation of the notion of a Shimura variety at the end of this chapter. Part of the reason that this case is so important is that it is one of the few cases where there is a direct relationship between automorphic representations and Galois representations. We point the reader to some of the vast body of work that has been done elucidating this relationship in §15.8.

Our real motivation for this chapter, however, is to explicitly discuss the relationship between distinguished representations and cycles on locally sym-

metric spaces. This was in fact the original motivation for Harder, Langlands, and Rapoport's introduction of the notion of a distinguished automorphic representation [HLR86]. In this paper, they related period integrals of automorphic representations to algebraic cycles on Hilbert modular varieties. Using this relation, they proved cases of the Tate conjecture for these varieties. There has been work generalizing their results, but even very basic questions remain. We point out one such question explicitly in §15.6. We hope that in the near future, the knowledge one now has about the étale cohomology of Shimura varieties can be combined with the relative trace formula to systematically study the Tate and Beilinson-Bloch conjectures for these varieties. It is beyond the scope of this book to make this completely precise, but we mention some promising results that have been obtained in §15.8.

Understanding the material in this chapter requires some knowledge of sheaves and cohomology. Not all of the schemes we discuss in this chapter are affine. More seriously, in §15.8 we even briefly discuss the étale cohomology of Shimura varieties. The results will not be used in the remainder of the book, so the reader should feel free to skip this chapter on a first reading. However, we encourage the reader to at some point investigate the fascinating interplay between arithmetic geometry and automorphic forms that is the subject of Langlands' quote in the epigraph.

15.2 Locally symmetric spaces

Unless otherwise specified, throughout this chapter G is a reductive group over $\mathbb{Q}$. We let $K_\infty \leq G(\mathbb{R})$ be a compact subgroup containing the maximal connected compact subgroup (in the real topology) and, as before, let

$$A_G \leq G(\mathbb{R})$$

be the identity component in the real topology of the maximal $\mathbb{Q}$ -split torus in the center of G . To ease notation, we write K for $K^\infty \leq G(\mathbb{A}_\mathbb{Q}^\infty)$, a compact open subgroup. Finally we set

$$X := X_G := A_G \backslash G(\mathbb{R}) / K_\infty. \quad (15.1)$$

Any connected component of X is a symmetric space. One can essentially take this to be the definition of a symmetric space, although it is not the most natural definition. We let

$$\mathrm{Sh}^K := \mathrm{Sh}(G, X)^K := G(\mathbb{Q}) \backslash (X \times G(\mathbb{A}_\mathbb{Q}^\infty)) / K \quad (15.2)$$

and refer to it as a **Shimura manifold**. As in explained (2.23), this is a finite union of locally symmetric spaces. Indeed, if we take a set of representatives $(g_i)_{i \in I}$ for the finite set $G(\mathbb{Q}) \backslash G(\mathbb{A}_\mathbb{Q}^\infty) / K$, then

$$\coprod_{i \in I} \Gamma_i \backslash X \xrightarrow{\sim} \mathrm{Sh}^K \quad (15.3)$$

$$\Gamma_i x \longmapsto G(\mathbb{Q})(x, g_i K),$$

where $\Gamma_i = g_i K g_i^{-1} \cap G(\mathbb{Q})$.

To relate this setting to the traditional perspective on locally symmetric spaces, it is useful to have in mind the notions of an arithmetic subgroup and a congruence subgroup. These were already introduced in §2.6, but we recall the definition for the convenience of the reader:

Definition 15.2.1. A subgroup $\Gamma \leq G(\mathbb{Q})$ is **arithmetic** if there is some faithful representation $\rho : G \rightarrow \mathrm{GL}_n$ such that $\rho(\Gamma)$ is commensurable with $\rho(G(\mathbb{Q})) \cap \mathrm{GL}_n(\mathbb{Z})$. The group $\Gamma \leq G(\mathbb{Q})$ is a **congruence** subgroup if it is of the form $G(\mathbb{Q}) \cap K$ for some compact open subgroup $K \leq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$.

Congruence subgroups are arithmetic (see Exercise 15.3). The famous congruence subgroup problem asks whether every arithmetic subgroup is congruence. The answer is in general no, but the failure of arithmetic subgroups to be congruence can be controlled via techniques that originated in the influential paper [BMS67] (see also [Men65]). In particular, every arithmetic subgroup of $\mathrm{SL}_n(\mathbb{Q})$ for $n \geq 3$ and $\mathrm{Sp}_{2n}(\mathbb{Q})$ for $n \geq 2$ is a congruence subgroup (see loc. cit.). The survey by Prasad and Rapinchuk in [Mil10] is a source for more up-to-date information.

A useful assumption on a subgroup $\Gamma \leq G(\mathbb{Q})$ is that it is torsion free. This implies, for example, that

$$\Gamma \backslash X \quad (15.4)$$

is a manifold, as opposed to an orbifold (see Exercise 15.2). Unfortunately, the property of being torsion free is not preserved under certain constructions, such as intersecting with a parabolic subgroup and mapping to the Levi quotient. A more robust condition is that of being neat:

Definition 15.2.2. An element $g \in \mathrm{GL}_n(\mathbb{Q})$ is **neat** if the subgroup of $\overline{\mathbb{Q}}^{\times}$ generated by its eigenvalues is torsion free. An arithmetic subgroup $\Gamma \leq G(\mathbb{Q})$ is **neat** if given any (equivalently, one faithful) representation $\rho : G \rightarrow \mathrm{GL}_n$, $\rho(g)$ is neat for all $g \in \Gamma$.

Clearly if Γ is neat then Γ is torsion free. Moreover, if Γ is neat, then all subgroups and homomorphic images of Γ are neat.

Lemma 15.2.3 *Any congruence subgroup $\Gamma \leq G(\mathbb{Q})$ contains a congruence subgroup of finite index that is neat.*

Proof. When $G = \mathrm{GL}_n$, we can take the neat subgroup to be of the form

$$\{g \in \mathrm{GL}_n(\mathbb{Z}) : g \equiv I \pmod{N}\}$$

for a sufficiently divisible integer N . The general case follows from this. $\square$

Definition 15.2.4. A compact open subgroup $K \leq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ is **neat** if $G(\mathbb{Q}) \cap K$ is neat.

An elaboration of the proof of Lemma 15.2.3 implies the following (see Exercise 15.4):

Lemma 15.2.5 *A compact open subgroup $K \leq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ contains a neat subgroup of finite index. If K is neat then so is gKg^{-1} for all $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$. $\square$*

The motivation for introducing Definition 15.2.4 is the following result:

Lemma 15.2.6 *If K is neat then Sh^K is a smooth manifold.*

Proof. Since K is neat, $gKg^{-1} \cap G(\mathbb{Q})$ is torsion free for all $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ by Lemma 15.2.5. It follows that the Γ_i occurring in the decomposition (15.3) are all torsion free. As mentioned above, this implies that the $\Gamma_i \backslash X$ are all manifolds. $\square$

In §5.2 we discussed the Hecke algebra $C_c^{\infty}(G(\mathbb{A}_{\mathbb{Q}}^{\infty}) // K)$. We recall that this is the space of $\mathbb{C}$ -linear combinations of characteristic functions of double cosets KgK for $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$. We now explain how these operators can be realized geometrically as correspondences. Let K and K' be compact open subgroups of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$, and $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ be such that $K' \leq gKg^{-1}$. Then we have a map

$$\begin{aligned} T_g : \mathrm{Sh}^{K'} &\longrightarrow \mathrm{Sh}^K \\ G(\mathbb{Q})(x, hK') &\longmapsto G(\mathbb{Q})(x, hgK). \end{aligned} \quad (15.5)$$

It is finite étale if K and K' are neat.

The geometric realization of the characteristic function of KgK is the correspondence

$$\begin{array}{ccc} & \mathrm{Sh}^{K \cap g^{-1}Kg} & \\ & \swarrow T_I \quad \searrow T_{g^{-1}} & \\ \mathrm{Sh}^K & & \mathrm{Sh}^K. \end{array} \quad (15.6)$$

It acts on functions via pullback T_I^* along T_I followed by pushforward $T_{g^{-1}*}$ along $T_{g^{-1}}$. In other words, for functions φ on Sh^K and φ' on $\mathrm{Sh}^{K \cap g^{-1}Kg}$ we have that

$$\begin{aligned} T_I^*(\varphi)(x, h(K \cap g^{-1}Kg)) &= \varphi(x, hK), \\ T_{g^{-1}*}(\varphi')(x, hK) &= \sum_{k \in K/(K \cap gKg^{-1})} \varphi'(x, hkg(K \cap g^{-1}Kg)). \end{aligned}$$

Set

$$T(g) := T_{g^{-1}*} \circ T_I^*. \quad (15.7)$$

Then for any function $\varphi : \mathrm{Sh}^K \rightarrow \mathbb{C}$,

$$T(g)\varphi = R(\mathbb{1}_{KgK})\varphi, \quad (15.8)$$

where

$$R(\mathbb{1}_{KgK})\varphi(x) := \int_{G(\mathbb{A}_{\mathbb{Q}}^{\infty})} \mathbb{1}_{KgK}(g')\varphi(xg')dg'$$

with the Haar measure dg' normalized so that $dg'(K) = 1$.

15.3 Local systems

A well-known classical fact is that if Γ is a congruence subgroup of $\mathrm{GL}_2\mathbb{Q}$ contained in the subgroup of matrices with positive determinant and $\mathfrak{H}$ is the upper half-plane, then the cohomology group

$$H^1(\Gamma \backslash \mathfrak{H}, \mathbb{C})$$

can be decomposed as a direct sum of three summands, one isomorphic to the space of weight 2 cusp forms $S_2(\Gamma)$ on Γ with trivial nebentypus (see §6.7 for the definition), one isomorphic to the space

$$\{z \mapsto f(\bar{z}) : f \in S_2(\Gamma)\}$$

of antiholomorphic forms, and one isomorphic to a certain space of Eisenstein series. In order to give a geometric interpretation of modular forms of weight bigger than 2, it is necessary to introduce certain local systems. We now explain their construction. Our treatment is largely modeled on that of the thesis of S. Morel [Mor08].

Until further notice, $\Gamma \leq G(\mathbb{Q})$ is a neat arithmetic subgroup. Let V_0 be a representation of the algebraic group G , let k be a characteristic zero field and let

$$V := V_0(k). \quad (15.9)$$

It is a representation of $G(k)$ and hence $G(\mathbb{Q})$ by restriction. We equip it with the discrete topology. A **local system** on a topological space is a locally constant sheaf of abelian groups on the topological space. The representation V defines a local system over $\Gamma \backslash X$ as follows. Form the quotient

$$\mathcal{V}^{\Gamma} := \Gamma \backslash (V \times X) \quad (15.10)$$

by the diagonal action of Γ on the product. Consider the projection

$$\mathcal{V}^\Gamma \longrightarrow \Gamma \backslash X. \quad (15.11)$$

This is the total space of a locally constant sheaf of vector spaces. For each open set $U \subseteq \Gamma \backslash X$, we let

$$\mathcal{V}^\Gamma|_U := \{\text{continuous sections } s : U \longrightarrow \mathcal{V}^\Gamma\} \quad (15.12)$$

be the abelian group of sections of the map (15.11). Then the functor

$$\begin{aligned} \{\text{open } U \subseteq \Gamma \backslash X\} &\longrightarrow \mathbf{Ab} \\ U &\longmapsto \mathcal{V}^\Gamma|_U \end{aligned} \quad (15.13)$$

from the category of open sets of $\Gamma \backslash X$ with morphisms given by inclusions to the category of abelian groups $\mathbf{Ab}$ is a sheaf. It is called the **sheaf of sections** of $\mathcal{V}^\Gamma$. Its sheaf cohomology is denoted by

$$H^\bullet(\Gamma \backslash X, \mathcal{V}^\Gamma). \quad (15.14)$$

To orient the reader, we point out that in the special case where $V = k$ is the trivial representation

$$H^\bullet(\Gamma \backslash X, \mathcal{V}^\Gamma) = H^\bullet(\Gamma \backslash X, k),$$

the usual singular or Betti cohomology group of $\Gamma \backslash X$ with coefficients in k .

We now turn to the adelic setting. Let V be as above. For $K \leq G(\mathbb{A}_{\mathbb{Q}}^\infty)$ a compact open neat subgroup, define

$$\mathcal{V}^K := G(\mathbb{Q}) \backslash (V \times X \times G(\mathbb{A}_{\mathbb{Q}}^\infty)/K), \quad (15.15)$$

where for $\gamma \in G(\mathbb{Q})$,

$$\gamma(v, x, gK) = (\gamma.v, \gamma.x, \gamma gK).$$

We thus have a natural map $\mathcal{V}^K \rightarrow \text{Sh}^K$. This is again the total space of a local system. We define the associated sheaf of sections and sheaf cohomology as above.

It is straightforward to relate the adelic and nonadelic constructions above. Fix $g \in G(\mathbb{A}_{\mathbb{Q}}^\infty)$ and let $\Gamma = gKg^{-1} \cap G(\mathbb{Q})$. We then have an embedding

$$\begin{aligned} \iota : \Gamma \backslash X &\longrightarrow \text{Sh}^K \\ \Gamma x &\longmapsto G(\mathbb{Q})(x, gK). \end{aligned} \quad (15.16)$$

One has that $\iota^{-1}\mathcal{V}^K \cong \mathcal{V}^\Gamma$ where $\iota^{-1}\mathcal{V}^K$ is the inverse image sheaf.

We now explain how for each $g \in G(\mathbb{A}_{\mathbb{Q}}^\infty)$ the correspondence $T(g^{-1})$ of (15.7) induces a homomorphism

$$T(g^{-1}) : H^\bullet(\text{Sh}^K, \mathcal{V}^K) \longrightarrow H^\bullet(\text{Sh}^K, \mathcal{V}^K). \quad (15.17)$$

We have written g^{-1} in places of g so that the formulae below have less inverses and the isomorphism (15.28) intertwines $T(g)$ with $\mathbb{1}_{KgK}$.

We first note that for any neat compact open subgroups $K, K' \leq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ with $K' \leq gKg^{-1}$, one has an isomorphism of sheaves

$$\begin{aligned} \theta : \mathcal{V}^{K'} &\xrightarrow{\sim} T_g^{-1} \mathcal{V}^K = \mathcal{V}^K \times_{\mathrm{Sh}^K} \mathrm{Sh}^{K'} \\ (v, x, hK') &\longmapsto ((v, x, hgK), (x, hK')). \end{aligned} \quad (15.18)$$

Here $T_g^{-1} \mathcal{V}^K$ is the inverse image sheaf and the object on the right is the fiber product over the canonical projection $\mathcal{V}^K \rightarrow \mathrm{Sh}^K$ and the morphism $T_g : \mathrm{Sh}^{K'} \rightarrow \mathrm{Sh}^K$. We have stipulated the isomorphism of sheaves by giving the map it induces on the total spaces. The isomorphism θ is called a **lift of the correspondence**. This isomorphism canonically induces the morphism (15.17). Concretely the procedure is to take a cocycle on Sh^K with values in $\mathcal{V}^K$, pull it back to $\mathrm{Sh}^{K'}$, use the isomorphism θ to produce a cocycle on $\mathrm{Sh}^{K'}$ with values in $T_g^{-1} \mathcal{V}^K$, and then sum over the fibers of $T_g^{-1} \mathcal{V}^K \rightarrow \mathcal{V}^K$ to obtain a cocycle on Sh^K .

Alternately, one can work at the level of sheaves as follows. By generalities on sheaf cohomology [Har11, (4.33)], one has a homomorphism

$$H^{\bullet}(\mathrm{Sh}^K, \mathcal{V}^K) \longrightarrow H^{\bullet}(\mathrm{Sh}^{K'}, T_I^{-1} \mathcal{V}^K) = H^{\bullet}(\mathrm{Sh}^{K'}, \mathcal{V}^{K'}). \quad (15.19)$$

Here I is the identity in $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ and the equality is a consequence of the identification $T_I^{-1} \mathcal{V}^K = \mathcal{V}^{K'}$. The isomorphism θ induces an isomorphism

$$\theta : H^{\bullet}(\mathrm{Sh}^{K'}, \mathcal{V}^{K'}) \xrightarrow{\sim} H^{\bullet}(\mathrm{Sh}^{K'}, T_g^{-1} \mathcal{V}^K). \quad (15.20)$$

To finish, we construct a “trace” homomorphism

$$H^{\bullet}(\mathrm{Sh}^{K'}, T_g^{-1} \mathcal{V}^K) \longrightarrow H^{\bullet}(\mathrm{Sh}^K, \mathcal{V}^K). \quad (15.21)$$

Composing (15.19), (15.20), and (15.21) then gives

$$T(g^{-1}) : H^{\bullet}(\mathrm{Sh}^K, \mathcal{V}^K) \longrightarrow H^{\bullet}(\mathrm{Sh}^K, \mathcal{V}^K). \quad (15.22)$$

To construct (15.21), we follow [Ive86, §VII.4] to which we refer for more details. We note that the assertions below all heavily depend on the fact that T_g is a covering map. For any sheaf $\mathcal{F}$ of abelian groups on Sh^K and any $y \in \mathrm{Sh}^K$, one has

$$(T_{g*} T_g^{-1} \mathcal{F})_y = \bigoplus_{x \in T_g^{-1}(y)} \mathcal{F}_x.$$

One can check that there is a unique morphism of sheaves

$$\mathrm{tr} : T_{g*} T_g^{-1} \mathcal{F} \longrightarrow \mathcal{F}$$

given on the fiber over y by taking the sum. Now let

$$\mathcal{V}^K \longrightarrow I^\bullet$$

be an injective resolution of $\mathcal{V}^K$. We then obtain a morphism

$$\mathrm{tr} : T_{g*} T_g^{-1} I^\bullet \longrightarrow I^\bullet. \quad (15.23)$$

We apply the “cohomology of global sections” functor to obtain a map

$$H^\bullet \Gamma(\mathrm{Sh}^K, T_{g*} T_g^{-1} I^\bullet) \longrightarrow H^\bullet(\mathrm{Sh}^K, \mathcal{V}^K).$$

But T_g^{-1} transforms injectives into injectives. Hence

$$H^\bullet \Gamma(\mathrm{Sh}^K, T_{g*} T_g^{-1} I^\bullet) = H^\bullet(\mathrm{Sh}^{K'}, T_g^{-1} \mathcal{V}^K).$$

15.4 $(\mathfrak{g}, K_\infty)$ -cohomology

A fundamental tool for describing $H^\bullet(\mathrm{Sh}^K, \mathcal{V}^K)$ as a Hecke module is $(\mathfrak{g}, K_\infty)$ -cohomology. We give its definition in this section. The basic reference is [BW00].

Let $\mathfrak{g} \geq \mathfrak{k}$ be the Lie algebras of $G_{\mathbb{R}}$ and K_∞ , respectively. Let $\mathcal{A}$ be a $\mathfrak{g}$ -module (not necessarily of finite dimension). For $q \in \mathbb{Z}_{\geq 0}$, let

$$C^q(\mathfrak{g}, \mathfrak{k}; \mathcal{A}) = \mathrm{Hom}_{\mathfrak{k}}(\wedge^q(\mathfrak{g}/\mathfrak{k}), \mathcal{A}). \quad (15.24)$$

Here the action of $\mathfrak{k}$ on $\wedge^q(\mathfrak{g}/\mathfrak{k})$ is the adjoint action. More precisely, $\mathrm{Hom}_{\mathfrak{k}}(\wedge^q(\mathfrak{g}/\mathfrak{k}), \mathcal{A})$ is the subspace of $\mathrm{Hom}_{\mathbb{C}}(\wedge^q(\mathfrak{g}/\mathfrak{k}), \mathcal{A})$ consisting of functions f satisfying

$$\sum_{i=1}^q f(x_1, \dots, [x, x_i], \dots, x_q) = x \cdot f(x_1, \dots, x_q)$$

for $x \in \mathfrak{k}$, where $[\ , \]$ is the Lie bracket of $\mathfrak{g}$. We define a differential

$$d : C^q(\mathfrak{g}, \mathfrak{k}; \mathcal{A}) \longrightarrow C^{q+1}(\mathfrak{g}, \mathfrak{k}; \mathcal{A})$$

on the complex by

$$\begin{aligned} (df)(x_0, \dots, x_q) &= \sum_{i=1}^q (-1)^i x_i \cdot f(x_0, \dots, \widehat{x}_i, \dots, x_q) \\ &\quad + \sum_{1 \leq i < j \leq q} (-1)^{i+j} f([x_i, x_j], x_0, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_q). \end{aligned}$$

Here a hat denotes an omitted argument. One checks that this is well defined (i.e. independent of the choice of representative of the cosets $x_i + \mathfrak{k}$) and that $d \circ d = 0$. We then let $H^\bullet(\mathfrak{g}, \mathfrak{k}; \mathcal{A})$ denote the cohomology of this complex:

$$H^q(\mathfrak{g}, \mathfrak{k}; \mathcal{A}) := H^q(C^\bullet(\mathfrak{g}, \mathfrak{k}; \mathcal{A}), d) := \frac{\ker(d : C^q(\mathfrak{g}, \mathfrak{k}; \mathcal{A}) \longrightarrow C^{q+1}(\mathfrak{g}, \mathfrak{k}; \mathcal{A}))}{\operatorname{im}(d : C^{q-1}(\mathfrak{g}, \mathfrak{k}; \mathcal{A}) \longrightarrow C^q(\mathfrak{g}, \mathfrak{k}; \mathcal{A}))}.$$

This is called the $(\mathfrak{g}, \mathfrak{k})$ -**cohomology** of $\mathcal{A}$. This cohomology is insensitive to the group of connected components of K_∞ . It is therefore desirable to refine it.

Assume now that $\mathcal{A}$ is a $(\mathfrak{g}, K_\infty)$ -module. The group K_∞ acts in a natural manner on $C^\bullet(\mathfrak{g}, \mathfrak{k}; \mathcal{A})$ via the action of K_∞ on $\mathcal{A}$ and the adjoint action on $\mathfrak{g}$. For this action, define

$$C^\bullet(\mathfrak{g}, K_\infty; \mathcal{A}) = C^\bullet(\mathfrak{g}, \mathfrak{k}; \mathcal{A})^{K_\infty}.$$

Denote by $H^\bullet(\mathfrak{g}, K_\infty; \mathcal{A})$ the cohomology of this complex.

Definition 15.4.1. The $(\mathfrak{g}, K_\infty)$ -**cohomology** of the $(\mathfrak{g}, K_\infty)$ -module $\mathcal{A}$ is $H^\bullet(\mathfrak{g}, K_\infty; \mathcal{A})$.

15.5 The cohomology of Shimura manifolds

Until otherwise stated we assume that V is a complex vector space (that is, $k = \mathbb{C}$ in (15.9)). We now relate $(\mathfrak{g}, K_\infty)$ -cohomology and the cohomology of Sh^K with coefficients in $\mathcal{V}^K$. We refer to [BW00, §VII.2] for proofs.

Let $m := \dim_{\mathbb{R}} X$ for $X = A_G \backslash G(\mathbb{R})/K_\infty$. Fix a basis ω^i , $1 \leq i \leq m$, of left $G(\mathbb{R})$ -invariant 1-forms on X . For

$$I = \{i_1, \dots, i_q\} \subseteq \{1, \dots, m\}$$

with $i_j < i_n$ for $j < n$, set

$$\omega^I := \omega^{i_1} \wedge \dots \wedge \omega^{i_q}.$$

Let

$$\mathcal{A} := C^\infty(G(\mathbb{Q}) \backslash G(\mathbb{A}_{\mathbb{Q}})).$$

Denote by

$$A^q(\operatorname{Sh}^K, \mathcal{V}^K)$$

the space of differential q -forms on Sh^K with coefficients in $\mathcal{V}^K$. Any element of $A^q(\operatorname{Sh}^K, \mathcal{V}^K)$ can be written as

$$\eta = \sum_I f_I \omega^I \tag{15.25}$$

with $f_I \in (\mathcal{A}^K \otimes V)^{A_G}$. Let

$$\mathfrak{a}_G := \text{Lie } A_G. \quad (15.26)$$

The equalities (15.25) as η vary yield an identification

$$A^\bullet(\text{Sh}^K, \mathcal{V}^K) = C^\bullet(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; (\mathcal{A}^K \otimes V)^{A_G}) \quad (15.27)$$

commuting with the differentials, which in turn yields an isomorphism

$$H^\bullet(\text{Sh}^K, \mathcal{V}^K) \cong H^\bullet(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; (\mathcal{A}^K \otimes V)^{A_G}). \quad (15.28)$$

This gives an explicit link between cohomology and automorphic representations. This isomorphism is Hecke equivariant in the sense that for $g \in G(\mathbb{A}_\mathbb{Q}^\infty)$, the action of the correspondence $T(g)$ is intertwined with the action of $\mathbb{1}_{KgK}$. We leave the proof of this Hecke equivariance assertion to the reader as Exercise 15.7.

The isomorphism (15.28) motivates the following definition:

Definition 15.5.1. A vector $\varphi \in \mathcal{A}^K$ is **cohomological** if there is a $v \in V$ and ω^I on X such that

$$\varphi \omega^I \otimes v \in C^\bullet(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; (\mathcal{A}^K \otimes V)^{A_G})$$

defines a nonzero class in $H^\bullet(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; (\mathcal{A}^K \otimes V)^{A_G})$.

If we wish to specify the representation V we say that φ is **V -cohomological**.

Now $\mathcal{A}$ is naturally a $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_\mathbb{Q}^\infty)$ -module. Let us decompose it under the action of $G(\mathbb{A}_\mathbb{Q}^\infty)$. We assume henceforth that there is a quasi-character

$$\xi : A_G \longrightarrow \mathbb{R}_{>0}$$

such that A_G acts on V via ξ^{-1} . Such a quasi-character exists when V is irreducible. Let

$$\mathcal{A}(\xi)^K := \{\varphi \in \mathcal{A}^K : \varphi(ag) = \xi(a)\varphi(g) \text{ for } (a, g) \in A_G \times G(\mathbb{A}_\mathbb{Q}^\infty)\}.$$

Then

$$(\mathcal{A}^K \otimes V)^{A_G} = \mathcal{A}(\xi)^K \otimes V.$$

As usual $[G] := A_G G(\mathbb{Q}) \backslash G(\mathbb{A}_\mathbb{Q})$. There is a $G(\mathbb{A}_\mathbb{Q})^1$ -intertwining map

$$\begin{aligned} \mathcal{A}(\xi)^K &\longrightarrow C^\infty([G])^K \\ \varphi &\longmapsto (g \mapsto \xi'(g)\varphi(g)) \end{aligned} \quad (15.29)$$

for an appropriate quasi-character $\xi' : G(\mathbb{A}_\mathbb{Q}) \rightarrow \mathbb{R}_{>0}$ (see Appendix D for more details). We let

$$\mathcal{A}(\xi)_{\text{cusp}}^K$$

be the inverse image of $L_{\text{cusp}}^2([G])^K \cap C^\infty([G])^K$ under the map (15.29). It follows from Theorem 9.7.3 that $\mathcal{A}(\xi)_{\text{cusp}}^K$ is also the inverse image of

$$(\mathcal{A}_{\text{cusp}}^{A_G})^K \leq L_{\text{cusp}}^2([G])^K \cap C^\infty([G])^K$$

under (15.29).

By Corollary 9.1.3,

$$\mathcal{A}(\xi)_{\text{cusp}}^K = \bigoplus_{\pi} (\pi^K)^{m(\pi)}$$

where the sum is over isomorphism classes of cuspidal automorphic representations π of $G(\mathbb{A}_{\mathbb{Q}})$ and $m(\pi)$ is the multiplicity of π in $\mathcal{A}(\xi)_{\text{cusp}}$. By loc. cit. $m(\pi)$ is finite for all π . By (15.28)

$$\begin{aligned} H_{\text{cusp}}^\bullet(\text{Sh}^K, \mathcal{V}^K) &:= H^\bullet(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; \mathcal{A}(\xi)_{\text{cusp}}^K \otimes V) \\ &= \bigoplus_{\pi} H^\bullet(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; \pi^K \otimes V)^{m(\pi)}. \end{aligned} \quad (15.30)$$

The group $H_{\text{cusp}}^\bullet(\text{Sh}^K, \mathcal{V}^K) \leq H^\bullet(\text{Sh}^K, \mathcal{V}^K)$ is known as the **cuspidal cohomology**. Its complement is described in terms of so-called **Eisenstein cohomology**. For some information about the decomposition of the whole of the cohomology, see [BLS96]. As a module under $C_c^\infty(G(\mathbb{A}_{\mathbb{Q}}^\infty) // K)$, one has that

$$H_{\text{cusp}}^\bullet(\text{Sh}^K, \mathcal{V}^K) = \bigoplus_{\pi} (H^\bullet(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; \pi_\infty \otimes V) \otimes \pi^\infty K)^{m(\pi)}. \quad (15.31)$$

This decomposition is sometimes called **Matsushima's formula**. For a cuspidal automorphic representation π of $G(\mathbb{A}_{\mathbb{Q}})$ we refer to

$$H_{\text{cusp}}^\bullet(\text{Sh}^K, \mathcal{V}^K)(\pi^\infty) := \bigoplus_{\pi': \pi'^\infty \cong \pi^\infty} (H^\bullet(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; \pi_\infty' \otimes V) \otimes \pi^\infty K)^{m(\pi')}$$

as the π^∞ -isotypic component of the cohomology.

Definition 15.5.2. A cuspidal automorphic representation π of $G(\mathbb{A}_{\mathbb{Q}})$ is **cohomological** if there is a representation V of as above such that

$$H^\bullet(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; \pi_\infty \otimes V) \neq 0.$$

A vector φ in the space of π is **cohomological** if there exists an embedding from the space of π into $L_{\text{cusp}}^2([G])$ such that the image of φ is cohomological.

Thus if there is a vector in the space of π that is cohomological, then π itself is cohomological. If we wish to be more specific, we can speak of V -cohomological representations or V -cohomological vectors.

The condition that π is cohomological is very restrictive. An easy way to quantify this is given by the following theorem [BW00, Corollary I.4.2]:

Theorem 15.5.3 *Assume V is irreducible. If $H^\bullet(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; \pi_\infty \otimes V) \neq 0$ then the infinitesimal character of $\pi_\infty^\vee$ is equal to the infinitesimal character of V .* $\square$

Corollary 15.5.4 *A cohomological cuspidal automorphic representation π of $G(\mathbb{A}_\mathbb{Q})$ is C -algebraic in the sense of Definition 12.8.5.*

Proof. The result follows from Theorem 15.5.3, Lemma 12.8.3 and Lemma 12.8.4. $\square$

The link between geometry and representation theory given by Matsushima's formula is key to a great deal of beautiful mathematics. It is of interest to see if this link can be generalized. There is an important generalization involving cohomology with coefficients in coherent sheaves in the special case where the symmetric space X is Hermitian [Har90, BHR94]. For applications to Galois representations see [PS16a, GK19] and for vanishing results see [Lan16]. In this theory the space under consideration does not change. It is always the Shimura manifold Sh^K .

We would like to point out a generalization where the space does change. Specifically, consider spaces of the form

$$G(\mathbb{Q}) \backslash (A_G \backslash G(\mathbb{R}) / K_\infty \times G(\mathbb{A}_\mathbb{Q}^\infty) / K), \quad (15.32)$$

where G is a reductive group over $\mathbb{Q}$ and $K_\infty < G(\mathbb{R})$ is a compact subgroup which is **not necessarily maximal**. It would be interesting to see if some analogue of Matsushima's formula holds in this level of generality, possibly employing a generalization of $(\mathfrak{g}, K_\infty)$ -cohomology.

If K_∞ satisfies certain desiderata the spaces $A_G \backslash G(\mathbb{R}) / K_\infty$ are known as **period domains** (see [CMSP17]). For suitable period domains, an analogue of Matsushima's formula exists [Wil91]. Using this analogue of Matsushima's formula, Carayol has obtained results suggesting ties between the arithmetic of certain automorphic representations and period domains [Car05, Car15].

15.6 The relation to distinction

Usually in the literature, one finds references to cohomological representations but no references to cohomological vectors. Despite this, the notion of a cohomological vector is of great importance.

Suppose that $H \leq G$ are reductive $\mathbb{Q}$ -groups. We assume for simplicity that the corresponding symmetric spaces can be chosen so that the inclusion of $H(\mathbb{R})$ into $G(\mathbb{R})$ induces an injection

$$X_H \hookrightarrow X := X_G.$$

Then there is an embedding

$$\mathrm{Sh}(H, X_H)^{K \cap H(\mathbb{A}_{\mathbb{Q}}^{\infty})} \longrightarrow \mathrm{Sh}(G, X)^K =: \mathrm{Sh}^K.$$

For the remainder of this section, we assume for simplicity that Sh^K is compact, which is to say that G^{der} has no nontrivial $\mathbb{Q}$ -split subtorus by Theorem 2.6.3. This implies that every automorphic representation of $G(\mathbb{A}_{\mathbb{Q}})$ is cuspidal and hence

$$H_{\mathrm{cusp}}^{\bullet}(\mathrm{Sh}^K, \mathcal{V}^K) = H^{\bullet}(\mathrm{Sh}^K, \mathcal{V}^K).$$

We also assume $\mathrm{Sh}(H, X_H)^{K \cap H(\mathbb{A}_{\mathbb{Q}}^{\infty})}$ and Sh^K are orientable.

Let V be a representation as in (15.9) and let $V^{\vee}$ be the dual representation with corresponding local system $(\mathcal{V}^{\vee})^K$. We denote by $\mathcal{V}_H$ the local system on $\mathrm{Sh}(H, X_H)^{K \cap H(\mathbb{A}_{\mathbb{Q}}^{\infty})}$ attached to $V|_{H(\mathbb{Q})}$. Let n be the real dimension of Sh^K and let q be the real codimension of $\mathrm{Sh}(H, X_H)^{K \cap H(\mathbb{A}_{\mathbb{Q}}^{\infty})}$ in Sh^K . There is a cycle class map

$$H^0(\mathrm{Sh}(H, X_H)^{K \cap H(\mathbb{A}_{\mathbb{Q}}^{\infty})}, \mathcal{V}_H) \longrightarrow H^q(\mathrm{Sh}^K, \mathcal{V}^K) \quad (15.33)$$

defined as follows. For a class $[Z] \in H^0(\mathrm{Sh}(H, X_H)^{K \cap H(\mathbb{A}_{\mathbb{Q}}^{\infty})}, \mathcal{V}_H)$, we have a linear functional

$$\begin{aligned} H^{n-q}(\mathrm{Sh}^K, (\mathcal{V}^{\vee})^K) &\longrightarrow \mathbb{C} \\ \eta &\longmapsto \int_{[Z]} \eta. \end{aligned} \quad (15.34)$$

Poincaré duality [Har11, Theorem 4.8.9] furnishes a perfect pairing

$$H^{n-q}(\mathrm{Sh}^K, (\mathcal{V}^{\vee})^K) \times H^q(\mathrm{Sh}^K, \mathcal{V}^K) \longrightarrow \mathbb{C} \quad (15.35)$$

so $[Z]$ defines an element of $H^q(\mathrm{Sh}^K, \mathcal{V}^K)$ yielding the cycle class map (15.33).

Describing the $C_c^{\infty}(G(\mathbb{A}_{\mathbb{Q}}^{\infty}) // K)$ -span of the image of (15.33) is an important (and mostly open) problem. This is especially true in the situation where Sh^K and $\mathrm{Sh}(H, X_H)^{K \cap H(\mathbb{A}_{\mathbb{Q}}^{\infty})}$ are Shimura varieties, not just manifolds, due to the connection with the Tate conjecture (see §15.8).

We now explain the precise connection between the image of (15.33) and distinguished representations and point out an important technical problem that deserves additional study. By Matsushima's formula (15.31), to describe the $C_c^{\infty}(G(\mathbb{A}_{\mathbb{Q}}^{\infty}) // K)$ -span of the image of (15.33), it suffices to describe its projection to

$$H^q(\mathrm{Sh}^K, \mathcal{V}^K)(\pi'^{\infty})$$

for all automorphic representations π' of $G(\mathbb{A}_{\mathbb{Q}})$.

Unwinding the considerations of §15.5 and the definition of the cycle class map, we see that the projection of the image of (15.33) to $H^{\bullet}(\mathrm{Sh}^K, \mathcal{V}^K)(\pi'^{\infty})$ is nonzero if and only if there is

- a (cuspidal) automorphic representation π of $G(\mathbb{A}_{\mathbb{Q}})$ with $\pi^{\infty} \cong \pi'^{\infty \vee}$,
- $(v^{\vee}, v) \in V^{\vee} \times H^0(\mathrm{Sh}(H, X_H)^{K \cap H(\mathbb{A}_{\mathbb{Q}}^{\infty})}, \mathcal{V}_H)$,
- a $V^{\vee}$ -cohomological vector φ in the space of π and an ω^I on X such that

$$\int_{[\mathrm{Sh}(H, X_H)^{K \cap H(\mathbb{A}_{\mathbb{Q}}^{\infty})}]} \varphi \omega^I v^{\vee}(v) \neq 0. \quad (15.36)$$

This motivates the following question:

Question 15.6.1 *If an automorphic representation π of $G(\mathbb{A}_{\mathbb{Q}})$ is H -distinguished and $V^{\vee}$ -cohomological, then is there*

$$(v^{\vee}, v) \in V^{\vee} \times H^0(\mathrm{Sh}(H, X_H)^{K \cap H(\mathbb{A}_{\mathbb{Q}}^{\infty})}, \mathcal{V}_H),$$

an ω^I on X , and a $V^{\vee}$ -cohomological vector φ in the space of π such that (15.36) holds?

Recall that π is H -distinguished if and only if

$$\int_{A_G H(\mathbb{Q}) \backslash A_G H(\mathbb{A}_{\mathbb{Q}})} \varphi(h) dh \neq 0$$

for some smooth $\varphi \in L^2(\pi)$, the π -isotypic subspace of $L^2([G])$. A priori, it could happen that no such φ satisfies the additional stipulations of the question. For example, it could be that no such φ is $V^{\vee}$ -cohomological.

We do not know, even conjecturally, the answer to Question 15.6.1 in any degree of generality. In important special cases, the question above can be answered in the affirmative (see [KS15, Sun17, Sun19] for example).

In this section we have assumed that Sh^K is compact. The considerations above are of course still of interest in the noncompact case, but even defining the cycle class map (15.33) in any level of generality is complicated. Extending the cycle class map to the noncompact case is another open problem that has not received the scrutiny it deserves. We refer to [GG12] for an example where the difficulties caused by noncompactness of Sh^K are overcome in a very special case.

15.7 More on $(\mathfrak{g}, K_{\infty})$ -cohomology

We have seen that the question of whether or not a given automorphic representation contributes to the cohomology of a Shimura manifold with coefficients in a local system is completely determined by the $(\mathfrak{g}, K_{\infty})$ -cohomology of its factor at infinity. Fortunately $(\mathfrak{g}, K_{\infty})$ -cohomology is a very pleasant object with which to work. In this section, we list some properties of these groups; the canonical reference is [BW00].

For $i = 1, 2$, let G_i be a reductive group over $\mathbb{R}$ and let $K_{i\infty} \leq G_i(\mathbb{R})$ be a maximal compact subgroup. Moreover let $\mathcal{A}_i$ be an admissible $(\mathfrak{g}_i, K_{i\infty})$ -module. Then one has a Künneth formula [BW00, §I.1.3]:

Theorem 15.7.1 (Künneth formula) *For any $n \in \mathbb{Z}_{\geq 0}$ one has natural isomorphisms*

$$H^n(\mathfrak{g}_1 \oplus \mathfrak{g}_2, K_{1\infty} \times K_{2\infty}; \mathcal{A}_1 \otimes \mathcal{A}_2) = \bigoplus_{p+q=n} H^p(\mathfrak{g}_1, K_{1\infty}; \mathcal{A}_1) \otimes H^q(\mathfrak{g}_2, K_{2\infty}; \mathcal{A}_2).$$

□

For the statement of the next theorem, let G be a reductive group over $\mathbb{Q}$ (we only use the $\mathbb{Q}$ -structure to define A_G). Let $\mathfrak{g}$ be the Lie algebra of $G_{\mathbb{R}}$ and let $K_\infty \leq G(\mathbb{R})$ be a maximal compact subgroup. Define X as in (15.1), then

$$\dim_{\mathbb{R}} X = \dim_{\mathbb{R}} \mathfrak{a}_G \backslash \mathfrak{g} - \dim_{\mathbb{R}} K_\infty.$$

Let $\mathcal{A}$ be an irreducible admissible $(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty)$ -module. The following is [BW00, Proposition I.7.6]:

Theorem 15.7.2 (Poincaré duality) *One has that*

$$H^q(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; \mathcal{A}) \cong H^{\dim_{\mathbb{R}} X - q}(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; \mathcal{A}^\vee)^\vee.$$

□

Combining Theorems 15.7.1 and 15.7.2 we see that to compute the $(\mathfrak{g}, K_\infty)$ -cohomology of a $(\mathfrak{g}, K_\infty)$ -module, it suffices to understand the case where $\mathfrak{g}$ is simple over $\mathbb{R}$ and when $q \leq \frac{1}{2} \dim_{\mathbb{R}} X$.

We now consider the cohomology of admissible representations of the form

$$\pi_\infty \otimes V$$

where π_∞ is an irreducible unitary $(\mathfrak{g}, K_\infty)$ -module and V is irreducible. Here when we say π_∞ is unitary we mean that π_∞ is the $(\mathfrak{g}, K_\infty)$ -module underlying a unitary representation of $G(\mathbb{R})$. Such a unitary representation is uniquely determined up to unitary equivalence by Theorem 4.4.10. Let

$$\theta : \mathfrak{g} \longrightarrow \mathfrak{g} \tag{15.37}$$

be the Cartan involution. Its fixed point subalgebra is $\mathfrak{k}$, the Lie algebra of K_∞ . Let $\mathfrak{p}$ be its -1 eigenspace.

When π_∞ is trivial, we have the following proposition [BW00, Corollary II.3.2]:

Proposition 15.7.3 *Assume as above that V is irreducible. One has that*

$$H^\bullet(\mathfrak{g}, K_\infty; V) = \begin{cases} 0 & \text{if } V \text{ is nontrivial,} \\ (\wedge^\bullet \mathfrak{p})^{K_\infty} & \text{if } V \text{ is trivial.} \end{cases}$$

□

We can interpret this proposition as saying that much of the cohomology vanishes when V is nondegenerate. This phenomenon persists for other π_∞ . We now state a vanishing theorem for cohomology in this context. Combined with Poincaré duality, it implies that the cohomology of Sh^K is concentrated around the degree equal to $\frac{1}{2}\dim_{\mathbb{R}}\mathrm{Sh}^K$.

For each θ -stable parabolic subalgebra $\mathfrak{q} \leq \mathfrak{g} \otimes \mathbb{C}$, let $\mathfrak{n}(\mathfrak{q})$ be its nilradical. Let

$$\mathcal{P}(V) := \left\{ \theta\text{-stable parabolic subalgebras } \mathfrak{q} : \dim V^{\mathfrak{n}(\mathfrak{q})} = 1 \right\}$$

and let

$$c(V) := \min \{ \dim(\mathfrak{n}(\mathfrak{q}) \cap \mathfrak{p}) : \mathfrak{q} \in \mathcal{P}(V) \}.$$

The following is the vanishing theorem mentioned above [BW00, Theorem II.10.1]:

Theorem 15.7.4 *Assume as above that π_∞ is an irreducible unitary $(\mathfrak{g}, K_\infty)$ -module and that V is irreducible. If $\mathfrak{g}$ is semisimple and the largest subalgebra of $\mathfrak{g}$ that acts trivially on the space of π_∞ is contained in $\mathfrak{k}$ then*

$$H^i(\mathfrak{g}, K_\infty; \pi_\infty \otimes V)$$

vanishes for $i < c(V^\vee)$.

□

We close this section by pointing out that Vogan and Zuckerman have provided a classification of all unitary $(\mathfrak{g}, K_\infty)$ -modules with nonzero cohomology and computed the cohomology of these modules (see [VZ84]).

15.8 Shimura varieties

In this section we define the notion of a Shimura variety. Although much of the theory of Shimura varieties was established by Shimura, a simpler formulation was introduced by Deligne in [Del71]. Deligne's formulation has been almost universally adopted in the literature and we will use it here. An introduction to [Del71] is contained in [Mil05]. Proofs for the results we state below can be found in these references.

For a reductive group G , let $G^{\mathrm{ad}} := G/Z_G$ be the adjoint group. The **Deligne torus** is

$$\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m). \quad (15.38)$$

Note that $\mathbb{S}(\mathbb{R}) = \mathbb{C}^\times$. The following definition is due to Deligne:

Definition 15.8.1. Let G be a reductive group over $\mathbb{Q}$ and let X be a $G(\mathbb{R})$ -conjugacy class of homomorphisms $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$. The pair (G, X) is a **Shimura datum** if

- (a) For $h \in X$, only the characters $z/\bar{z}$, 1, and $\bar{z}/z$ occur in the representation of $\mathbb{S}$ on $(\mathrm{Lie} G^{\mathrm{ad}})_{\mathbb{C}}$ defined by h .
- (b) $\theta = \mathrm{ad}(h(\sqrt{-1}))$ is a Cartan involution of G^{ad} .
- (c) G^{ad} has no $\mathbb{Q}$ -factors on which the projection of h is trivial.

Condition (b) is equivalent to the statement that

$$\{g \in G^{\mathrm{ad}}(\mathbb{C}) : g = \theta(\bar{g})\}$$

is compact. Conditions (a) and (b) together imply that X is a Hermitian symmetric space for G .

Since Sh^K is a complex manifold in this case, a natural question is whether Sh^K can be realized as the complex points of some variety. This is indeed the case by a basic theorem of Baily and Borel. It states that if K is neat then Sh^K can be given the structure of the complex points of smooth quasi-projective scheme over $\mathbb{C}$ in a canonical manner. One can say even more.

For each characteristic zero field k we may consider the set

$$\mathcal{C}(k) := G(k) \backslash \mathrm{Hom}(\mathbb{G}_{mk}, G_k)$$

of $G(k)$ -conjugacy classes of algebraic group morphisms from $\mathbb{G}_{mk}$ to G_k . This is in fact the k -points of a scheme over $\mathbb{Q}$ as follows from [DG74, §XI, Corollary 4.2] (see also [Bri22, Proposition 4.8]) combined with the theory of GIT quotients we develop in §17.2. Upon choosing an embedding $\bar{\mathbb{Q}} \rightarrow \mathbb{C}$ we obtain an isomorphism

$$\mathcal{C}(\bar{\mathbb{Q}}) \xrightarrow{\sim} \mathcal{C}(\mathbb{C}) \tag{15.39}$$

by [Mil05, Lemma 12.1] and the discussion following it. We use this isomorphism to identify $\mathcal{C}(\bar{\mathbb{Q}})$ and $\mathcal{C}(\mathbb{C})$. The utility of this identification lies in the fact that $\mathcal{C}(\bar{\mathbb{Q}})$ admits a natural action of $\mathrm{Gal}_{\bar{\mathbb{Q}}}$.

We identify $\mathbb{S}_{\mathbb{C}} = \mathbb{G}_m \times \mathbb{G}_m$ in such a way that the automorphism of $\mathbb{S}$ given by complex conjugation induces the automorphism

$$\begin{aligned} \mathbb{S}(\mathbb{C}) &\longrightarrow \mathbb{S}(\mathbb{C}) \\ (z_1, z_2) &\longmapsto (\bar{z}_2, \bar{z}_1). \end{aligned}$$

For each $x \in X$, we have a cocharacter

$$u_x(z) := h_{x_{\mathbb{C}}}(z, 1),$$

where $x_{\mathbb{C}}$ denotes the base change of x to $\mathbb{C}$. The image of u_x in

$$\mathcal{C}(\mathbb{C}) = \mathcal{C}(\bar{\mathbb{Q}})$$

is independent of the choice of x . Let $E(G, X)$ be the smallest field such that

$$u_x \in \mathcal{C}(\bar{\mathbb{Q}})^{\mathrm{Gal}_{E(G, X)}}.$$

It is a number field independent of the choice of $x \in X$. It is called the **reflex field** of (G, X) .

Theorem 15.8.2 *For each neat $K \leq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$, there exists a smooth quasi-projective scheme $M(G, X)^K$ over the reflex field $E(G, X)$ such that*

$$\mathrm{Sh}^K = M(G, X)^K(\mathbb{C})$$

and for all $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ the maps

$$T_g : \mathrm{Sh}^{gKg^{-1}} \longrightarrow \mathrm{Sh}^K$$

of (15.5) are induced by morphisms

$$T_g : M(G, X)^{gKg^{-1}} \longrightarrow M(G, X)^K \quad (15.40)$$

of quasi-projective schemes over $E(G, X)$. Furthermore, the scheme $M(G, X)^K$ is canonically characterized by the action of $\mathrm{Gal}_{E(G, X)}$ on certain special points. $\square$

Definition 15.8.3. The **Shimura variety** attached to (G, X) is the projective limit

$$M(G, X) := \varprojlim_K M(G, X)^K.$$

The maps (15.40) yield an action of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ on $M(G, X)$. This is one advantage of working with the projective limit. If one prefers to work with a fixed K as we did in the previous sections of this chapter, one can observe that the fact that (15.40) is defined implies that for any $K' \leq gKg^{-1}$ the map (15.5) is induced by a map $T_g : M(G, X)^{K'} \rightarrow M(G, X)^K$ of schemes over $E(G, X)$, and hence the maps in the definition of the correspondence $T(g)$ in (15.7) are all induced by maps of quasi-projective schemes over $E(G, X)$. We will use this observation to define the action of a Hecke algebra on certain étale cohomology groups below.

Definition 15.8.4. A **morphism of Shimura data** $(G, X) \rightarrow (G', X')$ is a morphism of algebraic groups $G \rightarrow G'$ such that the induced map $G(\mathbb{R}) \rightarrow G'(\mathbb{R})$ sends X to X' .

Definition 15.8.5. A **morphism of Shimura varieties** $M(G, X) \rightarrow M(G', X')$ is an inverse system of morphisms of schemes over $E(G, X)$ compatible with the actions of $G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ and $G'(\mathbb{A}_{\mathbb{Q}}^{\infty})$.

Theorem 15.8.6 *A morphism of Shimura data induces a morphism of Shimura varieties over the compositum $E(G, X)E(G', X')$. Moreover, it is a closed immersion if $G \rightarrow G'$ a closed immersion.* $\square$

Example 15.1. Let $G = \mathrm{GL}_2\mathbb{Q}$ and let $h(a + b\sqrt{-1}) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. In this case, each $\mathrm{Sh}(G, X)^K$ is a finite union of modular curves (see the end of §2.6).

Example 15.2. Let $G = \operatorname{Res}_{F/\mathbb{Q}} \operatorname{GL}_2$, where $F/\mathbb{Q}$ is totally real. Take

$$h(a + b\sqrt{-1}) = \prod_{\sigma} \begin{pmatrix} \sigma(a) & \sigma(b) \\ -\sigma(b) & \sigma(a) \end{pmatrix}.$$

Here the product is over all embeddings $\sigma : F \rightarrow \mathbb{R}$. Then the associated Shimura varieties are known as **Hilbert modular varieties**.

Example 15.3. Let $J'_{2n} = \begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}$ where J_n is the usual antidiagonal symmetric matrix (13.12). Let $G = \operatorname{GSp}_{2n}$. For a $\mathbb{Z}$ -algebra R , let

$$\operatorname{GSp}_{2n}(R) = \{g \in \operatorname{GL}_{2n}(R) : g^t J'_{2n} g = c(g) J'_{2n} \text{ for some } c(g) \in R^\times\}.$$

Let

$$h(a + b\sqrt{-1}) = \begin{pmatrix} aI_n & bJ_n \\ -bJ_n & aI_n \end{pmatrix}.$$

The associated Shimura varieties are known as **Siegel modular varieties**. This example can be generalized to totally real number fields as in the previous example yielding Hilbert-Siegel modular varieties.

Let D be an associative algebra with unit over a field F . We do not require D to be commutative. An involution of D is an F -linear bijection $*$: $D \rightarrow D$ such that $(yx)^* = x^*y^*$.

Example 15.4. Let F be a totally real number field and k/F a totally imaginary quadratic extension. Let D be a simple algebra with center k and assume D admits an involution $*$ that induces the nontrivial automorphism of k/F . For $\mathbb{Q}$ -algebras R , set

$$G(R) := \{x \in D \otimes_{\mathbb{Q}} R : xx^* \in R^\times\}.$$

Assume that we are given a homomorphism

$$h_0 : \mathbb{C} \longrightarrow D \otimes_{\mathbb{Q}} \mathbb{R}$$

such that $h_0(z)^* = h_0(\bar{z})$. The map

$$\begin{aligned} \iota : D &\longrightarrow D \\ x &\longmapsto h_0(\sqrt{-1})^{-1} x^* h_0(\sqrt{-1}) \end{aligned}$$

is then an involution of D . We assume it is positive in the sense that $\operatorname{tr}_{D/\mathbb{Q}}(x\iota(x)) > 0$ for $x \in D - \{0\}$. The restriction of h_0 to $\mathbb{C}^\times$ defines

$$h : \mathbb{S} \longrightarrow G_{\mathbb{R}}.$$

Let X be the $G(\mathbb{R})$ -conjugacy class of h . Then (G, X) is a Shimura datum. The associated Shimura varieties are sometimes called **Kottwitz varieties** since Kottwitz employed them to fantastic effect in the construction of Galois

representations attached to automorphic representations [Kot92]. There are two key properties that make these varieties pleasant objects to consider. First, if D is chosen appropriately, then Sh^K is compact. Second, the group G is a form of a unitary similitude group, and thus for essentially half the primes p , the group $G_{\mathbb{Q}_p}$ is a product of general linear groups. This fact was exploited in the original proofs of the local Langlands correspondence for GL_n [HT01, Hen00].

Example 15.5. For $n \geq 3$ there does not exist a Shimura datum of the form (GL_n, X) . However, it is shown in [GT03a, GT03b] that the real points of Siegel modular varieties may be naturally identified with finite unions of spaces of the form $\Gamma \backslash \mathrm{GL}_n(\mathbb{R}) / K_n$ for suitable arithmetic subgroups $\Gamma \leq \mathrm{GL}_n(\mathbb{Q})$. Here K_n is the standard maximal compact subgroup defined as in Example 4.2. As Goresky and Tai suggest, it would be interesting to investigate both generalizations of this result and arithmetic applications.

Suppose that (G, X) is a Shimura datum with reflex field E . Let $K \leq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ be a neat compact open subgroup. For simplicity, assume that $\mathrm{Sh}^{\bar{K}}$ is compact. We can then consider the étale cohomology groups

$$H_{\text{ét}}^{\bullet}(M(G, X)_{\bar{\mathbb{Q}}}^K, \bar{\mathbb{Q}}_{\ell}) := H_{\text{ét}}^{\bullet}(M(G, X)^K \times_E \bar{\mathbb{Q}}, \bar{\mathbb{Q}}_{\ell}). \quad (15.41)$$

We can also consider versions of these groups with coefficients in a local system, but let us take trivial coefficients for simplicity. We refer the reader to [Mil80] for proofs of the properties of étale cohomology we use below.

For fields k let

$$C_c(G(\mathbb{A}_{\mathbb{Q}}^{\infty}) // K, k) := \{f : G(\mathbb{A}_{\mathbb{Q}}^{\infty}) \rightarrow k : f(k_1 g k_2) = f(g)\} \quad (15.42)$$

for $g \in G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ and for $k_1, k_2 \in K$. Then the groups (15.41) are modules under

$$\bar{\mathbb{Q}}_{\ell}[\mathrm{Gal}_E] \otimes_{\bar{\mathbb{Q}}_{\ell}} C_c^{\infty}(G(\mathbb{A}_{\mathbb{Q}}^{\infty}) // K, \bar{\mathbb{Q}}_{\ell}),$$

where $\bar{\mathbb{Q}}_{\ell}[\mathrm{Gal}_E]$ is the group algebra. The action of $C_c^{\infty}(G(\mathbb{A}_{\mathbb{Q}}^{\infty}) // K, \bar{\mathbb{Q}}_{\ell})$ is defined by stipulating that the characteristic function $\mathbb{1}_{KgK}$ is sent to the correspondence $T(g)$.

Upon choosing an isomorphism $\iota : \bar{\mathbb{Q}}_{\ell} \xrightarrow{\sim} \mathbb{C}$ and an embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$, one obtains a comparison isomorphism

$$H_{\text{ét}}^{\bullet}(M(G, X)_{\bar{\mathbb{Q}}}^K, \bar{\mathbb{Q}}_{\ell}) \xrightarrow{\sim} H^{\bullet}(\mathrm{Sh}^K, \mathbb{C}). \quad (15.43)$$

Using ι we identify $C_c(G(\mathbb{A}_{\mathbb{Q}}^{\infty}) // K, \bar{\mathbb{Q}}_{\ell})$ and $C_c(G(\mathbb{A}_{\mathbb{Q}}^{\infty}) // K)$. Then (15.43) is $C_c^{\infty}(G(\mathbb{A}_{\mathbb{Q}}^{\infty}) // K)$ -equivariant.

Combining (15.43) with Matsushima's formula (15.31), we obtain a decomposition

$$H_{\text{ét}}^i(M(G, X)_{\bar{\mathbb{Q}}}^K, \bar{\mathbb{Q}}_{\ell}) = \bigoplus_{\pi^{\infty}} H_{\text{ét}}^i(\pi^{\infty}) \otimes \pi^{\infty K}$$

as $\overline{\mathbb{Q}}_\ell[\mathrm{Gal}_E] \otimes_{\overline{\mathbb{Q}}_\ell} C_c^\infty(G(\mathbb{A}_\mathbb{Q}^\infty) // K)$ -modules. Here $H_{\text{ét}}^i(\pi^\infty)$ is a continuous representation of Gal_E on which $C_c^\infty(G(\mathbb{A}_\mathbb{Q}^\infty) // K)$ acts trivially and Gal_E acts trivially on $\pi^{\infty K}$. The sum is over all isomorphism classes of irreducible admissible $G(\mathbb{A}_\mathbb{Q}^\infty)$ -representations π^∞ such that there exists a discrete automorphic representation π' of $A_G \backslash G(\mathbb{A}_\mathbb{Q})$ with $\pi'^\infty \cong \pi^\infty$. The Gal_E -representation $H_{\text{ét}}^i(\pi^\infty)$ is of dimension

$$\sum_{\pi_\infty} m(\pi_\infty \otimes \pi^\infty) \dim H^i(\mathfrak{a}_G \backslash \mathfrak{g}, K_\infty; \pi_\infty),$$

where the sum is over isomorphism classes of irreducible admissible representations of $A_G \backslash G(\mathbb{R})$ and $m(\pi_\infty \otimes \pi^\infty)$ is the multiplicity of $\pi_\infty \otimes \pi^\infty$ in $L^2([G])$.

One has a conjecture describing the $\overline{\mathbb{Q}}_\ell[\mathrm{Gal}_E] \otimes_{\overline{\mathbb{Q}}_\ell} C_c^\infty(G(\mathbb{A}_\mathbb{Q}^\infty) // K)$ -module

$$H_{\text{ét}}^i(\pi^\infty) \otimes \pi^{\infty K} \quad (15.44)$$

in terms of global L -parameters. A nice survey is given in [BR94b]. The description can be made independent of the existence of the Langlands group $\mathcal{L}_\mathbb{Q}$ using analogues of the considerations of §12.8. In more detail, one expects that packets of tempered automorphic representations of $G(\mathbb{A}_\mathbb{Q})$ that are related to cohomology correspond under the global Langlands correspondence to global L -parameters that factor through $\mathrm{Gal}_\mathbb{Q}$. Here we are ignoring the distinction between L -algebraic and C -algebraic representations for simplicity. In many cases one can use Shimura varieties both to construct the global L -parameters attached to cohomological automorphic representations and to prove the conjectural description of (15.44) mentioned above (see [CHLN11, Mor08, Mor10, Shi11] for instance).

Lest the reader be misled, the description of the étale cohomology group $H_{\text{ét}}^\bullet(M(G, X)_\mathbb{Q}^K, \overline{\mathbb{Q}}_\ell)$ in terms of automorphic representations is useful for more than proving cases of the global Langlands correspondence. In fact, this description is only the beginning of the story. These étale cohomology groups are intimately tied to two of the deepest conjectures in algebraic and arithmetic geometry, namely, the Tate [Tat94] and Beilinson-Bloch conjectures [Nek94, Sch88].

Let $CH^p(X)_{\overline{\mathbb{Q}}_\ell}$ denote the Chow group of codimension p cycles on X (with $\overline{\mathbb{Q}}_\ell$ coefficients). There is a cycle class map

$$\mathrm{cl} : CH^p(X)_{\overline{\mathbb{Q}}_\ell} \longrightarrow H_{\text{ét}}^{2p}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell(p))^{\mathrm{Gal}_E}. \quad (15.45)$$

Here (p) is the Tate twist.

Conjecture 15.8.7 (Tate) *The map (15.45) is surjective.*

The Beilinson-Bloch conjecture gives a description of the kernel of cl . We will not make the conjecture precise. To indicate the depth of the conjecture,

we point out that it contains the Birch and Swinnerton-Dyer conjecture as a special case.

Now for general X , it is difficult to compute $H_{\text{ét}}^{2p}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_{\ell}(p))^{\text{Gal}_E}$, and even more difficult to exhibit any elements in $CH^p(X)_{\overline{\mathbb{Q}}_{\ell}}$. Shimura varieties provide an example where one can compute the Galois representation $H_{\text{ét}}^{2p}(X_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_{\ell}(p))$ and produce at least some elements of $CH^p(X)_{\overline{\mathbb{Q}}_{\ell}}$ as we now explain.

Assume that (H, X_H) is a Shimura datum with reflex field $E_H := E(H, X_H)$ that is equipped with a morphism

$$(H, X_H) \longrightarrow (G, X).$$

For simplicity, assume that H is a subgroup of G and the map $X_H \rightarrow X$ is injective and induced by the inclusion $H(\mathbb{R}) \rightarrow G(\mathbb{R})$. For each compact open subgroup $K \leq G(\mathbb{A}_{\mathbb{Q}}^{\infty})$ we then have a closed immersion

$$M(H, X_H)_{EE_H}^{K_H} \longrightarrow M(G, X)_{EE_H}^K, \quad (15.46)$$

where $K_H := K \cap H(\mathbb{A}_{\mathbb{Q}}^{\infty})$. Here EE_H is the compositum of the reflex fields $E := E(G, X)$ and E_H .

Write q for the real codimension of the image of (15.46); it is independent of the choice of K . In the étale setting, we again have a cycle class map

$$H_{\text{ét}}^0(M(H, X_H)_{\overline{\mathbb{Q}}_{\ell}}^{K_H}, \overline{\mathbb{Q}}_{\ell}) \longrightarrow H_{\text{ét}}^q(M(G, X)_{\overline{\mathbb{Q}}_{\ell}}^K, \overline{\mathbb{Q}}_{\ell}(q/2))^{\text{Gal}_{EE_H}}. \quad (15.47)$$

The map (15.47) is compatible with the cycle class map of (15.33) under the comparison isomorphism (15.43) and its analogue for $M(H, X_H)$ [DMOS82, §I]. The image of (15.47) is contained in $\text{cl}(CH^{q/2}(M(G, X)_{EE_H}^K)_{\overline{\mathbb{Q}}_{\ell}})$. Motivated by the Tate conjecture, one is lead to the following question:

Question 15.8.8 *What subspace of*

$$H_{\text{ét}}^q(M(G, X)_{\overline{\mathbb{Q}}_{\ell}}^K, \overline{\mathbb{Q}}_{\ell}(q/2))^{\text{Gal}_{EE_H}}$$

is in the $C_c^{\infty}(G(\mathbb{A}_{\mathbb{Q}}^{\infty}) // K)$ -span of the image of (15.47)?

As explained in §15.6, this question can be rephrased in terms of the existence of distinguished representations satisfying certain local assumptions. This was, in fact, the original motivation for the definition of a distinguished representation as mentioned in the beginning of this chapter, and the answer is still not well understood. However, many interesting cases have been studied [GH14, GW14, Kud97, MR87, Ram04].

There is also a connection between distinguished representations and the Beilinson-Bloch conjecture, though it is not as direct. For many years, S. Kudla and his collaborators have investigated the conjecture for certain Shimura varieties using an arithmetic version of the theta correspondence [Kud04, KRY06]. Though the theory of distinguished representations is not

emphasized in their work, it is lurking in the background. In a different setting related to the Gan-Gross-Prasad conjecture W. Zhang conjectured a more direct connection between distinguished representations and the Beilinson-Bloch conjecture using a relative trace formula [Zha12a, Zha12b]. W. Zhang has since made decisive progress towards establishing his conjecture [Zha21]. In the function field setting, analogues of the relative trace formula can even be used to give the strongest results known towards the Birch and Swinnerton-Dyer conjecture [YZ17, YZ19].

Exercises

15.1. Let Y be a Hausdorff topological space and let Γ be a discrete group acting freely and properly discontinuously on it. Prove that there is a unique topology on the set-theoretic quotient $\Gamma \backslash Y$ such that the quotient map $Y \rightarrow \Gamma \backslash Y$ is a local homeomorphism. In other words, for every $y \in Y$, there is an open neighborhood of y mapping homeomorphically onto an open neighborhood of Γy .

15.2. Let Γ and X be as in (15.4). Prove that there is a unique manifold structure on $\Gamma \backslash X$ such that the quotient map $X \rightarrow \Gamma \backslash X$ is a local homeomorphism (see Exercise 15.1 above for the definition of the local homeomorphism).

15.3. Prove that a congruence subgroup is arithmetic.

15.4. Prove Lemma 15.2.5.

15.5. Compute the $(\mathfrak{gl}_2, \mathcal{O}_2(\mathbb{R}))$ -cohomology of all irreducible admissible $(\mathfrak{gl}_2, \mathcal{O}_2(\mathbb{R}))$ -modules.

15.6. Prove the formula (15.8).

15.7. Prove that the isomorphism (15.28) is Hecke equivariant.

15.8. Prove that the pairs (G, X) in Examples 15.1, 15.2, 15.3, and 15.4 are Shimura data.

Chapter 16

Spectral Sides of Trace Formulae



A felicitous but unproved conjecture may be of much more consequence for mathematics than the proof of many a respectable theorem.

A. Selberg

The Arthur-Selberg trace formula, or more generally, the relative trace formula, is an important tool in automorphic representation theory. We investigate the spectral side of these identities in this chapter.

16.1 The automorphic kernel function

The remaining chapters of this book are devoted to stating and proving simple versions of trace formulae and sketching some applications. These formulae all have a geometric side involving integrals of a test function along certain orbits and a spectral side involving period integrals of automorphic forms.

The first step is to employ a fundamental idea applied to the study of automorphic forms by Selberg [Sel56] which we now describe. Let G be a reductive group over a global field F and as in §2.6 let

$$[G] := A_G G(F) \backslash G(\mathbb{A}_F). \quad (16.1)$$

For $x \in A_G \backslash G(\mathbb{A}_F)$, one has the regular representation

$$\begin{aligned} R(x) : L^2([G]) &\longrightarrow L^2([G]) \\ \varphi &\longmapsto (g \mapsto \varphi(gx)). \end{aligned} \quad (16.2)$$

For

$$f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F)),$$

consider the operator

$$R(f) : L^2([G]) \longrightarrow L^2([G])$$

$$\varphi \longmapsto \left(g \mapsto \int_{AG \backslash G(\mathbb{A}_F)} f(x) \varphi(gx) dx \right). \quad (16.3)$$

Selberg observed that there is a simple formula for the kernel function of this operator. Before we give the formula, let us make precise what we mean by a kernel function. Let $V \leq L^2([G])$ be a closed subspace and let

$$A : V \longrightarrow V \quad (16.4)$$

be a $\mathbb{C}$ -linear operator. We say a function $K \in C^\infty([G] \times [G])$ is a **kernel function** for A if $y \mapsto K(x, y)$ is in $L^2([G])$ for all $x \in [G]$ and

$$\int_{[G]} K(x, y) \varphi(y) dy = A\varphi(x)$$

for all $\varphi \in V$ and almost every $x \in [G]$.

Lemma 16.1.1 *If a $\mathbb{C}$ -linear operator $A : V \rightarrow V$ admits a kernel function, then the kernel function is unique.*

Proof. Suppose that $K_1, K_2 \in C^\infty([G] \times [G])$ are kernel functions for A . Let $y_0 \in [G]$ and let $\{\varphi_n\}_{n=1}^\infty \in C_c^\infty([G])$ be a sequence of functions such that

$$\lim_{n \rightarrow \infty} \int_{[G]} \varphi(x) \varphi_n(x) dx = \varphi(y_0)$$

for all $\varphi \in C^\infty([G])$. We can assume that the support of φ_n is contained in a compact subset of $[G]$ independent of n . Then

$$\begin{aligned} K_1(x, y_0) &= \lim_{n \rightarrow \infty} \int_{[G]} K_1(x, y) \varphi_n(y) dy \\ &= \lim_{n \rightarrow \infty} \int_{[G]} K_2(x, y) \varphi_n(y) dy \\ &= K_2(x, y_0) \end{aligned}$$

for all $x \in [G]$. □

In view of the lemma, it makes sense to speak of “the” kernel function of an operator. Selberg observed the following:

Lemma 16.1.2 *The function*

$$K_f(x, y) := \sum_{\gamma \in G(F)} f(x^{-1}\gamma y) \quad (16.5)$$

is the kernel function for $R(f)$.

Proof. One has that

$$\begin{aligned}
 R(f)\varphi(x) &= \int_{A_G \backslash G(\mathbb{A}_F)} f(y)R(y)\varphi(x)dy \\
 &= \int_{A_G \backslash G(\mathbb{A}_F)} f(y)\varphi(xy)dy \\
 &= \int_{A_G \backslash G(\mathbb{A}_F)} f(x^{-1}y)\varphi(y)dy \\
 &= \int_{[G]} \sum_{\gamma \in G(F)} f(x^{-1}\gamma y)\varphi(y)dy.
 \end{aligned}$$

Here to justify the last step one uses unfolding (Lemma 9.2.5). To check that $K_f(x, y)$ is smooth as a function of x and y , we show that for (x, y) in a fixed compact set the sum is finite. Let

$$\Omega_1 \times \Omega_2 \subset A_G \backslash G(\mathbb{A}_F) \times A_G \backslash G(\mathbb{A}_F)$$

be compact subsets and let

$$(x, y^{-1}) \in \Omega_1 \times \Omega_2.$$

Then the only nonzero summands in $K_f(x, y)$ correspond to γ satisfying $\gamma \in \Omega_1 \text{Supp}(f)\Omega_2$. Finally, we check that $y \mapsto K(x, y)$ is in $L^2([G])$. One has

$$\begin{aligned}
 &\int_{[G]} K_f(x, y) \overline{K_f(x, y)} dy \\
 &= \int_{[G]} \left(\sum_{\gamma \in G(F)} f(x^{-1}\gamma y) \right) \left(\sum_{\gamma' \in G(F)} \overline{f(x^{-1}\gamma' y)} \right) dy.
 \end{aligned}$$

Unfolding, this is

$$\sum_{\gamma \in G(F)} \int_{A_G \backslash G(\mathbb{A}_F)} f(x^{-1}\gamma y) \overline{f(x^{-1}y)} dy. \quad (16.6)$$

For each x , the integral over y has compact support, and hence the sum over γ has finite support. Thus (16.6) is finite. $\square$

The function $K_f(x, y)$ is the **automorphic kernel function** attached to f .

The kernel $K_f(x, y)$ can also be expanded spectrally. It is easiest to describe the contribution of the cuspidal spectrum as we now explain. We will show in Theorem 16.2.3 below that the restriction $R_{\text{cusp}}(f)$ of $R(f)$ to $L^2_{\text{cusp}}([G])$ admits a kernel function

$$K_{f,\text{cusp}}(x, y) = \sum_{\pi} K_{\pi(f)}(x, y), \quad (16.7)$$

where the sum is over isomorphism classes of cuspidal automorphic representations π of $A_G \backslash G(\mathbb{A}_F)$ and $K_{\pi(f)}(x, y)$ is the kernel function of the restriction $\pi(f)$ of $R(f)$ to $L^2_{\text{cusp}}(\pi)$, the π -isotypical subspace of $L^2_{\text{cusp}}([G])$. Moreover, the same theorem states that $K_{\pi(f)}(x, y) \in L^2([G] \times [G])$, the sum in (16.7) converges in $L^2([G] \times [G])$ and the L^2 -expansion of $K_{\pi(f)}(x, y)$ is given by

$$K_{\pi(f)}(x, y) = \sum_{\varphi \in \mathcal{B}_{\pi}} \pi(f)\varphi(x)\overline{\varphi}(y) \quad (16.8)$$

for $\mathcal{B}_{\pi}$ an orthonormal basis of the space of π . We note that $\pi(f)$ need not be of finite rank in general, but it is if f is K_{∞} -finite by admissibility of π (see Exercise 16.1).

We thus arrive at the identity that underlies all trace formulae:

$$\sum_{\gamma \in G(F)} f(x^{-1}\gamma y) = \sum_{\pi} K_{\pi(f)}(x, y) + * \quad (16.9)$$

where the sum on the right is over isomorphism classes of cuspidal automorphic representations π of $A_G \backslash G(\mathbb{A}_F)$ and the $*$ denotes the contribution of the orthogonal complement of $L^2_{\text{cusp}}([G])$ in $L^2([G])$. This contribution will be made precise in §16.3. The left-hand side is the **geometric expansion** of the kernel. The right-hand side is the **spectral expansion** of the kernel.

The key point here is that the right-hand side manifestly contains all automorphic representations of $A_G \backslash G(\mathbb{A}_F)$, whereas the left-hand side, at least a priori, does not involve automorphic representations at all. It packages automorphic information in an entirely different manner. The goal is to play these two different manners of encoding automorphic representations off of each other.

One method of extracting information from (16.9) is taking traces as we now explain. Theorem 16.2.3 implies that $K_{\pi(f)}(x, y)$ is smooth in (x, y) . Moreover, it is the kernel function of a trace class operator by Theorem 9.1.2. Thus, by [Bri91, Corollary 3.2], one has

$$\sum_{\pi} \int_{[G]} K_{\pi(f)}(x, x) dx = \sum_{\pi} m(\pi) \text{tr } \pi(f), \quad (16.10)$$

where $m(\pi)$ is the multiplicity of π in $L^2_{\text{cusp}}([G])$.

Thus, (16.10), in principle, is equal to the integral of the left-hand side of (16.9) over the diagonal (minus the contribution of $*$). We say in principle because the integral of the left-hand side over the diagonal is rarely convergent. In any case, at the expense of a massive amount of work, one can make sense of the integral over the diagonal of the left-hand side of (16.9) and one obtains the Arthur-Selberg trace formula.

For the moment, we leave the geometric expansion of the kernel and focus on the spectral side. We return to the geometric side in §18.2 when we discuss trace formulae in simple settings.

16.2 Relative traces

Let π be a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$. As just mentioned, the integral of $K_{\pi(f)}(x, y)$ over the diagonal copy of $[G]$ gives the trace of the operator $m(\pi)\pi(f)$, where $m(\pi)$ is the multiplicity of π in $L^2_{\text{cusp}}([G])$.

There is no need to just integrate over the diagonal copy of $[G]$ however. Jacquet was the first to systematically study the integrals of $K_{\pi(f)}(x, y)$ over subgroups other than the diagonal copy of $[G]$ (apart from twisted versions of the diagonal embedding that appear in the twisted trace formula) [Jac97, Jac05a]. These new integrals are called **relative traces**. They are introduced in this section.

Let $\mathcal{HS}$ denote the $\mathbb{C}$ -vector space of Hilbert-Schmidt operators on $L^2([G])$. It becomes a Hilbert space with respect to the pairing

$$\begin{aligned} \mathcal{HS} \times \mathcal{HS} &\longrightarrow \mathbb{C} \\ (A, B) &\longmapsto \text{tr}(A \circ B^*). \end{aligned}$$

Here B^* is the adjoint of B and the pairing is well-defined because the convolution of two Hilbert-Schmidt operators is of trace class by Exercise 9.2.

Lemma 16.2.1 *The map*

$$\begin{aligned} \mathcal{HS} &\longrightarrow L^2([G] \times [G]) \\ A &\longmapsto \sum_{\varphi} A\varphi(x)\overline{\varphi}(y) \end{aligned}$$

is an isometric isomorphism, where the sum is over an orthonormal basis of $L^2([G])$. The map is independent of the choice of basis.

Proof. If A is of finite rank, then the sum $\sum_{\varphi} A\varphi(x)\overline{\varphi}(y)$ has only finitely many nonzero terms. With this in mind, it is easy to check that the map is an isometry when restricted to the subspace of finite rank operators. On the other hand, the space of finite rank operators is dense in $\mathcal{HS}$, so the map extends by continuity to an isometry. This implies in particular that the sum over φ converges in $L^2([G] \times [G])$. The inverse of the map is given by sending $K \in L^2([G] \times [G])$ to the operator

$$\varphi \longmapsto \left(x \mapsto \int_{[G]} K(x, y)\varphi(y)dy \right);$$

this operator is Hilbert-Schmidt by Exercise 9.4. The map in the lemma is independent of the choice of orthonormal basis because its inverse is independent of the choice of orthonormal basis. $\square$

For $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$, let

$$f^\vee(g) := f(g^{-1}) \quad \text{and} \quad f^*(g) = \overline{f(g^{-1})}. \quad (16.11)$$

Lemma 16.2.2 *Let $V \leq L^2([G])$ be a closed $A_G \backslash G(\mathbb{A}_F)$ -invariant subspace such that the restriction $R(f)|_V$ to V is Hilbert-Schmidt. Then*

$$(R(f)|_V)^* = R(f^*)|_V.$$

Proof. Let $\mathcal{B}_V$ be an orthonormal basis of V . If $\varphi_1, \varphi_2 \in \mathcal{B}_V$ then

$$\begin{aligned} & \int_{[G] \times [G]} \sum_{\varphi \in \mathcal{B}_V} R(f) \varphi(x) \overline{\varphi}(y) \overline{\varphi_2}(x) \varphi_1(y) dx dy \\ &= \int_{[G]} R(f) \varphi_1(x) \overline{\varphi_2}(x) dx \\ &= \int_{A_G \backslash G(\mathbb{A}_F)} \int_{[G]} f(g) \varphi_1(xg) \overline{\varphi_2}(x) dx dg \\ &= \int_{A_G \backslash G(\mathbb{A}_F)} \int_{[G]} \varphi_1(x) f(g) \overline{\varphi_2}(xg^{-1}) dx dg \\ &= \int_{[G]} \varphi_1(x) R(f^\vee) \overline{\varphi_2}(x) dx \\ &= \int_{[G] \times [G]} \sum_{\varphi \in \mathcal{B}_V} \varphi(x) \overline{\varphi}(y) R(f^\vee) \overline{\varphi_2}(x) \varphi_1(y) dx dy. \end{aligned}$$

Here the manipulation of integrals is justified by the fact that the sums converge in $L^2([G] \times [G])$ by Lemma 16.2.1. We deduce that

$$\sum_{\varphi \in \mathcal{B}_V} R(f) \varphi(x) \overline{\varphi}(y) = \sum_{\varphi \in \mathcal{B}_V} \varphi(x) R(f^\vee) \overline{\varphi}(y) \quad (16.12)$$

almost everywhere on $[G] \times [G]$. Therefore, the result follows from Lemma 16.2.1. $\square$

We now prove a theorem already employed in §16.1. To motivate the assumptions in the theorem, recall that the operator $R_{\text{cusp}}(f)$ is of trace class by Theorem 9.1.2, and (at least when F is a number field) the restriction of $R(f)$ to the whole discrete spectrum of $L^2([G])$ is of trace class by Theorem 9.1.4. Moreover, by Exercise 9.3, trace class operators are Hilbert-Schmidt.

Theorem 16.2.3 *Let $V \leq L^2([G])$ be a closed $A_G \backslash G(\mathbb{A}_F)$ -invariant subspace such that the restriction $R(f)|_V$ to V is Hilbert-Schmidt and let $\mathcal{B}_V$ be an*

orthonormal basis of V . For $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$, the restriction $R(f)|_V$ admits a kernel function

$$K_{f,V}(x, y) \in L^2([G] \times [G]) \cap C^\infty([G] \times [G])$$

with L^2 -expansion

$$\sum_{\varphi \in \mathcal{B}_V} R(f) \varphi(x) \overline{\varphi}(y). \quad (16.13)$$

Here when we say that (16.13) is an L^2 -expansion of $K_{f,V}(x, y)$, we mean that the sum converges in $L^2([G] \times [G])$ to $K_{f,V}(x, y)$.

Proof. Since V is closed in $L^2([G])$, we can extend the operator $R(f)|_V$ by 0 to obtain a Hilbert-Schmidt operator on $L^2([G])$. We apply Lemma 16.2.1 to conclude that (16.13) converges in $L^2([G] \times [G])$. If there is a smooth function $K_{f,V}(x, y)$ that is equal almost everywhere to (16.13), then it is easy to check that $K_{f,V}(x, y)$ is a kernel function for $R(f)|_V$.

By the Dixmier-Malliavin lemma (Theorem 4.2.9) and its obvious analogue in the non-Archimedean case, we can write f as a finite sum of functions of the form

$$f_1 * f_2 * f_3$$

for $f_1, f_2, f_3 \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$. It clearly suffices to prove the theorem for f of this special form, so we assume that $f = f_1 * f_2 * f_3$. Thus, using (16.12), one has that

$$\begin{aligned} & \sum_{\varphi \in \mathcal{B}_V} R(f_1 * f_2 * f_3) \varphi(x) \overline{\varphi}(y) \\ &= \sum_{\varphi \in \mathcal{B}_V} R(f_2 * f_3) \varphi(x) R(f_1^\vee) \overline{\varphi}(y) \\ &= (R(f_2) \times R(f_1^\vee)) \sum_{\varphi \in \mathcal{B}_V} R(f_3) \varphi(x) \overline{\varphi}(y). \end{aligned} \quad (16.14)$$

Here all equalities are of functions in $L^2([G] \times [G])$. The function on the bottom line of (16.14) is smooth as a function of (x, y) by Theorem 4.2.4, Proposition 4.2.7, and their obvious analogues in the non-Archimedean case. Thus, we take $K_{f,V}(x, y)$ to be the last line of (16.14). $\square$

We now prepare to define relative traces. Let

$$H \leq G \times G$$

be a subgroup. We assume that the neutral component of H is the direct product (not semidirect product) of a unipotent group and a reductive group. We set

$$A_{G,H} := H(\mathbb{A}_F) \cap (A_G \times A_G). \quad (16.15)$$

Then the natural map

$$A_{G,H} \backslash H(\mathbb{A}_F) \longrightarrow (A_G \times A_G) \backslash (A_G \times A_G) H(\mathbb{A}_F)$$

is an isomorphism. Implicit in the definition of $R(f)$ is a choice of measure on $A_G \backslash G(\mathbb{A}_F)$. To define relative traces, we additionally choose a Haar measure $d(h_\ell, h_r)$ on $A_{G,H} \backslash H(\mathbb{A}_F)$. It induces a right $H(\mathbb{A}_F)$ -invariant Radon measure on $A_{G,H} H(F) \backslash H(\mathbb{A}_F)$.

Let

$$\chi : H(\mathbb{A}_F) \longrightarrow \mathbb{C}^\times$$

be a quasi-character trivial on $A_{G,H} H(F)$, let π be a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ and let $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$. The **relative trace** of $\pi(f)$ with respect to H and χ is

$$\mathrm{rtr}_{H,\chi} \pi(f) := \int_{A_{G,H} H(F) \backslash H(\mathbb{A}_F)} K_{\pi(f)}(h_\ell, h_r) \chi^{-1}(h_\ell, h_r) d(h_\ell, h_r). \quad (16.16)$$

If H and χ are understood we often drop them from notation, writing

$$\mathrm{rtr} \pi(f) := \mathrm{rtr}_{H,\chi} \pi(f).$$

These relative traces are well-defined by the following lemma:

Lemma 16.2.4 *For any closed subspace $V \leq L_{\mathrm{cusp}}^2([G])$, one has that*

$$\int_{A_{G,H} H(F) \backslash H(\mathbb{A}_F)} |K_{f,V}(h_\ell, h_r) \chi^{-1}(h_\ell, h_r)| d(h_\ell, h_r) < \infty.$$

Proof. This is an immediate consequence of Theorem 16.2.3 and Corollary 14.3.4. $\square$

The relative trace of $\pi(f)$ is a generalization of the trace of $\pi(f)$. Indeed, if π is a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ and $H \leq G \times G$ is the group G embedded diagonally, then

$$\mathrm{rtr}_{H,\chi} \pi(f) = \mathrm{tr} \pi(f).$$

by Theorem 9.1.2 and [Bri91, Corollary 3.2].

The key property of relative traces is contained in the following proposition:

Proposition 16.2.5 *Suppose that π is a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$. The representation $\pi \otimes \pi^\vee$ of $(A_G \backslash G(\mathbb{A}_F))^2$ is (H, χ) -distinguished if and only if $\mathrm{rtr}_{H,\chi} \pi(f) \neq 0$ for some $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$.*

Proof. Recall the definition of the period integral $\mathcal{P}_\chi$ from (14.4). One has that

$$\mathcal{P}_\chi(K_{\pi(f)}) = \text{rtr}_{H,\chi} \pi(f),$$

where the convergence is justified by Lemma 16.2.4. Certainly, $K_{\pi(f)}$ lies in the $\pi \otimes \pi^\vee$ -isotypic subspace of $L^2_{\text{cusp}}([G] \times [G])$ so we deduce the “if” direction.

Now assume that $\pi \otimes \pi^\vee$ is (H, χ) -distinguished. By Lemma 14.3.5 and Exercise 16.4, there are K_∞ -finite smooth functions $\varphi, \varphi' \in L^2_{\text{cusp}}(\pi)$ such that $\mathcal{P}_\chi(\varphi' \otimes \bar{\varphi}) \neq 0$. Multiplying φ by a scalar, if necessary, we can assume that $\|\varphi\|_2 = 1$. Using Proposition 4.5.6 and Exercise 5.7, choose $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ such that $\pi(f)$ maps φ to φ' and sends any vector in $L^2_{\text{cusp}}(\pi)$ orthogonal to φ to zero. Then

$$\text{rtr}_{H,\chi} \pi(f) = \mathcal{P}_\chi(\varphi' \otimes \bar{\varphi}).$$

□

Thus investigating which representations of $(A_G \backslash G(\mathbb{A}_F))^2$ are (H, χ) -distinguished is equivalent to studying the linear functionals

$$\begin{aligned} C_c^\infty(A_G \backslash G(\mathbb{A}_F)) &\longrightarrow \mathbb{C} \\ f &\longmapsto \text{rtr}_{H,\chi} \pi(f). \end{aligned}$$

The main result of this chapter relates the kernel function $K_{f,\text{cusp}}(x, y)$ to these linear functionals. It amounts to the computation of the cuspidal contribution to the spectral side of the relative trace formula:

Theorem 16.2.6 *Let $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$. One has that*

$$\int_{A_{G,H} H(F) \backslash H(\mathbb{A}_F)} K_{f,\text{cusp}}(h_\ell, h_r) \chi^{-1}(h_\ell, h_r) d(h_\ell, h_r) = \sum_{\pi} \text{rtr}_{H,\chi} \pi(f),$$

where the sum on the right is over isomorphism classes of cuspidal automorphic representations π of $A_G \backslash G(\mathbb{A}_F)$. Moreover, the integral on the left and the sum on the right are absolutely convergent. In particular, if $R(f)$ has cuspidal image then

$$\int_{A_{G,H} H(F) \backslash H(\mathbb{A}_F)} K_f(h_\ell, h_r) \chi^{-1}(h_\ell, h_r) d(h_\ell, h_r) = \sum_{\pi} \text{rtr}_{H,\chi} \pi(f).$$

After the proof of Theorem 16.2.6, we describe the general spectral expansion of $K_f(x, y)$ and then describe one method for constructing functions f such that $R(f)$ has cuspidal image in §16.4.

Proof. We assume that F is a number field. The proof in the function field case is easier and is left as Exercise 16.2.

By Theorem 9.1.2, for any $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$, the operator $R_{\text{cusp}}(f)$ is of trace class, hence Hilbert-Schmidt and the same is true of $\pi(f)$ for any

cuspidal automorphic representation π of $A_G \backslash G(\mathbb{A}_F)$. These facts will be used several times below.

By the Dixmier-Malliavin lemma (Theorem 4.2.9), we can assume $f = f_1 * f_2 * f_3$. Using (16.12), we have

$$\sum_{\pi} |K_{\pi(f)}(x, y)| = \sum_{\pi} |(\pi(f_2) \times \pi(f_1^\vee)) K_{\pi(f_3)}(x, y)| \quad (16.17)$$

where the sum is over isomorphism classes of cuspidal automorphic representations of $A_G \backslash G(\mathbb{A}_F)$. Let $\mathfrak{S}(t)$ be a Siegel domain in $G(\mathbb{A}_F)$ such that $G(F)A_G\mathfrak{S}(t) = G(\mathbb{A}_F)$. By Proposition 9.4.7 and with the notation of that proposition, for $(x, y) \in \mathfrak{S}(t) \times \mathfrak{S}(t)$, the sum (16.17) is bounded by a constant depending on $\mathfrak{S}(t)$, $B_1, B_2 \in \mathbb{R}_{>0}$ and f_1, f_2 times

$$\begin{aligned} & \sum_{\pi} \min_{\alpha_1, \alpha_2 \in \Delta} \alpha_1(s_x)^{-B_1} \alpha_2(s_y)^{-B_2} \|K_{\pi(f_3)}\|_2 \\ &= \min_{\alpha_1, \alpha_2 \in \Delta} \alpha_1(s_x)^{-B_1} \alpha_2(s_y)^{-B_2} \|K_{f_3, \text{cusp}}\|_2. \end{aligned} \quad (16.18)$$

Here we have used the fact that $K_{f_3, \text{cusp}} \in L^2([G] \times [G])$ by Theorem 16.2.3 and the fact that $K_{\pi(f_3)}$ and $K_{\pi'(f_3)}$ are orthogonal in $L^2([G] \times [G])$ if $\pi \not\cong \pi'$. Thus the sum

$$\sum_{\pi} K_{\pi(f)}(x, y) \quad (16.19)$$

converges absolutely and uniformly in $(x, y) \in [G] \times [G]$ and hence is a continuous function on $[G] \times [G]$. On the other hand, $K_{f, \text{cusp}}(x, y)$ is smooth and equal to (16.19) almost everywhere by Theorem 16.2.3. We conclude that $K_{f, \text{cusp}}(x, y)$ is equal to (16.19) pointwise. In particular,

$$\begin{aligned} & \int_{A_{G,H}H(F) \backslash H(\mathbb{A}_F)} K_{f, \text{cusp}}(h_\ell, h_r) \chi^{-1}(h_\ell, h_r) d(h_\ell, h_r) \\ &= \int_{A_{G,H}H(F) \backslash H(\mathbb{A}_F)} \left(\sum_{\pi} K_{\pi(f)}(h_\ell, h_r) \right) \chi^{-1}(h_\ell, h_r) d(h_\ell, h_r). \end{aligned} \quad (16.20)$$

Combining (16.17) and (16.18) we have the estimate

$$\sum_{\pi} |K_{\pi(f)}(x, y)| \ll \min_{\alpha_1, \alpha_2 \in \Delta} \alpha_1(s_x)^{-B_1} \alpha_2(s_y)^{-B_2} \|K_{f_3, \text{cusp}}\|_2$$

where the implied constant depends on $\mathfrak{S}(t), f_1, f_2, B_1, B_2$. This implies that $\sum_{\pi} |K_{\pi(f)}(x, y)|$ is rapidly decreasing. Applying Proposition 14.3.3, we see that the integral in the top line of (16.20) is absolutely convergent, that we can bring the sum outside the integral in (16.20), and that the sums in the statement of the theorem are absolutely convergent. $\square$

16.3 The full expansion of the automorphic kernel

Using Langlands' decomposition of the entire spectrum of $L^2([G])$, we now give the full spectral expansion of $K_f(x, y)$, not just its restriction to the cuspidal subspace. Assume for this section that F is a number field.

We will use the notation and terminology developed in Chapter 10. In particular, we fix a minimal parabolic subgroup P_0 of G with Levi decomposition $P_0 = M_0N_0$ and assume that the Levi decomposition $P = MN$ of a standard parabolic subgroup P is chosen so that $M \geq M_0$.

Let $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$. The automorphic kernel function $K_f(x, y)$ has an expansion

$$K_f(x, y) = \sum_P n_P^{-1} \sum_\sigma \int_{i(\mathfrak{a}_M^G)^*} \sum_{\varphi \in \mathcal{B}(\sigma)} E(x, I(\sigma, \lambda)(f)\varphi, \lambda) \overline{E(y, \varphi, \lambda)} d\lambda \quad (16.21)$$

where the sum on P is over all standard parabolic subgroups of G , the sum on σ is over all isomorphism classes of irreducible $A_M \backslash M(\mathbb{A}_F)$ -subrepresentations of $L_{\text{disc}}^2([M])$, and $\mathcal{B}(\sigma)$ is an orthonormal basis of $\text{Ind}_P^G(L_{\text{disc}}^2([M])(\sigma))$ consisting of vectors in the subspace $\text{Ind}_P^G(L_{\text{disc}}^2([M])(\sigma))^0$ defined in (10.8). A priori, this expression only converges in the sense that, for each x , it is an L^2 -expansion of $K_f(x, y)$ as a function of y . However, one can make sense of it pointwise by an analogue of the argument in the proof of Theorem 16.2.3. See [Art78, §4] or below [Art05, Theorem 7.2].

16.4 Functions with cuspidal image

The general spectral expansion (16.21) of $K_f(x, y)$ is formidable, and it becomes more serious when one tries to integrate the kernel. In particular, it is not integrable over $[G]$ embedded diagonally into $[G] \times [G]$. To remedy this, Arthur introduced truncated versions of the kernel that are integrable over the diagonal and can be given spectral interpretations [Art05]. This work was used in the construction of the functorial transfers discussed in §13.4 and §13.8.

As we have seen, if we assume that $R(f)$ has cuspidal image, then the spectral expansion of $K_f(x, y)$ is much simpler. Happily, there are now several fairly flexible methods available to construct $R(f)$ with cuspidal image [BPLZZ21, LV07, Mai22]. We will content ourselves with explaining an older but simpler method. Although it has limitations that will be explained below, it is still quite useful in practice.

Definition 16.4.1. Let v be a non-Archimedean place of a global field F . A function $f_v \in C_c^\infty(G(F_v))$ is said to be **supercuspidal** if

$$\int_{N(F_v)} f_v(gnh)dn = 0$$

for all proper parabolic subgroups $P < G_{F_v}$ (defined over F_v) with unipotent radical N and all $g, h \in G(F_v)$.

Lemma 16.4.2 *Let v be a finite place of F , let $f^v \in C_c^\infty(A_G \backslash G(\mathbb{A}_F^v))$, and let $f_v \in C_c^\infty(G(F_v))$ be supercuspidal. Let $f(g) = f_v(g_v)f^v(g^v)$, so that $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$. Then $R(f)$ has cuspidal image.*

Proof. Let $P < G$ be a proper parabolic subgroup with unipotent radical N . As usual, let $[N] := N(F) \backslash N(\mathbb{A}_F)$. For $\varphi \in L^2([G])$, $R(f)\varphi$ is smooth and hence can be integrated over any compact subset. For all $x \in A_G \backslash G(\mathbb{A}_F)$, unfolding (see Lemma 9.2.5) we have

$$\begin{aligned} \int_{[N]} R(f)\varphi(nx)dn &= \int_{[N]} \int_{A_G \backslash G(\mathbb{A}_F)} f(g)\varphi(nxg)dgd n \\ &= \int_{[N]} \int_{A_G \backslash G(\mathbb{A}_F)} f(x^{-1}n^{-1}g)\varphi(g)dgd n \\ &= \int_{[N]} \int_{A_G N(F) \backslash G(\mathbb{A}_F)} \sum_{\delta \in N(F)} f(x^{-1}n^{-1}\delta g)\varphi(g)dgd n \\ &= \int_{A_G N(F) \backslash G(\mathbb{A}_F)} \int_{[N]} \sum_{\delta \in N(F)} f(x^{-1}n^{-1}\delta^{-1}g)\varphi(g)dndg \\ &= \int_{A_G N(F) \backslash G(\mathbb{A}_F)} \int_{N(\mathbb{A}_F)} f(x^{-1}n^{-1}g)\varphi(g)dndg \\ &= 0, \end{aligned}$$

where the last equality follows from the fact that the inner integral

$$\int_{N(\mathbb{A}_F)} f(x^{-1}n^{-1}g)\varphi(g)dn$$

vanishes since f_v is supercuspidal. □

Essentially all examples of supercuspidal functions are obtained using the following lemma (see Exercise 16.5):

Lemma 16.4.3 *Assume that $Z_G(F_v)$ is compact for some v and that (π_v, V) is an irreducible supercuspidal representation of $G(F_v)$. If f_v is a matrix coefficient of π_v then f_v is supercuspidal.*

The assumption that $Z_G(F_v)$ is compact is not essential; see the discussion of truncated matrix coefficients in [HL04]. This lemma indicates the intrinsic

limitation of supercuspidal functions: supercuspidal functions can only be used to study representations that are supercuspidal at some place. For a precise statement, see Exercise 16.6.

Proof. Let P be a proper parabolic subgroup of G_{F_v} with unipotent radical N . Let V_N and $V_N^\vee$ be the Jacquet modules defined as in (8.13) in §8.3. Assume that

$$\int_{N(F_v)} f_v(gnh)dn \neq 0$$

for some $g, h \in G(F_v)$. Realize $V \otimes V^\vee$ as a subspace of $C_c^\infty(G(F_v))$ as in the proof of Proposition 8.5.3. We then obtain a nonzero map

$$\begin{aligned} V \otimes V^\vee &\longrightarrow V_N \otimes V_N^\vee \\ f_v &\longmapsto \left((g, h) \mapsto \int_{N(F_v)} f_v(g^{-1}nh)dn \right) \end{aligned}$$

contradicting the supercuspidality of π_v . $\square$

Exercises

16.1. Let $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ and let K_∞ be a maximal compact subgroup of $G(F_\infty)$. If f is K_∞ -finite and π is a discrete automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ then prove that $\pi(f)$ has finite image.

16.2. Prove Theorem 16.2.6 when F is a function field.

16.3. Let v be a non-Archimedean place of a global field F . A function $f_v \in C_c^\infty(G(F_v))$ is said to be **F -supercuspidal** if

$$\int_{N(F_v)} f_v(gnh)dn = 0$$

for all proper parabolic subgroups $P < G$ (defined over F) with unipotent radical N and all $g, h \in G(F_v)$. Prove that the conclusion of Lemma 16.4.2 remains valid if we assume f_v is F -supercuspidal.

16.4. For $1 \leq i \leq 2$, let G_i be a reductive group over a number field F , let $K_{i\infty} \leq G_i(F_\infty)$ be maximal compact subgroup, and let π_i be a cuspidal automorphic representation of $A_{G_i} \backslash G_i(\mathbb{A}_F)$. Then a smooth function in

$$L_{\text{cusp}}^2([G_1] \times [G_2])(\pi_1 \otimes \pi_2) \tag{16.22}$$

need not be in the algebraic tensor product

$$L_{\text{cusp}}^2([G_1])(\pi_1) \otimes L_{\text{cusp}}^2([G_2])(\pi_2).$$

Despite this, show that a $K_{1\infty} \times K_{2\infty}$ -finite smooth function in (16.22) is a finite sum of functions of the form $\varphi_1 \otimes \varphi_2$ where φ_i is a $K_{i\infty}$ -finite smooth function in $L^2_{\text{cusp}}([G_i])(\pi_i)$ for $1 \leq i \leq 2$.

16.5. Let F be a non-Archimedean local field and let G be a reductive group over F . Assume that $Z_G(F)$ is compact. Prove that any supercuspidal function in $C_c^\infty(G(F))$ is a finite sum of matrix coefficients of supercuspidal representations of $G(F)$.

16.6. Under the assumptions of Exercise 16.5, let $f \in C_c^\infty(G(F))$ be a matrix coefficient of an irreducible supercuspidal representation π' . Prove that if π is an irreducible admissible representation of $G(F)$ such that $\pi(f) \neq 0$ then $\pi \cong \pi'$.

16.7. Let $H \leq G \times G$ be a subgroup such that H° is the direct product of a unipotent group and a reductive group. Let K_∞ be a maximal compact subgroup of $G(F_\infty)$. If f is K_∞ -finite, prove that

$$\text{rtr}_{H,\chi} \pi(f) = \sum_{\varphi \in \mathcal{B}_\pi} \mathcal{P}_\chi(\pi(f)\varphi \times \bar{\varphi}),$$

where we take the basis $\mathcal{B}_\pi$ to consist of K_∞ -finite vectors (which are smooth by Proposition 4.4.7).

Chapter 17

Orbital Integrals



When I suggested the term
endoscopy to Shelstad I didn't
know it had a medical meaning.

A. Ash

We define and study orbital integrals. In the next chapter, we explain how the geometric sides of trace formulae are written in terms of orbital integrals.

17.1 A refined study of orbits

The geometric side of any relative trace formula is a sum of orbital integrals, that is, integrals of a test function over the orbit of a group action, indexed by the set of all orbits. Thus it is not surprising that a robust understanding of orbital integrals is crucial in many applications of relative trace formulae.

The study of orbital integrals amounts to successively refining our understanding of orbits of algebraic group actions. At the first level of refinement one views orbits from the optic of algebraic geometry. Thus we review the theory of algebraic group actions in §17.2 and discuss quotients in the sense of geometric invariant theory in §17.3. This is necessary to give algebraic structure to the space of orbits. Out of necessity we pay particular attention to the description of the points of schemes over nonalgebraically closed fields.

Let H be an algebraic group over a field k . An essential feature of the theory is that the k -points of an orbit of H in the sense of algebraic geometry do not coincide with the orbits of the abstract group $H(k)$ in general. We discuss the local cohomological constructions necessary to come to grips with this phenomenon in §17.4. This is the second level of refinement of our study of orbits. It is an arithmetic refinement because it depends heavily on the structure of the fields involved. The final level of refinement comes when

we integrate test functions over orbits to form relative orbital integrals as in §17.5. This is a topological refinement; to prove convergence results we ultimately intersect compact and closed subsets of the points of schemes over local fields.

At the end of §17.5 we will have the basic features of the theory over a local field in hand. We then treat the adelic version. It is most satisfactory to proceed by employing Galois cohomology over the rings of integers of global fields and related Dedekind domains. For this purpose we discuss torsors for group schemes over rings in §17.6. We then turn to adelic points of orbits in §17.7. This is used to prove that global relative orbital integrals are convergent under appropriate assumptions (Proposition 17.8.2) and that the sum over all relative orbital integrals satisfying analogous assumptions is again convergent (Theorem 17.8.7).

Just as in the local case, these global convergence results boil down to topology. We endow Galois cohomology sets with topologies and prove various properness statements so that we can use the fact that our test functions are compactly supported. In particular, Proposition 17.8.2 is essentially a consequence of the properness of a certain map. Let F be a global field. Theorem 17.8.7 follows from the assertion that the space of closed orbits of a reductive group acting on an affine scheme is again an affine scheme together with the fact that for any affine F -scheme X , the subset $X(F) \subseteq X(\mathbb{A}_F)$ is closed and discrete.

In this chapter we use more serious algebraic geometry and Galois cohomology than in other chapters. Moving forward, the main concepts that we will require are various constructions and definitions related to relative classes contained in §17.2 and §17.3 and the definition of relative orbital integrals, given in §17.5 in the local setting and §17.8 in the adelic setting. If desired, the other sections of this chapter can be omitted.

However, all of the concepts in this chapter will be helpful for the reader desiring working knowledge of relative trace formulae. In particular, we have written the chapter so that the reader has the foundational results necessary to pre-stabilize (in the sense of [Lab99, §4.2]) any simple relative trace formula one may encounter.

17.2 Group actions, orbits, and stabilizers

We discuss the notion of an algebraic group action and some related constructions. Our references for the algebraic geometry in this chapter are [Mil17, MFK94, Poo17].

Let k be a commutative ring with identity, let H be a smooth affine group scheme over k , and let X be a separated scheme of finite type over k . We do not yet assume that X is affine, although this is the primary case of interest for us. A morphism

$$a : X \times H \longrightarrow X \tag{17.1}$$

is a (right) action of H on X if the diagram

$$\begin{array}{ccc} X \times H \times H & \xrightarrow{a \times 1_H} & X \times H \\ 1_X \times m \downarrow & & \downarrow a \\ X \times H & \xrightarrow{a} & X \end{array}$$

commutes and the composite

$$X \xrightarrow{1_X \times e} X \times H \xrightarrow{a} X$$

is the identity. Here m is the multiplication map and $e : \operatorname{Spec}(k) \rightarrow H$ is the identity section. These assumptions imply, in particular, that for every k -algebra R ,

$$a : X(R) \times H(R) \longrightarrow X(R)$$

is a right action. One can formulate the notion of a left action in the analogous manner.

Let $a : X \times H \rightarrow X$ and $a' : X' \times H \rightarrow X'$ be right actions of H on affine schemes X and X' of finite type over k , respectively. An **H -equivariant morphism** is a morphism of schemes $b : X \rightarrow X'$ such that

$$\begin{array}{ccc} X \times H & \xrightarrow{a} & X \\ b \times 1_H \downarrow & & \downarrow b \\ X' \times H & \xrightarrow{a'} & X' \end{array}$$

commutes. It is an isomorphism if b is an isomorphism.

For $\gamma \in X(k)$, we have a morphism

$$a(\gamma, \cdot) : H \longrightarrow X.$$

We denote by H_γ the stabilizer of γ . Thus H_γ is the subgroup scheme of H given by the fiber product

$$H_\gamma = \operatorname{Spec}(k) \times_X H$$

over $\gamma : \operatorname{Spec}(k) \rightarrow X$ and $a(\gamma, \cdot) : H \rightarrow X$. For k -algebras R one has that

$$H_\gamma(R) = \{h \in H(R) : a(\gamma, h) = \gamma\}. \quad (17.2)$$

When k is a field, the map $\operatorname{Spec}(k) \rightarrow X$ defined by γ is a closed immersion and hence H_γ is a closed subgroup scheme of H . The action a is often omitted from notation; in such a case one writes

$$\gamma h := \gamma.h := a(\gamma, h).$$

Assume for the remainder of this section that k is a field. As usual, we denote by $k^{\text{sep}} \leq \bar{k}$ a separable and algebraic closure of k , respectively.

If $I \leq H$ is a subgroup, then one can always define a quotient scheme $I \backslash H$ as in §1.10. The following is [Mil17, Proposition 7.17]:

Proposition 17.2.1 *The morphism*

$$H_\gamma \backslash H \longrightarrow X$$

induced by $a(\gamma, \cdot)$ is an immersion. □

For k -schemes Y denote by $|Y|$ the underlying topological space of Y . The morphism $a(\gamma, \cdot)$ induces a continuous map of topological spaces $|H| \rightarrow |X|$. The image

$$a(\gamma, |H|) \subseteq |X|$$

is locally closed (i.e. open in its closure). Let

$$O_H(\gamma) \subseteq X \tag{17.3}$$

be the subscheme with $a(\gamma, |H|)$ as its underlying topological space given the reduced induced scheme structure. The subscheme $O_H(\gamma)$ is the **H -orbit of γ** . If H is understood we omit it from notation, writing

$$O(\gamma) := O_H(\gamma). \tag{17.4}$$

Recall that we have assumed H is smooth.

Proposition 17.2.2 *The map $a(\gamma, \cdot)$ induces an isomorphism*

$$H_\gamma \backslash H \longrightarrow O(\gamma).$$

The scheme $O(\gamma)$ is smooth. Moreover, the map

$$H \longrightarrow O(\gamma)$$

is smooth if H_γ is smooth.

Proof. See [Mil17, Propositions 7.12] for the first assertion and see [Mil17, Proposition 7.4 and Proposition 7.15] for the last two assertions. □

For technical reasons later, we will require the following complementary results:

Lemma 17.2.3 *The quotient $H^\circ \backslash H$ is finite étale.*

Proof. See [Mil17, Proposition 5.58]. Milne takes étale to mean finite étale [Mil17, §A(i)]. □

Lemma 17.2.4 *If $I \leq H$ is a subgroup then the morphism*

$$I^\circ \backslash H \longrightarrow I \backslash H$$

is finite and flat and in particular proper.

Proof. This follows from Lemma 17.2.3 and [Mil17, Proposition 7.15]. $\square$

We say that a Y is a **reduced subscheme** of a k -scheme Z if Y is the reduced induced scheme structure on a locally closed subset of $|Z|$. Thus for all $\gamma \in X(k)$ the orbit $O(\gamma)$ is a reduced subscheme of X . If k is perfect and Y is a reduced subscheme of Z then $Y_{\bar{k}}$ is a reduced subscheme of $Z_{\bar{k}}$. However this is false for general k .

For any scheme Z there is a canonical reduced scheme Z_{red} equipped with a morphism $Z_{\text{red}} \rightarrow Z$ that induces a homeomorphism on the underlying topological spaces. If Y is a reduced subscheme of Z we identify $(Y_{\bar{k}})_{\text{red}}$ with the reduced induced subscheme structure on $|Y_{\bar{k}}|$. In particular, if k is perfect then $Y_{\bar{k}} = (Y_{\bar{k}})_{\text{red}}$.

Definition 17.2.5. A reduced subscheme O of X such that $(O_{\bar{k}})_{\text{red}} = O(\gamma)$ for some $\gamma \in X(\bar{k})$ is an **H -orbit**.

It is important to note that though $O(\bar{k})$ is nonempty for any H -orbit O , it may very well happen that $O(k)$ is empty.

To study this phenomenon we require a basic result from descent theory. Given a scheme Y over k^{sep} and $\sigma \in \text{Gal}_k$, let ${}^\sigma Y$ be the base change of Y along $\sigma : k^{\text{sep}} \rightarrow k^{\text{sep}}$. Thus, for instance, if

$$Y = \text{Spec}(k^{\text{sep}}[x_1, \dots, x_n]/(f_1, \dots, f_d))$$

for some $f_1, \dots, f_d \in k^{\text{sep}}[x_1, \dots, x_n]$ then

$${}^\sigma Y = \text{Spec}(k^{\text{sep}}[x_1, \dots, x_n]/(\sigma(f_1), \dots, \sigma(f_d))).$$

If Y is a subscheme of $X_{k^{\text{sep}}}$, then for all $\sigma \in \text{Gal}_k$ the scheme ${}^\sigma Y$ is naturally a subscheme of ${}^\sigma X = X$. Hence it is meaningful to ask whether ${}^\sigma Y = Y$.

The following theorem is a very special case of the general theory of Galois descent discussed in [Poo17, §4.4].

Theorem 17.2.6 *Assume that Y' is a quasi-projective subscheme of $X_{k^{\text{sep}}}$ such that ${}^\sigma Y' = Y'$ for all $\sigma \in \text{Gal}_k$. Then there is a subscheme Y of X such that $Y' = Y_{k^{\text{sep}}}$. $\square$*

Applying Galois descent to our situation we obtain the following:

Lemma 17.2.7 *Assume that X is affine. There is a bijection*

$$\begin{aligned} \{H\text{-orbits } O \subseteq X\} &\xrightarrow{\sim} \{\text{Aut}_k(\bar{k})\text{-stable } H(\bar{k})\text{-orbits in } X(\bar{k})\} \\ O &\longmapsto O(\bar{k}). \end{aligned}$$

Proof. We construct an inverse to the map in the lemma. Suppose we are given an $\text{Aut}_k(\bar{k})$ -stable $H(\bar{k})$ -orbit $O' \subset X(\bar{k})$. Choose an element $\gamma \in O'$. Let $O(\gamma) \subseteq X_{\bar{k}}$ be the $H_{\bar{k}}$ -orbit of γ . The scheme $O(\gamma)$ is independent of the choice of γ , and we have an isomorphism

$$a(\gamma, \cdot) : H_{\gamma\bar{k}} \backslash H_{\bar{k}} \xrightarrow{\sim} O(\gamma)$$

by Proposition 17.2.2.

The map $a(\gamma, \cdot) : H(\bar{k}) \rightarrow O(\gamma)(\bar{k})$ is surjective by Proposition 1.10.7. Hence $O(\gamma)(\bar{k}) = O'$. We may identify the topological spaces $|X_{k^{\text{sep}}}| = |X_{\bar{k}}|$ and hence O' may be identified with a locally closed subset of $|X_{k^{\text{sep}}}|$. Let $Y' \subset X_{k^{\text{sep}}}$ be the reduced induced scheme structure on O' . Then

$$(Y'_{\bar{k}})_{\text{red}} = O(\gamma).$$

The fact that $Y'(\bar{k}) = O'$ is $\text{Aut}_k(\bar{k})$ -invariant implies that for all $\sigma \in \text{Gal}_k$ we have ${}^{\sigma}Y' = Y'$. Moreover, Y' is a reduced subscheme of an affine scheme, hence it is quasi-affine (that is, an open subscheme of an affine scheme). This implies that Y is quasi-projective [Sta16, Tag 0B43]. Applying Theorem 17.2.6 we deduce that $Y' = Y_{k^{\text{sep}}}$ for some subscheme Y of X . Since Y' is reduced, Y is reduced. Thus Y is an H -orbit satisfying $Y(\bar{k}) = O'$ and we have constructed our desired inverse. $\square$

Definition 17.2.8. If $O(\gamma) \subseteq X$ is closed, we say that γ is **relatively semisimple**. We say that γ is **relatively regular** if the dimension of $O(\gamma)$ is maximal among all $\gamma \in X(\bar{k})$.

Here the adjective **relatively** is short for **relative to the action of H** .

When X is affine, the following theorem gives necessary conditions for an element of $X(k)$ to be relatively semisimple:

Theorem 17.2.9 *Assume that H° is reductive and that $\gamma \in X(k)$. The orbit $O(\gamma)$ is affine if and only if $(H_{\gamma})^{\circ}$ is reductive. In particular, if X is affine and $O(\gamma)$ is closed, then $(H_{\gamma})^{\circ}$ is reductive.* $\square$

Proof. If $I \leq H$ is a subgroup, then $I \backslash H$ is affine if and only if I° is reductive by the main result of [Hab78] or [Ric77], generalizing earlier work of Matsushima in characteristic zero [Mat60]. Combining this with Proposition 17.2.2 we deduce the theorem. $\square$

We point out that if the neutral component of H_{γ} is reductive then $O(\gamma)$ need not be closed (see Exercise 17.2).

The following theorem implies that if our smooth group H is unipotent and X is affine, then every $\gamma \in X(k)$ is relatively semisimple:

Theorem 17.2.10 (Kostant-Rosenlicht) *If H is a smooth unipotent algebraic group and X is affine then every H -orbit in X is closed.*

Proof. This is [Mil17, Theorem 17.64]. $\square$

Example 17.1. The multiplication $H \times H \longrightarrow H$ defines a (right) action of H on itself. For any subgroup $I \leq H$ this action descends to an action of H on $I \backslash H$.

Example 17.2. Let $X = H$ and let H act on X via conjugation. Thus for k -algebras R , the action is given by

$$\begin{aligned} X(R) \times H(R) &\longrightarrow X(R) \\ (\gamma, h) &\longmapsto h^{-1}\gamma h. \end{aligned}$$

We refer to this as the **group case**. In this setting, $O(\gamma)(\bar{k})$ is the $H(\bar{k})$ -conjugacy class of γ and H_γ is the centralizer of γ . Assume that k is perfect and H is reductive. Then an element $\gamma \in X(k)$ is relatively semisimple if and only if it is semisimple in the usual sense of Definition 1.5.4 (see [Ste65, Corollary 6.13]). Moreover, γ is relatively regular if and only if γ is regular in the sense of [Ste65]. This is the reason for the terminology relatively semisimple and relatively regular.

Example 17.3. Let $X = \mathfrak{h}$, where $\mathfrak{h} := \text{Lie } H$, and let H act on X via the adjoint action. Even if one is primarily interested in the group case, it has proven crucial to use this infinitesimal model of conjugation. For example, the proof of the fundamental lemma was reduced to a Lie algebra version by Waldspurger [Wal97, Wal06, Wal08, Wal09a, Wal09b]. It was this version that was proved by Ngô [Ngô10b].

Example 17.4. Let G be an algebraic group over k , let $X = G$, and let $H \leq G \times G$ be a subgroup. Then we have an action

$$\begin{aligned} G(R) \times H(R) &\longrightarrow G(R) \\ (g, (h_\ell, h_r)) &\longmapsto h_\ell^{-1}gh_r. \end{aligned} \tag{17.5}$$

We can recover Example 17.2 by taking H to be G embedded diagonally into $G \times G$.

17.3 Classes and quotients

We maintain the assumptions of the previous section. Thus k is a field and

$$a : X \times H \longrightarrow X$$

is an action of a smooth algebraic group H over k on a separated scheme X of finite type over k .

Definition 17.3.1. A **class** is an element of

$$\Gamma(k) := X(k)/H(k).$$

A **relatively semisimple class** is an element of

$$\Gamma_{\text{ss}}(k) := \{\gamma \in X(k) : \gamma \text{ is relatively semisimple}\} / H(k). \quad (17.6)$$

We use the notation $[\gamma]$ for the class of γ .

From the point of view of algebraic geometry, the notion of a class is too refined. In fact it turns out to be too refined even for some purposes in automorphic representation theory. One needs to consider the coarser notion of a geometric class. This entails some understanding of quotients of affine schemes by reductive group actions, so we digress to discuss this topic.

For the remainder of this section we assume that X is affine. The action of H on X corresponds to a morphism of k -algebras

$$\hat{a} : \mathcal{O}(X) \longrightarrow \mathcal{O}(X) \otimes_k \mathcal{O}(H), \quad (17.7)$$

where $\mathcal{O}(Y)$ is the coordinate ring of the affine scheme Y (see §1.2). In other words, $\hat{a}$ defined in (17.1) is just the map of affine schemes induced by the map $\hat{a}$ of k -algebras. Call an element $r \in \mathcal{O}(X)$ **invariant** if $\hat{a}(r) = r \otimes 1$. The set of invariant elements is a k -subalgebra, and by abuse of notation, we let

$$\mathcal{O}(X)^H \leq \mathcal{O}(X) \quad (17.8)$$

be this subalgebra.

If H° is reductive, we set

$$X/H := \text{Spec}(\mathcal{O}(X)^H). \quad (17.9)$$

It is an affine scheme of finite type over k [MFK94, Theorem 1.1]. This is the **Geometric Invariant Theory quotient** of X by H , or **GIT quotient** for short. The inclusion (17.8) induces a morphism

$$p : X \longrightarrow X/H.$$

Let us explain why X/H deserves to be regarded as a quotient. Let $p_1 : X \times H \rightarrow X$ denote the projection to the first factor. A **categorical quotient** of X by H is a k -scheme Y and a morphism $p : X \rightarrow Y$ such that

$$\begin{array}{ccc} X \times H & \xrightarrow{a} & X \\ p_1 \downarrow & & \downarrow p \\ X & \xrightarrow{p} & Y \end{array} \quad (17.10)$$

commutes and for any morphism $q : X \rightarrow Z$ from X to another k -scheme Z such that

$$\begin{array}{ccc} X \times H & \xrightarrow{a} & X \\ p_1 \downarrow & & \downarrow q \\ X & \xrightarrow{q} & Z \end{array} \quad (17.11)$$

commutes, there is a unique morphism $\chi : Y \rightarrow Z$ such that $q = \chi \circ p$. Concretely, the assertion that (17.11) commutes is a way to make precise the assertion that q is constant on H -orbits. The universal property states that any morphism constant on H -orbits factors through the categorical quotient. As usual with universal properties, the definition immediately implies that the categorical quotient is unique up to unique isomorphism if it exists. We emphasize that in this definition we are allowing Z to be any k -scheme; we do not assume that it is affine. Unless otherwise stated, all of our categorical quotients are understood to be in the category of k -schemes. This distinction is important. Indeed, if H° is not reductive, we may still consider the affine scheme $\mathrm{Spec}(\mathcal{O}(X)^H)$, though it may not be of finite type [Nag59]. Moreover, it is a categorical quotient in the category of affine schemes by Exercise 17.3, but not necessarily a categorical quotient in the category of schemes (see Example 17.5 below).

As the reader probably has guessed, the GIT quotient is a categorical quotient [MFK94, Theorem 1.1]. Thus it is reasonable to call it a quotient. We remark that the quotients $I \backslash H$ are also categorical quotients [Mil17, Definition B.6, Theorem B.37]. The formation of these quotients commutes with base change in the following sense. If k'/k is a field extension then there are canonical identifications

$$(X/H)_{k'} = X_{k'}/H_{k'} \quad \text{and} \quad (I \backslash H)_{k'} = I_{k'} \backslash H_{k'}. \quad (17.12)$$

The first identification is part of [MFK94, Theorem 1.1]. The second identification follows from the construction of the quotient $I \backslash H$ in terms of sheaves given in §1.10.

Example 17.5. Let $N \leq \mathrm{SL}_2$ be the subgroup of strictly upper triangular matrices, so $N(R) := \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in R \}$. Denote by $\mathbb{G}_a^2 - \{0\}$ the open complement of $\{0\}$ in $\mathbb{G}_a^2$. One has a commutative diagram

$$\begin{array}{ccc} N \backslash \mathrm{SL}_2 & \longrightarrow & \mathrm{Spec}(\mathcal{O}(\mathrm{SL}_2)^N) \\ \downarrow \sim & & \downarrow \sim \\ \mathbb{G}_a^2 - \{0\} & \longrightarrow & \mathbb{G}_a^2. \end{array}$$

Here the vertical arrows are isomorphisms induced by the maps

$$\begin{aligned} \mathrm{SL}_2(R) &\longrightarrow R^2 \\ g &\longmapsto (0, 1)g \end{aligned}$$

for k -algebras R , the top horizontal arrow is the map induced by the universal property of categorical quotients (in the category of k -schemes), and the bottom horizontal arrow is the canonical open immersion. In particular the top horizontal arrow is not an isomorphism.

Often in the literature the GIT quotient is denoted by $X // H$ and the stack theoretic quotient is denoted by X/H or $[X/H]$. In this book we will not make use of stack theoretic quotients so there is no danger of confusion. We only consider two types of quotients, namely GIT quotients of affine schemes over fields and quotients defined using representable fppf sheaves (see §1.10). In particular we only use the notation X/H for $\mathrm{Spec}(\mathcal{O}(X)^H)$ in the special case in which X is affine and H° is reductive.

Suppose that $I \leq H$ is a smooth subgroup with I° reductive. Then we can define $I \backslash H$ using fppf sheaves as in §1.10 or as the GIT quotient $\mathrm{Spec}(\mathcal{O}(H)^I)$. Happily these two definitions agree up to unique isomorphism since they are both categorical quotients. Analogous remarks apply in every other case we encounter in which there is a possibility of ambiguity.

A helpful technique for studying X/H is to search for a reductive subgroup $H' \leq H$ that stabilizes a large subscheme of X . In more detail, for k -algebras R and subgroups $H' \leq H$, let

$$X^{H'}(R) := \{x \in X(R) : H'_R \leq H_x\}.$$

This is a closed subscheme of X [Mil17, Theorem 7.1].

We have the following theorem, which is an amalgam of results of Luna and Richardson [MFK94, Appendix 1.D]:

Theorem 17.3.2 *Assume that X is normal and that k has characteristic zero. Let $H' \leq H$ be a subgroup with reductive neutral component. Then for $\gamma \in X^{H'}(k)$, the orbit of γ under H is closed if and only if the orbit of γ under the normalizer $N_H(H')$ is closed. The map*

$$X^{H'}/N_H(H') \longrightarrow X/H \tag{17.13}$$

is finite. If γ is a representative for a generic closed orbit in X , $H' = H_\gamma$, and $X^{H'}$ is irreducible, then the map (17.13) is an isomorphism. $\square$

There is a slightly more precise description of what is meant by a generic closed orbit in [LR79], but yet more precision is needed in practice. When we consider the action of H on itself ($X = H$) via conjugation, the **Chevalley restriction theorem** provides a definitive description of the quotient X/H [Ste65, §6]:

Theorem 17.3.3 *Assume H is reductive and let $T \leq H$ be a maximal torus. Then the inclusion $T \rightarrow H$ induces an isomorphism*

$$T/W(H, T) \xrightarrow{\sim} H/\sim,$$

where $H/\sim$ denotes the quotient by the conjugation action. $\square$

In this case, the stabilizer H_γ of $\gamma \in H(k)$ is called the **centralizer** of γ and is denoted $C_{\gamma,H}$. We refer to Definition 1.7.8 in §1.7 for the definition of the Weyl group $W(H,T)$ of T in H .

We refer to [Ric82] for a generalization to symmetric varieties in the sense of §14.5 and [HS01] for a generalization involving pairs of commuting involutions. Both of these generalizations are relevant to the conjectures of §14.6.

For our purposes we need to understand the points of X/H . We observe that for any $c \in (X/H)(k)$ the fiber $p^{-1}(c)$ of $p : X \rightarrow X/H$ is a closed subscheme of X . We describe X/H in terms of these fibers:

Proposition 17.3.4 *Assume that H° is reductive. There is a bijection*

$$\begin{aligned} \{\text{closed } H\text{-orbits in } X\} &\longrightarrow (X/H)(k) \\ O &\longmapsto p(O(k)). \end{aligned} \quad (17.14)$$

The inverse is given by sending $c \in (X/H)(k)$ to the unique closed H -orbit O in $p^{-1}(c)$.

If we identify the two sets in (17.14) using the given bijection then the map $X(k) \rightarrow (X/H)(k)$ sends an $H(k)$ -orbit $\gamma H(k)$ to the unique closed H -orbit in the Zariski closure of $O(\gamma)$ in X .

Proof. Assume first that $k = \bar{k}$. The morphism $p : X \rightarrow X/H$ is surjective (even universally submersive) [MFK94, Theorem 1.1]. Hence $p^{-1}(c)(k)$ is nonempty. Let $O \subseteq p^{-1}(c)$ be an H -orbit of minimal dimension. We claim that it is closed. Indeed, let $\bar{O}$ be the closure of O . Then $\bar{O} - O$, if nonempty, is a union of H -orbits of dimension strictly smaller than the dimension of O . Thus O is closed. Any two closed H -orbits in X can be separated by an element of $\mathcal{O}(X)^H$ (see Chapter 1.2, Corollary 1.2 and Appendix 1.C, Corollary A.1.3 of [MFK94]). Thus O is the unique closed H_k -orbit in $p^{-1}(c)$. This yields an injection

$$(X/H)(k) \longrightarrow \{\text{closed } H\text{-orbits } O \text{ in } X\} \quad (17.15)$$

given by sending $c \in (X/H)(k)$ to the unique closed H -orbit in $p^{-1}(c)$. The map (17.15) is the inverse of (17.14). This implies that (17.14) is a bijection when $k = \bar{k}$.

We now allow k to be an arbitrary field. We have a commutative diagram

$$\begin{array}{ccc} \{\text{closed } H\text{-orbits } O \text{ in } X\} & \xrightarrow{O \mapsto p(O(k))} & (X/H)(k) \\ \downarrow O \mapsto (O_{\bar{k}})_{\text{red}} & & \downarrow \\ \{\text{closed } H_{\bar{k}}\text{-orbits } O' \text{ in } X_{\bar{k}}\} & \xrightarrow{O' \mapsto p(O'(\bar{k}))} & (X/H)(\bar{k}). \end{array} \quad (17.16)$$

The right vertical arrow is the canonical map, both vertical arrows are injective, and we have already shown that the bottom arrow is a bijection. Thus it suffices to prove that the top horizontal arrow is surjective. Let $c \in (X/H)(k)$ and let O' be the unique closed $H_{\bar{k}}$ -orbit in $p^{-1}(c)_{\bar{k}}$. Since $p^{-1}(c)_{\bar{k}}$ is $\text{Aut}_k(\bar{k})$ -stable it follows that $O'_{\bar{k}}$ is $\text{Aut}_k(\bar{k})$ -stable. Applying Lemma 17.2.7 we deduce that there is an H -orbit O in X such that $(O_{\bar{k}})_{\text{red}} = O'$. Hence the top horizontal arrow is surjective.

We have already obtained the description of the inverse of (17.14) claimed in the proposition. The last assertion follows from the first. $\square$

Example 17.6. Let $X = \text{GL}_2$ and let $H = \text{GL}_2$ act on X by conjugation. For simplicity assume k is perfect. The map

$$\begin{aligned} \text{GL}_2(R) &\longrightarrow R \times R^\times \\ g &\longmapsto (\text{tr } g, \det g) \end{aligned}$$

induces an isomorphism $X/H \xrightarrow{\sim} \mathbb{G}_a \times \mathbb{G}_m$. The relatively semisimple regular classes in $\text{GL}_2(k)$ are conjugacy classes of semisimple matrices with distinct eigenvalues. The matrices

$$\begin{pmatrix} a & b \\ & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & \\ & a \end{pmatrix}$$

have the same image in the GIT quotient. The unique closed orbit in the Zariski closure of the conjugacy class of $\begin{pmatrix} a & b \\ & a \end{pmatrix}$ is $\{\begin{pmatrix} a & \\ & a \end{pmatrix}\}$ (which is its own conjugacy class).

We now drop our running assumption that H has reductive neutral component.

Definition 17.3.5. A **geometric class** is a nonempty set of the form $O(k)$ where $O \subseteq X$ is an H -orbit.

We observe that $\gamma_1, \gamma_2 \in X(k)$ are in the same geometric class if and only if $O(\gamma_1) = O(\gamma_2)$. The condition that γ is relatively semisimple depends only on the geometric class of γ by Exercise 17.17. It therefore makes sense to speak of the set

$$\Gamma_{\text{geo,ss}}(k) \tag{17.17}$$

of **relatively semisimple geometric classes**.

By Proposition 17.3.4, when H has reductive neutral component, one has a bijection

$$\Gamma_{\text{geo,ss}}(k) \xrightarrow{\sim} \text{im}(X(k) \rightarrow (X/H)(k)). \tag{17.18}$$

Still assuming H has reductive neutral component, one has a natural sequence of maps

$$\Gamma_{\text{ss}}(k) \longrightarrow X(k)/H(k) \longrightarrow \Gamma_{\text{geo,ss}}(k). \quad (17.19)$$

If $k = \bar{k}$ then the composite is an isomorphism by Proposition 17.3.4. However, even if $k = \bar{k}$, in general the left map is not surjective and the right map is not injective (see Exercise 17.5). If $k \neq \bar{k}$ then the right map is not in general surjective.

For use in §17.8 we record two lemmas.

Lemma 17.3.6 *If $\gamma \in X(k)$ is relatively semisimple with respect to H° then it is relatively semisimple with respect to H .*

Proof. We can assume $k = \bar{k}$. Let $O_H(\gamma)$ be the H -orbit of γ in X and let $O_{H^\circ}(\gamma)$ be the H° -orbit of γ in X . Then using $|\cdot|$ to denote underlying topological spaces,

$$|O_H(\gamma)| = \bigcup_{h \in H^\circ(k) \setminus H(k)} |O_{H^\circ}(\gamma)h|.$$

Therefore $|O_H(\gamma)|$ is a finite union of closed subspaces of $|X|$ and hence is closed. $\square$

Assume that

$$H^\circ := H_r \times H_u \quad (17.20)$$

with H_r reductive and H_u unipotent.

Lemma 17.3.7 *An element $\gamma \in X(k)$ is relatively semisimple with respect to the action of H if it is relatively semisimple with respect to the action of H_r .*

Proof. We can assume $k = \bar{k}$. By Lemma 17.3.6, it suffices to treat the case $H = H^\circ$. Throughout this proof, γ is relatively semisimple with respect to H_r . We first deal with the case where γ is additionally relatively regular with respect to the action of H_r .

Identify $X(k)$ with the set of closed points in $|X|$. The function on $|X|$ that assigns to a point the dimension of its stabilizer under the action of H_r is upper semicontinuous [MFK94, §0.3]. It follows that there is a unique open subset Z of $|X|$ such that $Z \cap X(k)$ is the set of elements of $X(k)$ that are relatively regular with respect to H_r . Let

$$X^{\text{reg}} \subseteq X$$

be the open subscheme of X with underlying topological space Z given the reduced induced scheme structure. Thus $X^{\text{reg}}(k)$ is the set of elements of $X(k)$ that are relatively regular with respect to H_r . It is easy to see that X^{reg} is an H -invariant subscheme of X .

Assume for the moment that $\gamma \in X^{\text{reg}}(k)$. Consider the GIT quotient

$$p : X \longrightarrow X/H_r.$$

Using the universal property of categorical quotients and the fact that H_u and H_r commute, we see that the action of H_u on X induces an action of H_u on X/H_r such that p is equivariant.

The fiber $p^{-1}(p(\gamma))$ is a union of H_r -orbits. There is a unique closed H_r -orbit in $p^{-1}(p(\gamma))$ by Proposition 17.3.4, namely the H_r -orbit $O_{H_r}(\gamma)$ of γ . By the proof of loc. cit. $O_{H_r}(\gamma)$ is also the H_r -orbit of minimal dimension in $p^{-1}(p(\gamma))$. Since γ is relatively regular with respect to H_r we deduce that $p^{-1}(p(\gamma)) = O_{H_r}(\gamma)$.

Let $O_{H_u}(p(\gamma))$ denote the H_u -orbit of $p(\gamma)$. If $x \in p^{-1}(O_{H_u}(p(\gamma)))(k)$ then $p(xh) = p(x)h = p(\gamma)$ for some $h \in H_u(k)$, which implies that

$$xh \in p^{-1}(p(\gamma))(k) = O_{H_r}(\gamma)(k).$$

Hence the H -orbit of γ is $p^{-1}(O_{H_u}(p(\gamma)))$. On the other hand, by Theorem 17.2.10, $O_{H_u}(p(\gamma)) \subseteq X/H_r$ is closed. Hence the H -orbit $p^{-1}(O_{H_u}(p(\gamma))) \subseteq X$ of γ is closed.

Thus we have proven the proposition for $\gamma \in X^{\text{reg}}(k)$, in other words, when γ is relatively regular with respect to H_r . We now reduce the general case to this one. Let $X^0 := X$, $X^1 := X - X^{\text{reg}}$, and for $i \geq 2$, let

$$X^i := X^{i-1} - (X^{i-1})^{\text{reg}}.$$

These subschemes are all preserved by H . Thus we have a sequence of closed subschemes

$$X = X^0 \supseteq X^1 \supseteq \dots.$$

Since X is Noetherian, this sequence stabilizes. It follows that there exists an n such that $(X^n)^{\text{reg}} = X^n$. Thus our sequence is

$$X = X^0 \supsetneq X^1 \supsetneq \dots \supsetneq X^n \supsetneq \emptyset$$

where for each i , one has $X^{i-1} - X^i = (X^{i-1})^{\text{reg}}$. Let i be the largest index such that $\gamma \in X^i(k)$. Then γ is relatively regular with respect to the action of H_r on X^i . Since $O_{H_r}(\gamma)$ is closed in X it is also closed in X^i , which is to say that γ is relatively semisimple with respect to the action of H_r on X^i . Thus we have reduced to the case where γ is relatively regular with respect to H_r , as desired. $\square$

17.4 Local geometric classes

We continue to assume that k is a field. Suppose that the stabilizer H_γ of $\gamma \in X(k)$ is smooth (we do not assume that H has reductive neutral component). We have an injection

$$H_\gamma(k) \backslash H(k) \longrightarrow (H_\gamma \backslash H)(k)$$

and we can identify the set on the right with $O(\gamma)(k)$ by Proposition 17.2.2. We now explain how to describe $H_\gamma(k) \backslash H(k)$ and its complement within $(H_\gamma \backslash H)(k)$ using Galois cohomology. Our primary reference is [Ser02, §I.5].

For a smooth affine algebraic group H over k , define $Z^1(k, H)$, the set of **1-cocycles** of Gal_k in $H(k^{\text{sep}})$, to be the set of continuous maps

$$c : \text{Gal}_k \longrightarrow H(k^{\text{sep}})$$

that satisfy the **cocycle condition**

$$c(\sigma_1 \sigma_2) = c(\sigma_1) \sigma_1(c(\sigma_2)) \quad (17.21)$$

for all $\sigma_1, \sigma_2 \in \text{Gal}_k$. Here we give $H(k^{\text{sep}})$ the discrete topology. Two 1-cocycles c and c' are called **cohomologous** if there is an $h \in H(k^{\text{sep}})$ such that $c'(\sigma) = h^{-1}c(\sigma)\sigma(h)$ for all $\sigma \in \text{Gal}_k$. This is an equivalence relation and the set of equivalence classes is denoted by

$$H^1(k, H). \quad (17.22)$$

It is the **first Galois cohomology** set of Gal_k in H (or more precisely in $H(k^{\text{sep}})$). This is not a group unless H is commutative, but it does come with a distinguished element, called the **neutral element**. It is the class of the 1-cocycle sending every element of Gal_k to 1. These cohomology sets are functorial in H in the obvious sense. We also mention that, though this is not at all obvious from the definition, the sets $H^1(k, H)$ can be computed in many interesting cases. We refer to [Ser02] for more details.

When H is commutative, the set $H^1(k, H)$ has a group structure, and using this fact, Borovoi defined a group structure on $H^1(k, H)$ for all reductive H when k is of characteristic zero [Bor98b]. Though we will not use Borovoi's work, we point out that it can be used to give an elegant treatment of the stabilization of the trace formula [Lab99]. In fact, Borovoi's definition is motivated by constructions occurring in Kottwitz's work on the stabilization of the trace formula [Kot84, Kot86b].

If $I \rightarrow H$ is a morphism of smooth group schemes over k , let

$$\mathcal{D}(k, I, H) := \ker(H^1(k, I) \longrightarrow H^1(k, H)). \quad (17.23)$$

Here the kernel is defined to be all classes that are sent to the neutral element of $H^1(k, H)$.

Now suppose that $I \leq H$ is a smooth subgroup scheme. We leave the following lemma as Exercise 17.10:

Lemma 17.4.1 *The map*

$$\text{cl} : (I \backslash H)(k) \longrightarrow \mathcal{D}(k, I, H)$$

$$I(k^{\text{sep}})h \longmapsto (\sigma \mapsto h\sigma(h^{-1}))$$

defines a bijection

$$(I \backslash H)(k) / H(k) \xrightarrow{\sim} \mathcal{D}(k, I, H)$$

between the $H(k)$ -orbits in $(I \backslash H)(k)$ and $\mathcal{D}(k, I, H)$. $\square$

In other words, the set of classes in the geometric class of γ is in bijection with $\mathcal{D}(k, I, H)$. This observation is the germ of the theory of endoscopy. The groundwork of the theory was laid by Kottwitz, Langlands, and Shelstad [Kot84, Kot86b, Lan83, LS87, She82] and more recent useful references include [KS99, Lab99].

We refer to the map cl in the lemma as the **class map**. The lemma implies that there is an exact sequence of pointed sets

$$1 \longrightarrow I(k) \backslash H(k) \longrightarrow (I \backslash H)(k) \xrightarrow{\text{cl}} \mathcal{D}(k, I, H) \longrightarrow 1. \quad (17.24)$$

The fact that (17.24) is exact only gives us a description of the fiber of cl over the neutral element. It is natural to ask for a description of the other fibers. By Proposition 1.10.7, any element of $(I \backslash H)(k)$ is a coset $I(k^{\text{sep}})h$ for some $h \in H(k^{\text{sep}})$ satisfying $h\sigma(h^{-1}) \in I(k^{\text{sep}})$ for all $\sigma \in \text{Gal}_k$. Let I_h be the stabilizer of $I(k^{\text{sep}})h$ under the action of H . It is independent of the choice of $h \in H(k^{\text{sep}})$ representing the coset $I(k^{\text{sep}})h$. Then we have

$$\begin{aligned} (I \backslash H)(k) &= \coprod_{I(k^{\text{sep}})h} I(k^{\text{sep}})hH(k) \\ &= \coprod_{I(k^{\text{sep}})h} \text{im}(I_h(k) \backslash H(k) \longrightarrow (I \backslash H)(k)), \end{aligned} \quad (17.25)$$

where the disjoint union is over a set of representatives for the $H(k)$ -orbits in $(I \backslash H)(k)$ and the implicit maps are the injections given by

$$\begin{aligned} I_h(k) \backslash H(k) &\longrightarrow (I \backslash H)(k) \\ I_h(k)x &\longmapsto I(k^{\text{sep}})hx. \end{aligned} \quad (17.26)$$

We note that

$$\text{im}(I_h(k) \backslash H(k) \longrightarrow (I \backslash H)(k)) = \text{cl}^{-1}([\sigma \mapsto h\sigma(h^{-1})]) \quad (17.27)$$

where $[\sigma \mapsto h\sigma(h^{-1})]$ is the class of the 1-cocycle $\sigma \mapsto h\sigma(h^{-1})$.

The groups I_h are not necessarily isomorphic as h varies, but they are related in an important manner. Two smooth algebraic groups Q and $\tilde{Q}$ over k are said to be **forms** of each other if there is an isomorphism

$$\varphi : Q_{k^{\text{sep}}} \xrightarrow{\sim} \tilde{Q}_{k^{\text{sep}}}. \quad (17.28)$$

The concept of a form arises naturally in the classification of algebraic groups. One can proceed in many circumstances by classifying groups of a certain type over k^{sep} and then classifying all of the groups over k that base change to a given group over k^{sep} . A pair $(\tilde{Q}, \varphi)$ consisting of an algebraic group $\tilde{Q}$ over k and an isomorphism φ as in (17.28) is called an **inner form** of Q if for all $\sigma \in \text{Gal}_k$, the automorphism $\varphi^{-1} \circ \sigma \circ \varphi \circ \sigma^{-1}$ of $Q_{k^{\text{sep}}}$ is an inner automorphism. Usually the isomorphism φ is omitted from notation. It is easy to check that the I_h are all inner forms of I (see Exercise 17.12).

Now suppose that k is a local field. In keeping with our usual notation for local and global fields, we let $k = F$. We record the following useful result (see [Poo17, Proposition 3.5.73]):

Proposition 17.4.2 *If $X \rightarrow Y$ is a smooth map of separated schemes of finite type over F , then $X(F) \rightarrow Y(F)$ is open.* $\square$

Using this proposition, we prove the following lemma:

Lemma 17.4.3 *Let $I \leq H$ be a smooth algebraic subgroup. The map*

$$\text{cl} : (I \backslash H)(F) \longrightarrow \mathcal{D}(F, I, H)$$

is continuous if we give $\mathcal{D}(F, I, H)$ the discrete topology. In particular, the image of $I(F) \backslash H(F)$ in $(I \backslash H)(F)$ is closed.

Proof. The last assertion follows from the first and Lemma 17.4.1. To prove the first assertion, it suffices to show that the preimage under cl of any point in $\mathcal{D}(F, I, H)$ is open. By (17.27), this is equivalent to the assertion that the maps

$$I_h(F) \backslash H(F) \longrightarrow (I \backslash H)(F) \tag{17.29}$$

of (17.26) have open image. Here $h \in H(F^{\text{sep}})$ satisfies $h\sigma(h^{-1}) \in I(F^{\text{sep}})$ for $\sigma \in \text{Gal}_F$.

Let $b_h = I(F^{\text{sep}})h \in (I \backslash H)(F)$. Let $a : I \backslash H \times H \rightarrow I \backslash H$ be the canonical action. Then the image of (17.29) is the image of

$$a(b_h, \cdot) : H(F) \longrightarrow (I \backslash H)(F).$$

This map is the F -points of the map $a(b_h, \cdot) : H \rightarrow I \backslash H$, and hence by Proposition 17.4.2 it suffices to check that $a(b_h, \cdot)$ is smooth. But I_h is an inner form of the smooth group I , hence it is smooth. Thus we can apply Proposition 17.2.2 to conclude that $a(b_h, \cdot)$ is smooth. $\square$

The following theorem gives conditions under which $\mathcal{D}(F, I, H)$ is finite:

Theorem 17.4.4 *Assume I is a smooth algebraic group over the local field F . If I is reductive or F has characteristic zero, then $H^1(F, I)$ is finite.*

Proof. If F has characteristic zero, this is contained in [Ser02, §III.4.2 and 4.3]. In the reductive case, see [Ser02, §III.4.3, Remarks]. $\square$

17.5 Local relative orbital integrals

Let F be a local field and let $X \times H \rightarrow X$ be an action of a smooth algebraic group H over F on a separated scheme X of finite type over F . We define local relative orbital integrals attached to such an action in this section.

Assume we are given a quasi-character $\chi : H(F) \rightarrow \mathbb{C}^\times$.

Definition 17.5.1. An element $\gamma \in X(F)$ is χ -**relevant** if χ is trivial on $H_\gamma(F)$.

If χ is understood, we often omit it from notation and speak simply of relevant elements. We require the notion of relevance to define local orbital integrals, and only relevant γ will contribute to the global trace formula (see Theorem 17.8.7 below). If χ is trivial then all elements of $X(F)$ are relevant.

Assume X is smooth. When F is Archimedean, smooth functions on the real manifold $X(F)$ are defined. If F is non-Archimedean, we say a function $f : X(F) \rightarrow \mathbb{C}$ is smooth if it is locally constant. In either case we denote by $C^\infty(X(F))$ the space of smooth functions on $X(F)$ and by $C_c^\infty(X(F)) \leq C^\infty(X(F))$ the subspace of compactly supported smooth functions.

For arbitrary (not necessarily smooth) X we denote by $C_c(X(F))$ the space of compactly supported continuous functions on $X(F)$. We point out that when F is non-Archimedean a function $f : X(F) \rightarrow \mathbb{C}$ is continuous if and only if it is locally constant.

We require one more assumption on $\gamma \in X(F)$ before defining local relative orbital integrals.

Definition 17.5.2. We say $\gamma \in X(F)$ is **relatively unimodular** if

$$\delta_H|_{H_\gamma} = \delta_{H_\gamma}.$$

See §3.2 for definition of the modular character δ_H . If H° is reductive and X is affine then relatively semisimple γ are unimodular by Theorem 17.2.9.

Now suppose we are given $f \in C_c(X(F))$ and a relevant relatively semisimple and relatively unimodular $\gamma \in X(F)$. We fix right Haar measures $d_r h$ on $H(F)$ and $d_r h_\gamma$ on $H_\gamma(F)$. Because γ is relatively unimodular, the quotient measure $\frac{d_r h}{d_r h_\gamma}$ is defined by Theorem 3.2.3. We can then form the **local relative orbital integral**

$$\mathrm{RO}_\gamma(f) := \mathrm{RO}_\gamma^\chi(f) = \int_{H_\gamma(F) \backslash H(F)} f(\gamma h) \chi^{-1}(h) \frac{d_r h}{d_r h_\gamma}. \quad (17.30)$$

Note that we have used the fact that γ is χ -relevant to define this integral. The integral depends on the choice of measures, but traditionally they are not encoded in the notation. In some settings it may be necessary to replace H_γ by some subgroup of H_γ containing $(H_\gamma)^\circ$.

Theorem 17.5.3 *If $\gamma \in X(F)$ is relevant, relatively unimodular and relatively semisimple and H_γ is smooth then the integral defining $\mathrm{RO}_\gamma(f)$ is absolutely convergent.*

Proof. The measure $\frac{d_r h}{d_r h_\gamma}$ is a Radon measure on $H_\gamma(F) \backslash H(F)$. Thus to prove the integral is absolutely convergent, it is enough to show that

$$\begin{aligned} H_\gamma(F) \backslash H(F) &\longrightarrow X(F) \\ H_\gamma(F)h &\longmapsto \gamma h \end{aligned} \tag{17.31}$$

induces a pull-back map

$$C_c(X(F)) \longrightarrow C_c(H_\gamma(F) \backslash H(F)).$$

For this it suffices to show that (17.31) is proper. We will in fact prove (17.31) is a closed embedding.

The orbit $O(\gamma)$ of γ is closed in X . Hence $O(\gamma)(F)$ is closed in $X(F)$ by Exercise 2.3. It therefore suffices to treat the case where $X = O(\gamma) = H_\gamma \backslash H$. In this case, we invoke Lemma 17.4.3 to see that the image of $H_\gamma(F) \backslash H(F)$ in $(H_\gamma \backslash H)(F)$ is closed. $\square$

17.6 Torsors

In this section we discuss torsors, which play a key role in §17.7. It is convenient to return to the setting of §17.2. Thus H is a smooth affine group scheme over a commutative ring k acting (on the right) on a separated scheme X of finite type over k :

$$a : X \times H \longrightarrow X.$$

For example, one could take $X = H$ with the natural right H -action. This is known as the **trivial H -torsor**. This example can be profitably generalized as we now explain.

A scheme Y over k is said to be **fppf** over k if it is faithfully flat and locally of finite presentation over k . This notion was discussed in the affine case in §1.10.

Definition 17.6.1. The scheme X is an **H -torsor** if it is fppf over k and the map

$$1_X \times a : X \times H \longrightarrow X \times X$$

is an isomorphism. Here $1_X : X \rightarrow X$ is the identity map.

An equivalent (and more intuitive) way to define an H -torsor is to say that it is a scheme fppf over k equipped with an action of H that becomes isomorphic to the trivial H -torsor after an fppf base extension. For more details, we refer

the reader to [Poo17, §6.5.1]. The set of all isomorphism classes of H -torsors is denoted by

$$H_{\text{fppf}}^1(k, H) = H_{\text{fppf}}^1(\text{Spec}(k), H). \quad (17.32)$$

This is the **first fppf cohomology** set of H . It is a pointed set with the class of the trivial torsor as its neutral element. An H -torsor X is isomorphic to the trivial torsor if and only if $X(k) \neq \emptyset$ (see [Poo17, Proposition 6.5.3]). Since we have assumed H is smooth, H -torsors will in fact be smooth over $\text{Spec}(k)$ [Poo17, Remark 6.5.2].

We now discuss functoriality of these fppf cohomology sets. Assume that k' is a k -algebra. Then one has a natural map

$$H_{\text{fppf}}^1(k, H) \longrightarrow H_{\text{fppf}}^1(k', H_{k'}) \quad (17.33)$$

given by sending an H -torsor Y to its base extension $Y_{k'}$.

To understand functoriality in H we require contracted products. Suppose that I and H are smooth over k and that $\phi : I \rightarrow H$ is a morphism of group schemes over k . Let Y be an I -torsor. One has an action of $I \times H$ on $Y \times H$ given on points in a k -algebra R by

$$\begin{aligned} Y(R) \times H(R) \times I(R) \times H(R) &\longrightarrow Y(R) \times H(R) \\ (y, h, i, h') &\longmapsto (yi, \phi(i)^{-1}hh'). \end{aligned} \quad (17.34)$$

The actions of I and H commute. We now wish to take the quotient under the action of I . For this we use some terminology from §1.10. Certainly we have a presheaf, that is, a set-valued functor, on the category of k -algebras $\mathbf{Alg}_k$ given on objects by

$$R \longmapsto (Y(R) \times H(R))/I(R).$$

It turns out that the sheafification of this presheaf is representable by a k -scheme $(Y \times H)/I$ [Poo17, §6.5.6.3].

Definition 17.6.2. The **contracted product**

$$Y \wedge^I H := (Y \times H)/I$$

is the quotient of $Y \times H$ by the action of I above.

Since $Y \wedge^I H$ is defined as the sheafification of a presheaf quotient, it is a categorical quotient of $Y \times H$ by I . Moreover, for any k -algebra k' there is a canonical identification

$$(Y \wedge^I H)_{k'} = Y_{k'} \wedge^{I_{k'}} H_{k'}. \quad (17.35)$$

The contracted product $Y \wedge^I H$ is an H -torsor, where the action is inherited from the action of H in (17.34). It provides us with a morphism of pointed

sets

$$H_{\text{fppf}}^1(k, I) \longrightarrow H_{\text{fppf}}^1(k, H) \quad (17.36)$$

sending the isomorphism class of an I -torsor Y to the isomorphism class of $Y \wedge^I H$.

Assume momentarily that k is a field. Then there is a canonical bijection

$$H_{\text{fppf}}^1(k, H) \xrightarrow{\sim} H^1(k, H). \quad (17.37)$$

It is constructed as follows. Suppose we are given an H -torsor X . Then, since X is smooth, we can choose $x \in X(k^{\text{sep}})$ by Lemma 1.10.6. Since the map

$$1_X \times a : X \times H \longrightarrow X \times X \quad (17.38)$$

is an isomorphism, we see that the map $a(x, \cdot) : H(k^{\text{sep}}) \rightarrow X(k^{\text{sep}})$ is a bijection. In particular, for each $\sigma \in \text{Gal}_k$, there is a $c(\sigma) \in H(k^{\text{sep}})$ such that $\sigma(x) = x \cdot c(\sigma)$. One checks that $c(\sigma)$ is a 1-cocycle and that replacing X by an equivalent H -torsor X' and x by $x' \in X'(k^{\text{sep}})$ yields a cohomologous 1-cocycle. The map (17.37) sends the equivalence class of X to the class of c . We omit the construction of the inverse map (see [Poo17, §5.12.4]). If c is a 1-cocycle in $H(k^{\text{sep}})$ constructed from X and a choice of $x \in X(k^{\text{sep}})$ as above we say that c is a **1-cocycle in $H(k^{\text{sep}})$ attached to X** . Using (17.37), we occasionally allow ourselves to identify $H_{\text{fppf}}^1(k, H)$ and $H^1(k, H)$ when k is a field.

Lemma 17.6.3 *Assume k is a field and that $\phi : I \rightarrow H$ is a morphism of smooth group schemes over k . One has a commutative diagram*

$$\begin{array}{ccc} H_{\text{fppf}}^1(k, I) & \longrightarrow & H_{\text{fppf}}^1(k, H) \\ \sim \downarrow & & \downarrow \sim \\ H^1(k, I) & \longrightarrow & H^1(k, H) \end{array}$$

where the vertical arrows are given by the bijection (17.37), the top arrow is given by (17.36), and the bottom arrow is given by sending a 1-cocycle c to $\phi \circ c$.

Proof. The universal property of categorical quotients provides a Gal_k -equivariant map

$$(Y(k^{\text{sep}}) \times H(k^{\text{sep}}))/I(k^{\text{sep}}) \longrightarrow (Y \wedge^I H)(k^{\text{sep}}). \quad (17.39)$$

We claim that this map is bijective.

To see this we employ the useful technique of reduction to the trivial torsor. Choose $y \in Y(k^{\text{sep}})$ using Lemma 1.10.6. There is then an isomorphism $(I \times H)_{k^{\text{sep}}} \xrightarrow{\sim} (Y \times H)_{k^{\text{sep}}}$ given on points in a k^{sep} -algebra R by

$$I(R) \times H(R) \xrightarrow{\sim} Y(R) \times H(R)$$

$$(i, h) \longmapsto (yi, \phi(i)^{-1}h).$$

This intertwines the right action of $I_{k^{\text{sep}}}$ on $(I \times H)_{k^{\text{sep}}}$ induced by its right action on $I_{k^{\text{sep}}}$ with the right action of $I_{k^{\text{sep}}}$ on $(Y \times H)_{k^{\text{sep}}}$. Using the universal property of categorical quotients we deduce that there is a commutative diagram

$$\begin{array}{ccc} I_{k^{\text{sep}}} \times H_{k^{\text{sep}}} & \xrightarrow{\sim} & Y_{k^{\text{sep}}} \times H_{k^{\text{sep}}} \\ \downarrow & & \downarrow \\ (I_{k^{\text{sep}}} \times H_{k^{\text{sep}}})/I_{k^{\text{sep}}} & \xrightarrow{\sim} & Y_{k^{\text{sep}}} \wedge^{I_{k^{\text{sep}}}} H_{k^{\text{sep}}} \end{array}$$

where the horizontal arrows are the quotient maps. The fact that (17.39) is bijective now follows from (17.35) and Proposition 1.10.7 applied to

$$(I_{k^{\text{sep}}} \times H_{k^{\text{sep}}})/I_{k^{\text{sep}}}.$$

Let $y \in Y(k^{\text{sep}})$ and for each $\sigma \in \text{Gal}_k$, let $c(\sigma) \in I(k^{\text{sep}})$ be the element such that $\sigma(y) = yc(\sigma)$. Thus c is a 1-cocycle in $I(k^{\text{sep}})$ attached to Y . Now $(y \times 1)I(k^{\text{sep}}) \in (Y \wedge^I H)(k^{\text{sep}})$ and

$$\sigma(y \times 1)I(k^{\text{sep}}) = (yc(\sigma) \times 1)I(k^{\text{sep}}) = (y \times \phi(c(\sigma)))I(k^{\text{sep}}).$$

Thus $\phi \circ c$ is 1-cocycle in $H(k^{\text{sep}})$ attached to $Y \wedge^I H$. □

In the next section we will make use of the following finiteness result:

Theorem 17.6.4 *Assume that $\mathcal{O}$ is a Henselian local ring with finite residue field κ and H is a smooth affine group scheme of finite type over $\mathcal{O}$. Then the base change map*

$$H_{\text{fppf}}^1(\mathcal{O}, H) \xrightarrow{\sim} H_{\text{fppf}}^1(\kappa, H_\kappa) = H^1(\kappa, H_\kappa)$$

is a bijection. In particular $H_{\text{fppf}}^1(\mathcal{O}, H)$ is finite.

Proof. The fact that the base change map is an isomorphism is implied by [C22, Proposition 6.1.1]. The set $H^1(\kappa, H_\kappa)$ is finite by [Ser02, §III.4.3]. □

Corollary 17.6.5 *Assume that $\mathcal{O}$ is a Henselian local ring with finite residue field κ and H is a smooth affine group scheme of finite type over $\mathcal{O}$ with connected special fiber. Then*

$$H_{\text{fppf}}^1(\mathcal{O}, H) \xrightarrow{\sim} H_{\text{fppf}}^1(\kappa, H_\kappa) = H^1(\kappa, H_\kappa)$$

is trivial.

Proof. In view of Theorem 17.6.4 this is a consequence of a theorem of Lang [Poo17, Theorem 5.12.19(a)]. □

17.7 Adelic classes

Let F be a global field and let H be a smooth algebraic group over F acting on a separated scheme X of finite type over F on the right. Thus we are in the setting of §17.2 in the special case in which $F = k$. We now develop topological results on $H(\mathbb{A}_F)$ -classes in $X(\mathbb{A}_F)$ that are necessary to study global relative orbital integrals (see Proposition 17.8.2). Our exposition is based on the treatment in [GH15] and generalizes some of the results of op. cit.

Theorem 17.7.1 *Suppose that γ is relatively semisimple and H_γ is smooth. Then the map*

$$H_\gamma(\mathbb{A}_F) \backslash H(\mathbb{A}_F) \longrightarrow X(\mathbb{A}_F)$$

is proper.

For γ as in the statement of the theorem, one has that $O(\gamma)(\mathbb{A}_F)$ is closed in $X(\mathbb{A}_F)$ by Theorem 2.4.14. It is therefore no loss of generality to assume that $X = O(\gamma)$. Thus we consider $X = I \backslash H$ where $I \leq H$ is a smooth subgroup of H . Recall that for any finite set S of places of F including the infinite places we set

$$\widehat{\mathcal{O}}_F^S = \prod_{v \notin S} \mathcal{O}_{F_v} \quad \text{and} \quad \mathcal{O}_F^S := F \cap \widehat{\mathcal{O}}_F^S.$$

Thus $\mathcal{O}_F^S < F$ is the set of elements of F that are integral outside of the finite places in S .

We will make use of the notion of a model of an affine scheme; for our conventions we refer to §2.4. For a sufficiently large set S of places of F , we can choose a model $\mathcal{H}$ of H over $\mathcal{O}_F^S$ and a separated scheme $\mathcal{X}$ of finite type over $\mathcal{O}_F^S$ with $\mathcal{X}_F = X := I \backslash H$ such that there is a map

$$\mathcal{H} \longrightarrow \mathcal{X}$$

whose generic fiber is the canonical map $H \rightarrow X$ [Poo17, Theorem 3.2.1]. We let $\mathcal{I}$ be the schematic closure of I in $\mathcal{H}$. By loc. cit., upon enlarging S if necessary we can assume that $\mathcal{H}$ and $\mathcal{I}$ are smooth over $\mathcal{O}_F^S$.

The map $\widehat{\mathcal{O}}_F^S \rightarrow \mathbb{A}_F^S$ induces an open topological embedding

$$\mathcal{X}(\widehat{\mathcal{O}}_F^S) \longrightarrow \mathcal{X}(\mathbb{A}_F^S) = X(\mathbb{A}_F^S) \tag{17.40}$$

by [Con12b, §4] (or Exercise 2.2 if X is affine). Similarly by Lemma 2.4.13 for every finite place v one has open topological embeddings

$$\mathcal{X}(\mathcal{O}_{F_v}) \longrightarrow \mathcal{X}(F_v) = X(F_v). \tag{17.41}$$

By Proposition 2.4.10 we have a homeomorphism of topological spaces

$$X(\mathbb{A}_F) \cong \prod'_v X(F_v), \quad (17.42)$$

where the restricted direct product is taken with respect to the open sets $\mathcal{X}(\mathcal{O}_{F_v})$. This discussion is also valid with X and $\mathcal{X}$ replaced by H (resp. I) and $\mathcal{H}$ (resp. $\mathcal{I}$). In fact we discussed the algebraic group case in §2.3. Thus there are isomorphisms of topological groups

$$H(\mathbb{A}_F) \cong \prod'_v H(F_v) \quad \text{and} \quad I(\mathbb{A}_F) \cong \prod'_v I(F_v), \quad (17.43)$$

where the restricted direct product is taken with respect to the subgroups $\mathcal{H}(\mathcal{O}_{F_v})$ (resp. $\mathcal{I}(\mathcal{O}_{F_v})$) for all finite places v . These isomorphisms are compatible with the group actions $X(F_v) \times H(F_v) \rightarrow X(F_v)$ and $X(\mathbb{A}_F) \times H(\mathbb{A}_F) \rightarrow X(\mathbb{A}_F)$ because the generic fiber of $\mathcal{H} \rightarrow \mathcal{X}$ is $H \rightarrow X$. It is tempting to try to define the quotient $\mathcal{I} \backslash \mathcal{H}$ and relate it to $\mathcal{X}$, but we do not wish to discuss representability of quotients over Dedekind rings. Fortunately, coming to grips with this issue is unnecessary for our purposes.

Lemma 17.7.2 *Assume that I is connected. For a large enough finite set $S' \supseteq S$ of places of F the inverse image of $\mathcal{X}(\hat{\mathcal{O}}_F^{S'})$ in $H(\mathbb{A}_F^{S'})$ is equal to $I(\mathbb{A}_F^{S'})\mathcal{H}(\hat{\mathcal{O}}_F^{S'})$ and the morphism $\mathcal{H} \rightarrow \mathcal{X}$ induces a bijection*

$$\mathcal{I}(\hat{\mathcal{O}}_F^{S'}) \backslash \mathcal{H}(\hat{\mathcal{O}}_F^{S'}) \xrightarrow{\sim} \mathcal{X}(\hat{\mathcal{O}}_F^{S'}).$$

Proof. Since I is connected and the map $H \rightarrow X$ is smooth by Proposition 17.2.2, the map

$$H(\mathbb{A}_F) \longrightarrow X(\mathbb{A}_F)$$

is open by part (e) of Theorem 2.4.14. For any finite set $S' \supseteq S$ of places of F the subset $\mathcal{H}(\hat{\mathcal{O}}_F^{S'}) \leq H(\mathbb{A}_F^{S'})$ is open and compact and so its image is open and compact in $X(\mathbb{A}_F^{S'})$. Upon enlarging S' if necessary we see that this image must be $\mathcal{X}(\hat{\mathcal{O}}_F^{S'})$ by definition of the restricted direct topology on $X(\mathbb{A}_F)$ (see Exercise 17.15).

Let $v \notin S'$ be a place of F . Two elements $h, h' \in H(F_v)$ have the same image in $X(F_v)$ if and only if $h \in I(F_v)h'$. Applying the conclusion of the previous paragraph, an element $h \in H(F_v)$ is in the inverse image of $\mathcal{X}(\mathcal{O}_{F_v})$ if and only if there is an $h' \in \mathcal{H}(\mathcal{O}_{F_v})$ such that $h \in I(F_v)h'$. Thus we deduce the first assertion of the lemma.

Similarly $h, h' \in \mathcal{H}(\mathcal{O}_{F_v})$ have the same image in $\mathcal{X}(\mathcal{O}_{F_v})$ if and only if $hh'^{-1} \in I(F_v)$. Since $\mathcal{H}(\mathcal{O}_{F_v}) \cap I(F_v) = \mathcal{I}(\mathcal{O}_{F_v})$ this occurs if and only if $hh'^{-1} \in \mathcal{I}(\mathcal{O}_{F_v})$. This implies the second assertion of the lemma. $\square$

Proof of Theorem 17.7.1: As mentioned below the statement of Theorem 17.7.1, we can assume that $X = O(\gamma) = H_\gamma \backslash H$. Since the map

$$H_\gamma^\circ \backslash H \longrightarrow H_\gamma \backslash H$$

is proper by Lemma 17.2.4, the map

$$(H_\gamma^\circ \backslash H)(\mathbb{A}_F) \longrightarrow (H_\gamma \backslash H)(\mathbb{A}_F)$$

is proper by part (d) of Theorem 2.4.14.

Thus we are reduced to showing that for any connected smooth subgroup $I \leq H$, the map

$$I(\mathbb{A}_F) \backslash H(\mathbb{A}_F) \longrightarrow (I \backslash H)(\mathbb{A}_F)$$

is proper. To keep notation consistent with Lemma 17.7.2, we write $X = I \backslash H$ and let $\mathcal{X}$ and $\mathcal{H}$ be defined as above that lemma.

Let $\Omega \subseteq X(\mathbb{A}_F)$ be a compact set. If we choose a sufficiently large finite set $S' \supseteq S$ of places of F then Ω is contained in $\Omega_{S'} \times \mathcal{X}(\widehat{\mathcal{O}}_F^{S'})$ where $\Omega_{S'} \subseteq X(F_{S'})$ is compact. The inverse image of $\Omega_{S'}$ in $I(F_{S'}) \backslash H(F_{S'})$ is compact by Lemma 17.4.3. Upon enlarging S' if necessary, Lemma 17.7.2 implies that the inverse image of $\mathcal{X}(\widehat{\mathcal{O}}_F^{S'})$ in $I(\mathbb{A}_F^{S'}) \backslash H(\mathbb{A}_F^{S'})$ is the compact subset $I(\mathbb{A}_F^{S'})\mathcal{H}(\widehat{\mathcal{O}}_F^{S'})$. Hence the inverse image of $\Omega_{S'} \times \mathcal{X}(\widehat{\mathcal{O}}_F^{S'})$ in $I(\mathbb{A}_F) \backslash H(\mathbb{A}_F)$ is compact. This implies that the inverse image of Ω in $I(\mathbb{A}_F) \backslash H(\mathbb{A}_F)$ is the intersection of a closed set and a compact set. In particular it is compact. $\square$

We observe the following:

Lemma 17.7.3 *If Y is an affine scheme of finite type over F then $Y(F)$ is discrete and closed in $Y(\mathbb{A}_F)$.*

Proof. The set $F < \mathbb{A}_F$ is discrete and closed by Lemma 2.1.8 and hence we conclude by Exercise 2.2. $\square$

In the local setting, we analyzed classes in a geometric class using the class map of Lemma 17.4.1. We now provide the adelic analogue.

We maintain the notation introduced after the statement of Theorem 17.7.1. Thus $I \leq H$ is a smooth subgroup and $X := I \backslash H$. Moreover S is a finite set of places of F including the infinite places and $\mathcal{H}$ is a model of H defined over $\mathcal{O}_F^S$, etc. Let $S_0 \subseteq S$. We allow S_0 to be empty.

Recall the cohomology sets of (17.32). For $v \notin S$ set

$$\mathcal{D}_{\text{fppf}}(\mathcal{O}_{F_v}, \mathcal{I}, \mathcal{H}) := \ker(H_{\text{fppf}}^1(\mathcal{O}_{F_v}, \mathcal{I}) \longrightarrow H_{\text{fppf}}^1(\mathcal{O}_{F_v}, \mathcal{H})), \quad (17.44)$$

where the map is given as in (17.36). We have a diagram

$$\begin{array}{ccc} H_{\text{fppf}}^1(\mathcal{O}_{F_v}, \mathcal{I}) & \longrightarrow & H_{\text{fppf}}^1(\mathcal{O}_{F_v}, \mathcal{H}) \\ \downarrow & & \downarrow \\ H^1(F_v, I) & \longrightarrow & H^1(F_v, H), \end{array} \quad (17.45)$$

where the vertical arrows are given by functoriality (17.33) and (17.37), the top arrow is given by (17.36), and the bottom arrow is the canonical map. The commutativity of the diagram follows from Lemma 17.6.3. Recall from (17.23) that

$$\begin{aligned}\mathcal{D}(F, I, H) &:= \ker(H^1(F, I) \longrightarrow H^1(F, H)), \\ \mathcal{D}(F_v, I, H) &:= \ker(H^1(F_v, I) \longrightarrow H^1(F_v, H)).\end{aligned}$$

We let

$$\mathcal{D}(\mathcal{O}_{F_v}, \mathcal{I}, \mathcal{H}) := \text{im}(\mathcal{D}_{\text{fppf}}(\mathcal{O}_{F_v}, \mathcal{I}, \mathcal{H}) \longrightarrow \mathcal{D}(F_v, I, H)) \quad (17.46)$$

and set

$$\mathcal{D}(\mathbb{A}_F^{S_0}, I, H) := \prod'_{v \notin S_0} \mathcal{D}(F_v, I, H), \quad (17.47)$$

where the prime indicates that we take the restricted direct product with respect to $\mathcal{D}(\mathcal{O}_{F_v}, \mathcal{I}, \mathcal{H}) \subseteq \mathcal{D}(F_v, I, H)$ for all $v \notin S$.

By Theorem 17.6.4 the set $\mathcal{D}(\mathcal{O}_{F_v}, \mathcal{I}, \mathcal{H})$ is finite for all $v \notin S$. Consider sets of the form

$$\Omega_{S'-S_0} \times \prod_{v \notin S'} \mathcal{D}(\mathcal{O}_{F_v}, \mathcal{I}, \mathcal{H}) \quad (17.48)$$

for some finite set S' of places of F containing S , where $\Omega_{S'-S_0}$ is any subset of $\prod_{v \in S'-S_0} \mathcal{D}(F_v, I, H)$. They form a base of open sets for a topology on $\mathcal{D}(\mathbb{A}_F^{S_0}, I, H)$. With respect to this topology, a set of the form (17.48) with $\Omega_{S'-S_0}$ finite is a compact open set. As the notation indicates, the set $\mathcal{D}(\mathbb{A}_F^{S_0}, I, H)$ and its topology is independent of the choice of models $\mathcal{I}$ and $\mathcal{H}$ (see Exercise 17.13).

The most important properties of $\mathcal{D}(\mathbb{A}_F^{S_0}, I, H)$ are contained in the following theorem:

Theorem 17.7.4 *The image of the diagonal map*

$$\mathcal{D}(F, I, H) \longrightarrow \prod_{v \notin S_0} \mathcal{D}(F_v, I, H)$$

is contained in $\mathcal{D}(\mathbb{A}_F^{S_0}, I, H)$. If $H^1(F_v, I/I^\circ)$ is finite for all v , then

$$\mathcal{D}(F, I, H) \longrightarrow \mathcal{D}(\mathbb{A}_F^{S_0}, I, H)$$

is proper if we give $\mathcal{D}(F, I, H)$ the discrete topology.

Proof. Using Lemma 17.6.3 we may regard an element of $\mathcal{D}(F, I, H)$ as the isomorphism class of an I -torsor Y such that $Y \wedge^I H$ is isomorphic to the

trivial H -torsor. By spreading out [Poo17, Theorem 3.2.1], for a sufficiently large finite set of places $S' \supseteq S$, the I -torsor Y is isomorphic as an I -torsor to the generic fiber of an $\mathcal{I}_{\mathcal{O}_{\bar{F}}^{S'}}$ -torsor $\mathcal{Y}$ such that $\mathcal{Y} \wedge^{\mathcal{I}_{\mathcal{O}_{\bar{F}}^{S'}}} \mathcal{H}_{\mathcal{O}_{\bar{F}}^{S'}}$ is isomorphic to a trivial $\mathcal{H}_{\mathcal{O}_{\bar{F}}^{S'}}$ -torsor. This implies the first assertion of the theorem. Consider the natural map

$$H^1(F, I) \longrightarrow \prod_{v \notin S'} H^1(F_v, I).$$

Let $H_{S'}^1(F, \mathcal{I})$ be the inverse image of

$$\prod_{v \notin S'} \text{im}(H_{\text{fppf}}^1(\mathcal{O}_{F_v}, \mathcal{I}) \longrightarrow H^1(F_v, I))$$

in $H^1(F, I)$. Our assumption that $H^1(F_v, I/I^\circ)$ is finite for all v and [Poo17, Theorem 6.5.13(a)] together imply that $H_{S'}^1(F, \mathcal{I})$ is finite. Thus the inverse image of

$$\prod_{v \in S' - S_0} \mathcal{D}(F_v, I, H) \times \prod_{v \notin S'} \mathcal{D}(\mathcal{O}_{F_v}, \mathcal{I}, \mathcal{H}) \quad (17.49)$$

in $\mathcal{D}(F, I, H)$ is finite for all $S' \supseteq S$. Any compact subset of $\mathcal{D}(\mathbb{A}_F^{S_0}, I, H)$ is contained in a set of the form (17.49) for some S' , so we deduce the second assertion of the theorem. $\square$

The following lemma is a general version of Hensel's lemma [BLR90, §2.3, Proposition 5]:

Lemma 17.7.5 (Hensel's Lemma) *Let $\mathcal{O}$ be a local Henselian ring with residue field κ and let Y be a smooth scheme over $\mathcal{O}$. Then the map*

$$Y(\mathcal{O}) \longrightarrow Y(\kappa)$$

is surjective. $\square$

For every non-Archimedean place v of F , let $\mathcal{O}_{F_v}^{\text{nr}}$ be the ring of integers of the maximal unramified extension F_v^{nr} of F_v in some separable closure. We will use the following lemma several times below:

Lemma 17.7.6 *Let v be a non-Archimedean place of F . If Y is a smooth scheme over $\mathcal{O}_{F_v}$ then*

$$Y(\mathcal{O}_{F_v}^{\text{nr}}) \neq \emptyset.$$

Proof. Let κ be the residue field of $\mathcal{O}_{F_v}$ and let $\bar{\kappa}$ be the residue field of $\mathcal{O}_{F_v}^{\text{nr}}$. It is an algebraic closure of κ . The surjection $\mathcal{O}_{F_v}^{\text{nr}} \rightarrow \bar{\kappa}$ induces a surjection

$$Y(\mathcal{O}_{F_v}^{\text{nr}}) \longrightarrow Y(\bar{\kappa})$$

by Lemma 17.7.5. The set $Y(\bar{\kappa})$ is nonempty. $\square$

Lemma 17.7.7 *A class in $H^1(F_v, H)$ is in the image of the map from $H^1_{\text{fppf}}(\mathcal{O}_{F_v}, \mathcal{H})$ if and only if it is represented by a 1-cocycle in $\mathcal{H}(\mathcal{O}_E)$ for some finite unramified Galois extension E/F_v .*

Proof. Let Y be an $\mathcal{H}_{\mathcal{O}_{F_v}}$ -torsor. It is smooth over $\mathcal{O}_{F_v}$, and hence there is a finite unramified Galois extension E/F_v such that $Y(\mathcal{O}_E)$ is nonempty by Lemma 17.7.6. The 1-cocycle defined using a point $y \in Y(\mathcal{O}_E)$ as in the definition of (17.37) will take values in $\mathcal{H}(\mathcal{O}_E)$. This implies the “only if” implication.

Conversely, let $c : \text{Gal}_{F_v} \rightarrow \mathcal{H}(\mathcal{O}_E)$ be a 1-cocycle, where E/F_v is an unramified Galois extension. This defines an effective descent datum on $\mathcal{H}_{\mathcal{O}_E}$ [Poo17, Proposition 4.4.4(i), Corollary 4.4.6, Remark 4.4.7]. In particular the descent datum is obtained from an $\mathcal{H}$ -torsor Y by pullback along $\text{Spec}(\mathcal{O}_E) \rightarrow \text{Spec}(\mathcal{O}_{F_v})$. The image of this torsor under the map $H^1_{\text{fppf}}(\mathcal{O}_{F_v}, \mathcal{H}) \rightarrow H^1(F_v, H)$ is cohomologous to c , so we deduce the converse assertion. $\square$

For each place v , we have a class map $\text{cl} : (I \backslash H)(F_v) \rightarrow \mathcal{D}(F_v, I, H)$ as in Lemma 17.4.1.

Lemma 17.7.8 *The local class maps induce a continuous map*

$$\text{cl} : (I \backslash H)(\mathbb{A}_F^{S_0}) \longrightarrow \mathcal{D}(\mathbb{A}_F^{S_0}, I, H).$$

Proof. Let $\mathcal{O}_{F_v}^{\text{nr}}$ be the ring of integers of the maximal unramified extension F_v^{nr} of F_v in F_v^{sep} . For a finite set of places $S' \supseteq S$ that is large enough, the map $\mathcal{H}_{\mathcal{O}_{F_v}^{S'}} \rightarrow \mathcal{X}_{\mathcal{O}_{F_v}^{S'}}$ is smooth [Poo17, Theorem 3.2.1]. Hence the map $\mathcal{H}(\mathcal{O}_{F_v}^{\text{nr}}) \rightarrow \mathcal{X}(\mathcal{O}_{F_v}^{\text{nr}})$ is surjective for $v \notin S'$ by Lemma 17.7.6 applied to the fibers of the morphism $\mathcal{H}_{\mathcal{O}_{F_v}} \rightarrow \mathcal{X}_{\mathcal{O}_{F_v}}$ for $v \notin S'$. Thus for $v \notin S'$ and given $x \in \mathcal{X}(\mathcal{O}_{F_v})$, we can choose $h \in \mathcal{H}(\mathcal{O}_{F_v}^{\text{nr}})$ such that $I(F_v^{\text{sep}})h = x$. This implies $\text{cl}(x)$ is contained in $\mathcal{D}(\mathcal{O}_{F_v}, \mathcal{I}, \mathcal{H})$ by Lemma 17.7.7. This implies that the map cl is well-defined.

To prove continuity, let S' be a finite set of places of F containing S and consider the set

$$\Omega_{S'-S_0} \times \prod_{v \notin S'} \mathcal{D}(\mathcal{O}_{F_v}, \mathcal{I}, \mathcal{H}) \quad (17.50)$$

where $\Omega_{S'-S_0} \subseteq \prod_{v \in S'-S_0} \mathcal{D}(F_v, I, H)$ is any subset. As S' and $\Omega_{S'-S_0}$ vary, these sets form a basis for the open sets in $\mathcal{D}(\mathbb{A}_F^{S_0}, I, H)$. On the other hand, $(I \backslash H)(\mathbb{A}_F^{S_0})$ is the union of the open subsets

$$(I \backslash H)(F_{S''-S_0}) \times \mathcal{X}(\widehat{\mathcal{O}}_F^{S''}) \quad (17.51)$$

as S'' ranges over all finite sets of places containing S . The inverse image of (17.50) in (17.51) is open by the previous paragraph and the corresponding local assertion, Lemma 17.4.3. This implies that cl is continuous. $\square$

Proposition 17.7.9 *One has a commutative diagram*

$$\begin{array}{ccccccc}
 1 & \longrightarrow & I(F) \backslash H(F) & \longrightarrow & (I \backslash H)(F) & \xrightarrow{\text{cl}} & \mathcal{D}(F, I, H) \\
 \downarrow & & \downarrow & & \downarrow \alpha & & \downarrow \beta \\
 1 & \longrightarrow & I(\mathbb{A}_F^{S_0}) \backslash H(\mathbb{A}_F^{S_0}) & \longrightarrow & (I \backslash H)(\mathbb{A}_F^{S_0}) & \xrightarrow{\text{cl}} & \mathcal{D}(\mathbb{A}_F^{S_0}, I, H).
 \end{array}$$

The top row is an exact sequences of pointed sets, and the map β is proper if we give $\mathcal{D}(F, I, H)$ the discrete topology and $H^1(F_v, I/I^\circ)$ is finite for all v .

Proof. The commutativity of the diagram is clear, the top line is exact since (17.24) is exact, and β is proper by Theorem 17.7.4. $\square$

We now prove that suitable subsets of the set of relative classes inside a geometric class are finite:

Theorem 17.7.10 *Assume that $H^1(F_v, I/I^\circ)$ is finite for all v . Let $\Omega \subseteq (I \backslash H)(\mathbb{A}_F^{S_0})$ be a compact set. There are only finitely many $H(F)$ -orbits $xH(F) \subseteq (I \backslash H)(F)$ such that $xH(\mathbb{A}_F^{S_0})$ intersects Ω .*

Proof. We use the diagram in Proposition 17.7.9. Let $x \in (I \backslash H)(F)$. The bottom cl map is constant on $H(\mathbb{A}_F^{S_0})$ -orbits. Thus if $xH(\mathbb{A}_F^{S_0}) \cap \Omega$ is nonempty then

$$\text{cl} \circ \alpha(x) \in \text{cl}(\Omega).$$

By commutativity of the diagram this implies

$$\beta \circ \text{cl}(x) \in \text{cl}(\Omega)$$

and hence

$$\text{cl}(x) \in \beta^{-1}(\text{cl}(\Omega)).$$

Thus by Lemma 17.4.1 the set of $H(F)$ -orbits $xH(F)$ in $(I \backslash H)(F)$ such that $xH(\mathbb{A}_F^{S_0})$ intersects Ω is mapped injectively via cl into $\beta^{-1}(\text{cl}(\Omega))$. The set $\text{cl}(\Omega)$ is compact by Lemma 17.7.8. Since β is proper, $\beta^{-1}(\text{cl}(\Omega))$ is finite. $\square$

Previously in this section we replaced X by $I \backslash H$ with no loss of generality. For the following lemma we consider more general actions:

Lemma 17.7.11 *Let H be a smooth algebraic group over F such that H° is reductive. Let X be an affine scheme of finite type over F equipped with an action of H and let $\Omega \subseteq X(\mathbb{A}_F)$ be a compact set. There are only finitely many closed H -orbits $O \in (X/H)(F)$ such that $\gamma H(\mathbb{A}_F) \cap \Omega$ is nonempty for some $\gamma \in O(F)$.*

Proof. Let

$$p : X(\mathbb{A}_F) \longrightarrow (X/H)(\mathbb{A}_F)$$

be the canonical map. Let $O \in (X/H)(F)$ be a closed H -orbit. If $\gamma H(\mathbb{A}_F) \cap \Omega \neq \emptyset$ for some $\gamma \in O(F)$, then $O \in p(\Omega)$. Thus the set of closed H -orbits

$O \in (X/H)(F)$ such that $\gamma H(\mathbb{A}_F) \cap \Omega$ is nonempty for some $\gamma \in O(F)$ injects into

$$(X/H)(F) \cap p(\Omega). \quad (17.52)$$

But $p(\Omega)$ is compact and $(X/H)(F)$ is discrete and closed in $(X/H)(\mathbb{A}_F)$ by Lemma 17.7.3, so (17.52) is finite. $\square$

We observe that if S_0 is nonempty and $\Omega \subseteq X(\mathbb{A}_F^{S_0})$ is compact then there may very well be infinitely many closed H -orbits $O \in (X/H)(F)$ such that $\gamma H(\mathbb{A}_F^{S_0}) \cap \Omega$ is nonempty for some $\gamma \in O(F)$. For example, one can take H to be the trivial group acting on $X = \mathbb{G}_a$. This is in marked contrast to the situation in Theorem 17.7.10.

17.8 Global relative orbital integrals

We continue to assume that F is a global field. Let $X \times H \rightarrow X$ be an action of a smooth affine algebraic group H over F on a separated scheme X of finite type over F . In this section we define and study global relative orbital integrals attached to such an action. The main result, Theorem 17.8.7, will be applied in the proof of Theorem 18.2.5 to derive the geometric side of the simple relative trace formula.

Fix a separated scheme of finite type over $\mathcal{O}_F^S$ with generic fiber X and denote it again by X by abuse of notation. Here S is a sufficiently large set of places of F including all infinite places.

We begin by making the spaces of functions involved in the definition of global relative orbital integrals precise. For all $v \notin S$, the characteristic function $\mathbb{1}_{X(\mathcal{O}_{F_v})}$ of $X(\mathcal{O}_{F_v})$ is an element of $C_c(X(F_v))$. We define

$$C_c(X(\mathbb{A}_F^\infty)) = \otimes_{v \nmid \infty} C_c(X(F_v)),$$

where the restricted tensor product is taken with respect to the functions $\mathbb{1}_{X(\mathcal{O}_{F_v})}$ for $v \notin S$. It turns out that $C_c(X(\mathbb{A}_F^\infty))$, defined in this manner, is equal to the $\mathbb{C}$ -vector space of compactly supported locally constant functions on $X(\mathbb{A}_F^\infty)$ (see Exercise 17.14). When X is smooth we write $C_c^\infty(X(\mathbb{A}_F^\infty))$ for $C_c(X(\mathbb{A}_F^\infty))$. We use analogous notation when the set of infinite places ∞ is replaced by a set of places of F including the infinite places.

If F is a function field and X is smooth we write $C_c^\infty(X(F_\infty)) := C_c(X(F_\infty))$. If F is a number field and X is smooth, we let $C_c^\infty(X(F_\infty))$ be the space of smooth compactly supported functions on $X(F_\infty)$, viewed as a real manifold. We use analogous notation when the set of infinite places ∞ is replaced by a subset of the set of infinite places.

For the remainder of this section S' is an arbitrary set of places of F (possibly empty). Let

$$C_c(X(\mathbb{A}_F^{S'})) := C_c(X(F_{\infty-S'})) \otimes C_c(X(\mathbb{A}_F^{S' \cup \infty}))$$

and when X is smooth let

$$C_c^\infty(X(\mathbb{A}_F^{S'})) := C_c^\infty(X(F_{\infty-S'})) \otimes C_c^\infty(X(\mathbb{A}_F^{S' \cup \infty})).$$

We now define adelic analogues of two concepts previously introduced in the local setting in §17.5. First, an element $\gamma \in X(F)$ is **relatively unimodular** if its image in $X(F_v)$ is relatively unimodular for every place v of F .

For the second concept, assume that

$$\chi : H(\mathbb{A}_F) \longrightarrow \mathbb{C}^\times$$

is a quasi-character trivial on $H(F)$. We continue to denote by χ the quasi-character

$$H(\mathbb{A}_F^{S'}) \hookrightarrow H(\mathbb{A}_F) \xrightarrow{\chi} \mathbb{C}^\times.$$

Here the first arrow is the canonical injection.

Definition 17.8.1. A relatively semisimple element $\gamma \in X(F)$ is **χ -relevant** if χ is trivial on $H_\gamma(\mathbb{A}_F)$.

As in the local setting, if χ is understood we omit it from notation.

For $f^{S'} \in C_c(X(\mathbb{A}_F^{S'}))$ and relevant, relatively unimodular $\gamma \in X(F)$ we define the **global relative orbital integral**

$$\mathrm{RO}_\gamma(f^{S'}) := \mathrm{RO}_\gamma^\chi(f^{S'}) = \int_{H_\gamma(\mathbb{A}_F^{S'}) \backslash H(\mathbb{A}_F^{S'})} f^{S'}(\gamma h) \chi^{-1}(h) \frac{d_r h}{d_r h_\gamma}. \quad (17.53)$$

Here $d_r h$ is a right Haar measure on $H(\mathbb{A}_F^{S'})$ and $d_r h_\gamma$ is a right Haar measure on $H_\gamma(\mathbb{A}_F^{S'})$.

Proposition 17.8.2 *If $\gamma \in X(F)$ is relevant, relatively semisimple, relatively unimodular, and H_γ is smooth then the integral defining $\mathrm{RO}_\gamma(f^{S'})$ is absolutely convergent.*

Proof. As in the proof of Theorem 17.5.3, it suffices to show that the map

$$H_\gamma(\mathbb{A}_F^{S'}) \backslash H(\mathbb{A}_F^{S'}) \longrightarrow X(\mathbb{A}_F^{S'})$$

is proper. This is implied by Theorem 17.7.1. $\square$

Global relative orbital integrals factor into local relative orbital integrals. In more detail, suppose that $f = \otimes_{v \notin S'} f_v \in C_c(X(\mathbb{A}_F^{S'}))$ where $f_v \in C_c(X(F_v))$ for all places $v \notin S'$. Thus $f_v = \mathbb{1}_{X(\mathcal{O}_{F_v})}$ for all but finitely many places v . Assume that $\gamma \in X(F)$ is relevant, relatively semisimple and relatively unimodular. Normalize the Haar measures $d_r h_\gamma$, $d_r h_{\gamma,v}$, $d_r h$, and $d_r h_v$ on $H_\gamma(\mathbb{A}_F^{S'})$, $H_\gamma(F_v)$, $H(\mathbb{A}_F^{S'})$ and $H(F_v)$, respectively, so that

$$d_r h_\gamma = \prod_{v \notin S'} d_r h_{\gamma, v} \quad \text{and} \quad d_r h = \prod_{v \notin S'} d_r h_v.$$

We moreover assume without loss of generality that there are compact open subgroups

$$K_\gamma^\infty = \prod_{v \nmid \infty} K_{\gamma, v} \leq H_\gamma(\mathbb{A}_F^\infty) \quad \text{and} \quad K^\infty = \prod_{v \nmid \infty} K_v \leq H(\mathbb{A}_F^\infty)$$

such that $\text{meas}_{d_r h_{\gamma, v}}(K_{\gamma, v}) = \text{meas}_{d_r h_v}(K_v) = 1$ for $v \in S' - \infty$. By Exercise 17.15 one has $K_{\gamma, v} = K_v \cap H_\gamma(F_v)$ for all but finitely many v .

Lemma 17.8.3 *Under the assumptions of Proposition 17.8.2 one has that*

$$\text{RO}_\gamma(f^{S'}) = \prod_{v \notin S'} \text{RO}_\gamma(f_v), \quad (17.54)$$

where for each v the local orbital integral $\text{RO}_\gamma(f_v)$ is defined as in (17.30). In fact, for all but finitely many v , $\text{RO}_\gamma(\mathbb{1}_{X(\mathcal{O}_{F_v})}) = 1$.

Proof. Let $S'' \supseteq S'$ be a set of places of F such that $S'' - S'$ is finite and let $f^{S''} = \otimes_{v \notin S''} f_v$. Then by Theorem 17.5.3, Proposition 17.8.2, and the Fubini-Tonelli theorem we have

$$\text{RO}_\gamma(f^{S'}) = \text{RO}_\gamma(f^{S''}) \prod_{v \in S'' - S'} \text{RO}_\gamma(f_v). \quad (17.55)$$

Thus to prove the lemma we are free to replace S' by any finite set of places of F .

Assume $S' \supseteq S$ so that $X(\widehat{\mathcal{O}}_F^{S'})$ is defined. By Theorem 17.7.1 the map $H_\gamma(\mathbb{A}_F^{S'}) \backslash H(\mathbb{A}_F^{S'}) \rightarrow X(\mathbb{A}_F^{S'})$ is proper. This implies that the inverse image Ω of $X(\widehat{\mathcal{O}}_F^{S'})$ in $H_\gamma(\mathbb{A}_F^{S'}) \backslash H(\mathbb{A}_F^{S'})$ is compact and open. One has a canonical homeomorphism

$$H_\gamma(\mathbb{A}_F^{S'}) \backslash H(\mathbb{A}_F^{S'}) \xrightarrow{\sim} \prod'_{v \notin S'} H_\gamma(F_v) \backslash H(F_v) \quad (17.56)$$

where the restricted direct product is with respect to the sets $H_\gamma(F_v)K_v$ for all $v \notin S'$. Applying Exercise 17.15 and replacing S' by a larger set of places if necessary we may assume that the image of Ω under (17.56) is

$$\prod_{v \notin S'} H_\gamma(F_v)K_v.$$

We can moreover assume that S' is chosen large enough that χ is trivial on K_v and $K_{\gamma, v} = K_v \cap H_\gamma(F_v)$ for all $v \notin S'$. Then

$$\text{RO}_\gamma(\mathbb{1}_{X(\mathcal{O}_{F_v})}) = 1$$

for all $v \notin S'$. The lemma follows. $\square$

In practice, one requires more than Proposition 17.8.2. More precisely, one needs conditions under which the integral

$$\int_{H(F) \backslash H(\mathbb{A}_F)} \sum_{\gamma} f(\gamma h) \chi^{-1}(h) d_r h \quad (17.57)$$

converges. This is a very delicate question in general and we will only address it in special cases. We first make the assumption that

$$H^\circ := H_r \times H_u \quad (17.58)$$

where H_r is reductive and H_u is unipotent. We also restrict to relatively unimodular elements so that our orbital integrals are well-defined, and to elements whose H_r -orbit is closed, because otherwise even the contribution of a single orbital integral can diverge [Art05, p. 21].

Let C_X be the stabilizer of X under H_r . We assume that $C_X \leq Z_H$ and let

$$A := A_{C_X} \quad (17.59)$$

where A_{C_X} is defined as in (2.20). Clearly if A is nontrivial then (17.57) is not absolutely convergent. Therefore we consider instead

$$\int_{AH(F) \backslash H(\mathbb{A}_F)} \sum_{\gamma} f(\gamma h) \chi^{-1}(h) d_r h, \quad (17.60)$$

where $f \in C_c^\infty(X(\mathbb{A}_F))$. There is an obvious obstruction to the absolute convergence of this expression. Namely, it is possible that $A_{H_\gamma^\circ}$ strictly contains A for some γ , which implies that the contribution of the class of γ to (17.60) is not absolutely convergent by the unfolding argument in the proof of Theorem 17.8.7 below. We exclude this possibility using the notion of an elliptic element. Let $d_r h_\gamma$ be a right Haar measure on $A \backslash H_\gamma(\mathbb{A}_F)$. We abuse notation and denoted again by $d_r h_\gamma$ the $A \backslash H_\gamma(\mathbb{A}_F)$ -invariant Radon measure on $AH_\gamma(F) \backslash H_\gamma(\mathbb{A}_F)$ induced by this right Haar measure.

Definition 17.8.4. A $\gamma \in X(F)$ is **relatively elliptic** if

$$\tau(H_\gamma) := \text{meas}_{d_r h_\gamma}(AH_\gamma(F) \backslash H_\gamma(\mathbb{A}_F)) \quad (17.61)$$

is finite.

If H_γ has reductive neutral component then $\gamma \in X(F)$ is relatively elliptic if and only if $A_{H_\gamma} \leq A$ by Theorem 2.6.3.

The following lemma will be used to show that only relevant γ contribute to (17.60):

Lemma 17.8.5 *If γ is relatively elliptic and χ is trivial on A then*

$$\int_{AH_\gamma(F) \backslash H_\gamma(\mathbb{A}_F)} \chi^{-1}(h_\gamma) d_r h_\gamma$$

converges absolutely. It is equal to $\tau(H_\gamma)$ if $\chi|_{H_\gamma(\mathbb{A}_F)}$ is trivial and equal to 0 otherwise. $\square$

We leave the proof as Exercise 17.18. We emphasize that we do not assume that $\chi : H(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ is unitary in Lemma 17.8.5.

Since we have assumed (17.58), the group $H(\mathbb{A}_F)$ is unimodular. Hence the assumption that γ is relatively unimodular is equivalent to the assertion that $H_\gamma(\mathbb{A}_F)$ is unimodular. If $H^\circ = H_r$, X is affine, and the H_r -orbit of γ is closed then the group $H_\gamma(\mathbb{A}_F)$ is unimodular by Theorem 17.2.9 and Lemma 3.6.4.

In the function field case, there are serious difficulties having to do with the fact that stabilizers may be nonsmooth and $H^1(F_v, I)$ can be infinite for nonreductive I . This motivates the following definition:

Definition 17.8.6. An element $\gamma \in X(F)$ is **gcf** (Galois cohomologically finite) if H_γ is smooth and for all places v of F one has $|H^1(F_v, H_\gamma/H_\gamma^\circ)| < \infty$.

We emphasize that in the number field case, all γ are gcf by theorems 1.5.3 and 17.4.4. Clearly γ is gcf if and only if every element of its class is gcf so it makes sense to speak of a class being gcf.

Upon restricting our attention to the summands in (17.60) that satisfy the various desiderata above, the corresponding integral does indeed converge absolutely:

Theorem 17.8.7 *Assume (17.58), that X is affine, and that χ is trivial on A . The expression*

$$\int_{AH(F) \backslash H(\mathbb{A}_F)} \sum_{\gamma} |f(\gamma h) \chi^{-1}(h)| dh \quad (17.62)$$

is convergent. Here the sum is over $\gamma \in X(F)$ that are gcf, relatively unimodular and relatively elliptic, and such that the H_r -orbit of γ is closed. Moreover

$$\int_{AH(F) \backslash H(\mathbb{A}_F)} \sum_{\gamma} f(\gamma h) \chi^{-1}(h) dh = \sum_{[\gamma]} \tau(H_\gamma) \text{RO}_\gamma(f) \quad (17.63)$$

where the sum over γ on the left is as before and the sum on the right is over all classes that are relevant, gcf, relatively unimodular, and relatively elliptic and such that the H_r -orbit of γ is closed.

Two remarks are in order. First, the assumption that the H_r -orbit of γ is closed implies that γ is relatively semisimple with respect to H (see Lemma

17.3.7). We will use this fact without further mention below. Second, in the definition of $\text{RO}_\gamma(f)$, we use the measure $\frac{dh}{dh_\gamma}$ on

$$H_\gamma(\mathbb{A}_F) \backslash H(\mathbb{A}_F) = (A \backslash H_\gamma(\mathbb{A}_F)) \backslash (A \backslash H(\mathbb{A}_F))$$

where dh_γ is a Haar measure on $A \backslash H_\gamma(\mathbb{A}_F)$. We use the same measure dh_γ in the definition of $\tau(H_\gamma)$.

Proof. We first prove that (17.62) is convergent. We reduce to the case $H = H^\circ$ as follows. The natural map

$$AH^\circ(F) \backslash H^\circ(\mathbb{A}_F) \longrightarrow AH(F) \backslash H(F)H^\circ(\mathbb{A}_F)$$

is a homeomorphism. Hence

$$\begin{aligned} & \int_{AH(F) \backslash H(\mathbb{A}_F)} \sum_{\gamma} |f(\gamma h) \chi^{-1}(h)| dh \\ &= \int_{H(F)H^\circ(\mathbb{A}_F) \backslash H(\mathbb{A}_F)} \left(\int_{AH^\circ(F) \backslash H^\circ(\mathbb{A}_F)} \sum_{\gamma} |f(\gamma h^\circ h) \chi^{-1}(h^\circ h)| dh^\circ \right) \frac{dh}{dh^\circ} \end{aligned}$$

where dh° is induced by a Haar measure on $H^\circ(\mathbb{A}_F)$. The set $H^\circ(\mathbb{A}_F) \backslash H(\mathbb{A}_F)$ is compact by Proposition 14.3.2, so to prove that (17.62) is convergent, we can assume $H^\circ = H$.

A $\gamma \in X(F)$ has closed H_r -orbit if and only if any element of its H -orbit $O(\gamma)$ has closed H_r -orbit. Moreover, by Exercise 17.17, a $\gamma \in X(F)$ is gcf (resp. relatively unimodular, relatively elliptic) if and only if any element of $O(\gamma)$ is gcf (resp. relatively unimodular, relatively elliptic). Thus we can talk about a geometric class having closed H_r -orbit, etc. With this in mind, one has that

$$\begin{aligned} & \int_{AH(F) \backslash H(\mathbb{A}_F)} \sum_{\gamma} |f(\gamma h) \chi^{-1}(h)| dh \\ &= \sum_O \int_{AH(F) \backslash H(\mathbb{A}_F)} \sum_{\gamma \in O(F)} |f(\gamma h) \chi^{-1}(h)| dh. \end{aligned} \tag{17.64}$$

Here the outer sum is over closed H -orbits $O \subseteq X$ such that any $\gamma \in O(F)$ has closed H_r -orbit and is gcf, relatively unimodular and relatively elliptic. Unfolding the inner integral for a fixed O , we obtain

$$\begin{aligned} & \int_{AH(F) \backslash H(\mathbb{A}_F)} \sum_{\gamma \in O(F)} |f(\gamma h) \chi^{-1}(h)| dh \\ &= \sum_{[\gamma]} \int_{AH_\gamma(F) \backslash H(\mathbb{A}_F)} |f(\gamma h) \chi^{-1}(h)| dh \end{aligned}$$

where the bottom sum is over the relative classes $[\gamma]$ in $O(F)$. By Lemma 17.8.5 the sum above is

$$\sum_{[\gamma]} \text{meas}_{dh_\gamma}(AH_\gamma(F) \backslash H_\gamma(\mathbb{A}_F)) \int_{H_\gamma(\mathbb{A}_F) \backslash H(\mathbb{A}_F)} |f(\gamma h) \chi^{-1}(h)| \frac{dh}{dh_\gamma} \quad (17.65)$$

where for each relatively unimodular γ , the measure dh_γ is a Haar measure on $A \backslash H_\gamma(\mathbb{A}_F)$. The measure

$$\tau(H_\gamma) = \text{meas}_{dh_\gamma}(AH_\gamma(F) \backslash H_\gamma(\mathbb{A}_F))$$

is finite for all γ in (17.65) since all such γ are relatively elliptic. Thus (17.65) is finite by Proposition 17.8.2 and Theorem 17.7.10 with $S_0 = \infty$.

It therefore suffices to show that there are only finitely many closed H -orbits $O \subseteq X$ that can contribute a nonzero summand to (17.64). The sum (17.64) is equal to

$$\begin{aligned} & \sum_O \int_{AH_r(F) \backslash H_r(\mathbb{A}_F) \times [H_u]} \sum_{\gamma \in O(F)} |f(\gamma h_r h_u) \chi^{-1}(h_r h_u)| dh_r dh_u \\ &= \sum_O \int_{AH_r(F) \backslash H_r(\mathbb{A}_F)} \sum_{\gamma \in O(F)} |\tilde{f}(\gamma h_r) \chi^{-1}(h_r)| dh_r \end{aligned} \quad (17.66)$$

where $\tilde{f}(x) := \int_{\mathcal{F}} |f(x h_u) \chi^{-1}(h_u)| dh_u \in C_c(X(\mathbb{A}_F))$ with $\mathcal{F}$ a compact fundamental domain for $H_u(F)$ acting on $H_u(\mathbb{A}_F)$ (see Theorem 2.6.3). Here the sum over O is as in (17.64). Let $\Omega \subset X(\mathbb{A}_F)$ be the support of $\tilde{f}$; it is a compact set. If the summand corresponding to O in (17.66) is nonzero, then for some H_r -orbit

$$O_{H_r} \in (O/H_r)(F) \subseteq (X/H_r)(F)$$

and some $\gamma \in O_{H_r}(F)$ one has $\gamma H_r(\mathbb{A}_F) \cap \Omega \neq \emptyset$. But there are only finitely many such H_r -orbits O_{H_r} by Lemma 17.7.11. Thus there are only finitely many H -orbits O contributing a nonzero summand to (17.66), which is equal to (17.64).

We now prove the identity in the theorem. All of the manipulations with integrals occurring in this proof are justified by the convergence of (17.62). Consider

$$\int_{AH(F) \backslash H(\mathbb{A}_F)} \sum_{\gamma} f(\gamma h) \chi^{-1}(h) dh \quad (17.67)$$

where the sum is over γ with closed H_r -orbit that are relatively elliptic, relatively unimodular and gcf. We unfold the integral to obtain

$$\begin{aligned}
& \sum_{[\gamma]} \int_{AH_\gamma(F) \backslash H(\mathbb{A}_F)} f(\gamma h) \chi^{-1}(h) dh \\
&= \sum_{[\gamma]} \int_{AH_\gamma(F) \backslash H_\gamma(\mathbb{A}_F)} \chi^{-1}(h_\gamma) dh_\gamma \text{RO}_\gamma(f) \\
&= \sum_{[\gamma]} \tau(H_\gamma) \text{RO}_\gamma(f)
\end{aligned}$$

where the first and second sums over $[\gamma]$ are over $H(F)$ -orbits in $X(F)$ with closed H_r -orbit that are gcf, relatively elliptic, and relatively unimodular. By Lemma 17.8.5, the third sum is over relevant, gcf, relatively elliptic, relatively unimodular classes in $X(F)$ with closed H_r -orbit. $\square$

Exercises

17.1. Let $X \times H \rightarrow X$ be an action of an algebraic group H on an affine scheme X over a field k . Let $\gamma \in X(k)$. Prove that the stabilizer $H_\gamma \leq H$ is a closed subgroup scheme and deduce that it is again an affine algebraic group.

17.2. Give an example of a reductive group H acting on an affine scheme X over a field k and a $\gamma \in X(k)$ such that $O(\gamma)$ is not closed but H_γ is reductive.

17.3. Let $X \times H \rightarrow X$ be an action of an algebraic group H on an affine scheme X over a field k . Prove that $\text{Spec}(\mathcal{O}(X)^H)$ is a categorical quotient in the category of affine schemes (but see Example 17.5).

17.4. Let $\sigma : H \rightarrow H$ be an automorphism of the smooth algebraic group H over a field k . For any k -algebra R let

$$H^\sigma(R) := \{h \in H(R) : \sigma(h) = h\}.$$

Prove that H^σ is an algebraic subgroup of H .

17.5. Give examples to show that the map $\Gamma_{\text{ss}}(k) \rightarrow H(k) \backslash X(k)$ need not be surjective and that the map

$$H(k) \backslash X(k) \longrightarrow \Gamma_{\text{geo,ss}}(k)$$

need not be injective.

17.6. Let k be an algebraically closed field. Assume that $\gamma \in \text{GL}_n(k)$ has finite order and that either k is of characteristic zero or that the order of γ is coprime to the characteristic of k . Prove that γ is semisimple. Give a counterexample to show that if the characteristic of k is positive and divides the order of γ then γ need not be semisimple.

17.7. Let k be a field with algebraic closure $\bar{k}$. Prove that two elements of $\mathrm{GL}_n(k)$ are $\mathrm{GL}_n(k)$ -conjugate if and only if they are $\mathrm{GL}_n(\bar{k})$ -conjugate.

17.8. Let k be a field with separable closure k^{sep} . Prove that a conjugacy class in $\mathrm{GL}_n(k^{\mathrm{sep}})$ that is stable under $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ contains an element in $\mathrm{GL}_n(k)$.

17.9. Let G be an algebraic group over a field k and let H be the image of the diagonal map $G \rightarrow G \times G$. Construct a bijection between $G(k)$ -conjugacy classes in $G(k)$ and

$$H(k) \backslash (G(k) \times G(k)) / H(k).$$

17.10. Using Proposition 1.10.7, prove Lemma 17.4.1.

17.11. Assume that k is a perfect field, that H is reductive, and that $\gamma \in H(k)$ is semisimple. Prove that γ is relatively regular with respect to the conjugation action of H on itself if and only if $C_{\gamma, H}^{\circ}$ is a maximal torus in H .

17.12. Prove that the groups I_h appearing in (17.25) are all inner forms of I .

17.13. Let S_0 be a finite set of places of the global field F . Prove that the set $\mathcal{D}(\mathbb{A}_F^{S_0}, I, H)$ and its topology are both independent of the choice of models of I and H used to define them.

17.14. Let S be a set of places of the global field F including the infinite places. Prove that the space $C_c(X(\mathbb{A}_F^S))$, defined as a restricted direct product as in §17.8, is equal to the space of locally constant compactly supported functions on $X(\mathbb{A}_F^S)$.

17.15. Let $\{X_{\alpha}\}_{\alpha \in A}$ be a set of locally compact topological spaces indexed by a countable set A , and for all α outside a finite subset S_0 of A , let $K_{\alpha} \subseteq X_{\alpha}$ be a compact open subset of X_{α} . Let

$$X := \prod'_{\alpha \in A} X_{\alpha}$$

be the restricted direct product with respect to the K_{α} . For all nonempty compact open subsets $\Omega \subset X$, prove that there is a finite subset A' of A such that

$$\Omega = \Omega_{A'} \times \prod_{\alpha \in A - A'} K_{\alpha},$$

where $\Omega_{A'} \subseteq \prod_{\alpha \in A'} X_{\alpha}$ is a compact open set.

17.16. Prove that if G is a commutative smooth algebraic group over a field k and c_1, c_2 are 1-cocycles, then

$$\sigma \longmapsto c_1(\sigma)c_2(\sigma)$$

is a 1-cocycle. This defines a binary operation on $H^1(k, G)$. Deduce that this binary operation endows $H^1(k, G)$ with an abelian group structure with the class of the neutral element as the identity element.

17.17. Let F be a global field and let H be a smooth algebraic group over F acting on an affine scheme X of finite type over F on the right. Prove that $\gamma \in X(F)$ is relatively semisimple if and only if any element of $O(\gamma)(F)$ is relatively semisimple. Prove that the following conditions depend only on the relative geometric class of a $\gamma \in X(F)$:

- (a) γ is relatively unimodular.
- (b) γ is relatively elliptic.
- (c) γ is gcf.

17.18. Prove Lemma 17.8.5.

Chapter 18

Simple Trace Formulae



It is rare that you can take a proof in the geometric setting and convert it to the genuine number theoretic setting. That is what has transpired through Ngô's achievement (the proof of the fundamental lemma). Ngô has provided a bridge and now everybody is using this bridge.

P. Sarnak

In this chapter we prove a simple general relative trace formula and overview its various specializations. This recovers (simple versions of) many trace formulae that have appeared in the literature. In particular, one such specialization yields the Petersson-Bruggeman-Kuznetsov trace formula. Due to its continuing importance in analytic number theory we discuss this case in some detail.

18.1 A brief history of trace formulae

The trace formula is a powerful tool for understanding automorphic representations of reductive groups. It is an expression for the trace of the integral operator $R(f)$ of §16.1 on the space of cusp forms in terms of orbital integrals. Relative trace formulae, which include the trace formula as a special case, are formulae that describe period integrals of cuspidal automorphic forms in terms of the relative orbital integrals of Chapter 17.

The original trace formula was introduced by Selberg in his seminal 1956 paper [Sel56]. Let G be a semisimple group over the real numbers and let $\Gamma \leq G(\mathbb{R})$ be a discrete subgroup. Selberg proved a trace formula valid whenever $\Gamma \backslash G(\mathbb{R})$ is compact, and also treated in detail certain special noncompact

quotients such as $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R})$ using the theory of Eisenstein series. He succeeded in proving the existence of cusp forms on $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{SL}_2(\mathbb{R})$ invariant under $\mathrm{SO}_2(\mathbb{R})$ on the right. In fact, he proved an asymptotic for the number of such forms as a function of their Laplacian eigenvalue. Results of this type are known as Weyl laws since the corresponding asymptotics were proven by Weyl for general compact manifolds [Wey12]. This is discussed in §19.2 in the next chapter.

A key difficulty in extending Selberg's work to arbitrary reductive groups G over number fields F was proving the analytic continuation of Eisenstein series in general and using them to decompose $L^2([G])$. Langlands achieved this in the groundbreaking work [Lan76] summarized in Chapter 10. Langlands' study of Eisenstein series in turn led him to the formulation of his functoriality conjecture. All of this earned him an Abel prize in 2018.

Giving a decomposition of $L^2([G])$ in terms of Eisenstein series is a first step to generalizing Selberg's trace formula, but it is not enough by itself. Over a lifetime of work, Arthur used the theory of Eisenstein series as the basis for a completely general trace formula for an arbitrary reductive group. The technical difficulties he overcame were so immense that the trace formula is often called the **Arthur-Selberg trace formula** in honor of his contributions.

Let $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$. The basic idea behind the proof of the trace formula is to integrate the automorphic kernel function $K_f(x, y)$ of (16.5) along the diagonal copy of $[G]$ and evaluate the expression in two different manners. Later, in [Jac86] (see also [Jac05a]), Jacquet realized that the trace formula could be generalized in a profitable manner. The idea behind Jacquet's **relative trace formula** is to replace the diagonal integration in the usual trace formula by integration along a pair of subgroups $H_1 \times H_2 \leq G \times G$. Exercises 18.12-18.16 below explain why this is indeed a generalization.

Our first goal in this chapter is to prove Theorem 18.2.5, the general simple relative trace formula. This is a matter of collecting our preparatory work in Chapter 16 and Chapter 17. Our setup is a variant of Jacquet's in that we work with a general subgroup $H \leq G \times G$ that need not decompose as a product of the projections to the two factors. For example, one can take H to be the diagonal copy of G , in which case one obtains the usual simple trace formula as explained in §18.4.

In the proof, we impose assumptions on the test function f to ensure that various analytic difficulties disappear. This is the reason we call the formula **simple**. The disadvantage is that in most cases, these assumptions restrict the class of automorphic representations one can treat. In the context of the usual trace formula, Arthur's work mentioned above removes these assumptions. In the relative setting, this is the subject of ongoing work. See [Lap06, Zyd19, Zyd20].

After proving Theorem 18.2.5 we devote the remainder of the chapter to illustrating it via examples, recovering various (simple) trace formulae in the literature.

18.2 A general simple relative trace formula

For the rest of this chapter we let G be a reductive group over a number field F . We omit a discussion of the function field case to avoid distracting subtleties. We have already laid the foundation of the theory in chapters 16 and 17. Thus the interested reader should be able to deduce a function field analogue of the simple general relative trace formula of Theorem 18.2.5 below if desired.

Before proceeding we must come to grips with certain technicalities involving the subgroup A_G .

Lemma 18.2.1 *If $H \leq G$ then one has that*

$$A_H \cap A_G = H(\mathbb{A}_F) \cap A_G = A_{H \cap Z_G}.$$

Proof. By applying Weil restriction of scalars we can assume that $F = \mathbb{Q}$. Let T_G be the largest split subtorus of Z_G and let T_H be the largest split subtorus in Z_H . Theorem 4.4.1 implies that the groups $T_H(\mathbb{R}) \cap T_G(\mathbb{R})$ and $H(\mathbb{R}) \cap T_G(\mathbb{R})$ have finitely many components in the Hausdorff topology, and hence the same is true of the groups $A_H \cap A_G$ and $H(\mathbb{A}_{\mathbb{Q}}) \cap A_G = H(\mathbb{R}) \cap A_G$. By Exercise 18.1 we deduce that the groups $A_H \cap A_G$ and $H(\mathbb{A}_{\mathbb{Q}}) \cap A_G$ are connected.

The group $(H \cap T_G)^\circ$ is a split torus by Exercise 18.2. Thus $T_H \cap T_G \geq (H \cap T_G)^\circ$. On the other hand, $T_H \cap T_G \leq H \cap T_G$, so we deduce

$$(T_H \cap T_G)^\circ(\mathbb{R}) = (H \cap T_G)^\circ(\mathbb{R}). \quad (18.1)$$

By the comments in the first paragraph we deduce that the neutral component of $(T_H \cap T_G)^\circ(\mathbb{R})$ is $A_H \cap A_G$ and the neutral component of $(H \cap T_G)^\circ(\mathbb{R})$ is $H(\mathbb{A}_{\mathbb{Q}}) \cap A_G$, proving the first equality.

On the other hand $(T_H \cap T_G)^\circ$ is the maximal split torus in $H \cap Z_G$. Hence the neutral component of $(T_H \cap T_G)^\circ(\mathbb{R})$ is $A_{H \cap Z_G}$. $\square$

We now let

$$H \leq G \times G$$

be a subgroup. We always assume that H° is the direct product of a reductive group and a unipotent group. This implies in particular that $H(\mathbb{A}_F)$ is unimodular. Consider the action of H on G given on points in an F -algebra R by

$$\begin{aligned} a : G(R) \times H(R) &\longrightarrow G(R) \\ (g, (h_1, h_2)) &\longrightarrow h_1^{-1}gh_2. \end{aligned} \quad (18.2)$$

Let

$$\begin{aligned} A_{G,H} &:= H(\mathbb{A}_F) \cap (A_G \times A_G), \\ A &:= H(\mathbb{A}_F) \cap \Delta(A_G) \end{aligned} \quad (18.3)$$

where

$$\Delta : G \longrightarrow G \times G$$

is the diagonal embedding. To ease notation we sometimes write

$$G^2 = G \times G,$$

so $A_{G^2} = A_G \times A_G$ and

$$G^2(\mathbb{A}_F) = A_{G^2} G^2(\mathbb{A}_F)^1.$$

Since H° is the direct product of a unipotent and a reductive group, there is a canonical homomorphism from H° to the unipotent radical H_u of H° . The kernel of this homomorphism is the unique maximal reductive subgroup H_r of H . We assume

$$H_r(\mathbb{A}_F) = (H_r(\mathbb{A}_F) \cap A_{G^2})(H_r(\mathbb{A}_F) \cap G^2(\mathbb{A}_F)^1). \quad (18.4)$$

This is not automatic by Exercise 18.6. However, it is no essential loss of generality for our purposes as we will explain after stating Theorem 18.2.5 below.

Lemma 18.2.2 *The assumption (18.4) implies that*

$$H(\mathbb{A}_F) = (H(\mathbb{A}_F) \cap A_{G^2})(H(\mathbb{A}_F) \cap G^2(\mathbb{A}_F)^1).$$

Proof. Let

$$p : G^2(\mathbb{A}_F) \longrightarrow A_{G^2} \xrightarrow{\sim} \mathbb{R}_{>0}^n$$

be the composite of the canonical quotient map and a choice of isomorphism. It suffices to show that

$$p(H(\mathbb{A}_F)) = p(H(\mathbb{A}_F) \cap A_{G^2}).$$

It is clear that $p(H(\mathbb{A}_F)) \supseteq p(H(\mathbb{A}_F) \cap A_{G^2})$ so we are left with the reverse inclusion.

We claim that in fact

$$p(H(\mathbb{A}_F)) = p(H_r(\mathbb{A}_F) \cap A_{G^2}). \quad (18.5)$$

The decomposition (18.4) implies that

$$p(H_r(\mathbb{A}_F)) = p(H_r(\mathbb{A}_F) \cap A_{G^2}) \xrightarrow{\sim} \mathbb{R}_{>0}^m \quad (18.6)$$

for some $n \geq m \geq 0$. Here the isomorphism is the restriction of the isomorphism in the definition of p . Clearly $p(H^\circ(\mathbb{A}_F)) = p(H_r(\mathbb{A}_F))$. Thus there is a continuous injection

$$p(H(\mathbb{A}_F))/p(H^\circ(\mathbb{A}_F)) \hookrightarrow \mathbb{R}_{>0}^{n-m}. \quad (18.7)$$

Since $H(\mathbb{A}_F)/H^\circ(\mathbb{A}_F)$ is compact by Proposition 14.3.2 the group on the left in (18.7) is trivial. Hence

$$p(H(\mathbb{A}_F)) = p(H^\circ(\mathbb{A}_F)) = p(H_r(\mathbb{A}_F)) = p(H_r(\mathbb{A}_F) \cap A_{G^2})$$

by (18.6), proving (18.5). $\square$

Lemma 18.2.3 *Let $C_G \leq H_r$ be the stabilizer of G under H_r . One has $C_G \leq Z_H$ and $A = A_{C_G}$.*

Thus the definition of A in (18.3) is consistent with the definition (17.59) in the special case $X = G$.

Proof. By Lemma 18.2.1 we have

$$A = H(\mathbb{A}_F) \cap \Delta(A_G) = (H \cap \Delta(G))(\mathbb{A}_F) \cap A_{\Delta(G)} = A_{H \cap \Delta(Z_G)}.$$

Moreover $A_{H \cap \Delta(Z_G)} = A_{H_r \cap \Delta(Z_G)}$. On the other hand

$$C_G(R) = \{(z, z) \in H_r(R) : z \in Z_G(R)\} = H_r(R) \cap \Delta(Z_G)(R).$$

Thus $C_G \leq Z_H$ and $A_{C_G} = A_{H_r \cap \Delta(Z_G)} = A$. $\square$

Let

$$\chi : A_{G,H} \backslash H(\mathbb{A}_F) \longrightarrow \mathbb{C}^\times$$

be a quasi-character trivial on $H(F)$, let π be a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$, and let $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$. As in (16.16), we define the relative trace of $\pi(f)$ with respect to H and χ to be

$$\text{rtr } \pi(f) = \int_{A_{G,H} H(F) \backslash H(\mathbb{A}_F)} K_{\pi(f)}(h_1, h_2) \chi^{-1}(h_1, h_2) d(h_1, h_2). \quad (18.8)$$

This depends on the choice of Haar measure on $A_{G,H} \backslash H(\mathbb{A}_F)$ used to define $d(h_1, h_2)$.

We now define relative orbital integrals in this context. For all $\gamma \in G(F)$ one has that

$$A_{G,H} \cap H_\gamma(\mathbb{A}_F) = A. \quad (18.9)$$

Hence

$$A_{G,H} H_\gamma(\mathbb{A}_F) \backslash H(\mathbb{A}_F) = (A \backslash H_\gamma(\mathbb{A}_F)) \backslash (A_{G,H} \backslash H(\mathbb{A}_F)). \quad (18.10)$$

Assume that γ is relevant (i.e. $\chi|_{H_\gamma(\mathbb{A}_F)} = 1$) relatively semisimple (i.e. has closed H -orbit) and relatively unimodular (i.e. $H_\gamma(\mathbb{A}_F)$ is unimodular). We can then define an orbital integral

$$\text{RO}_\gamma(f) = \int f(h_1^{-1} \gamma h_2) \chi^{-1}(h_1, h_2) \frac{d(h_1, h_2)}{dh_\gamma}, \quad (18.11)$$

where the integral is over

$$(A \backslash H_\gamma(\mathbb{A}_F)) \backslash (A_{G,H} \backslash H(\mathbb{A}_F)).$$

This is slightly different from the definition in (17.53). The root of this is that it is more convenient for the spectral side of the trace formula to work with functions on $A_G \backslash G(\mathbb{A}_F)$ instead of $G(\mathbb{A}_F)$. In (18.11) dh_γ is a Haar measure on $A \backslash H_\gamma(\mathbb{A}_F)$. Thus $\frac{d(h_1, h_2)}{dh_\gamma}$ is an $A_{G,H} \backslash H(\mathbb{A}_F)$ -invariant Radon measure on

$$(A \backslash H_\gamma(\mathbb{A}_F)) \backslash (A_{G,H} \backslash H(\mathbb{A}_F)).$$

In order for Theorem 18.2.5 below to be valid, it is important that we use the same Haar measure $d(h_1, h_2)$ on $A_{G,H} \backslash H(\mathbb{A}_F)$ used in the definition of the relative trace.

We now check that (18.11) is well-defined by slightly modifying the proof of Proposition 17.8.2:

Proposition 18.2.4 *If γ is relatively unimodular and relatively semisimple then one has that*

$$\int_{(A \backslash H_\gamma(\mathbb{A}_F)) \backslash (A_{G,H} \backslash H(\mathbb{A}_F))} |f(h_1^{-1} \gamma h_2) \chi^{-1}(h_1, h_2)| \frac{d(h_1, h_2)}{dh_\gamma} < \infty.$$

Proof. By Proposition 17.8.2 with $S' = \infty$ it suffices to check that

$$\int_{(A \backslash H_\gamma(F_\infty)) \backslash (A_{G,H} \backslash H(F_\infty))} |f(h_1^{-1} \gamma h_2) \chi^{-1}(h_1, h_2)| \frac{d(h_1, h_2)}{dh_\gamma} < \infty. \quad (18.12)$$

Using Lemma 18.2.2 one checks that the stabilizer of $A_G \gamma \in A_G \backslash G(F_\infty)$ in $H(F_\infty)$ is $A_{G,H} H_\gamma(F_\infty)$. In view of (18.10) this implies that we have a commutative diagram

$$\begin{array}{ccc} H_\gamma(F_\infty) \backslash H(F_\infty) & \hookrightarrow & G(F_\infty) \\ \downarrow & & \downarrow p \\ (A \backslash H_\gamma(F_\infty)) \backslash (A_{G,H} \backslash H(F_\infty)) & \hookrightarrow & A_G \backslash G(F_\infty) \end{array} \quad (18.13)$$

where the vertical arrows are the quotient maps and the horizontal arrows are induced by the action of $H(F_\infty)$ on $G(F_\infty)$. We observe that p has a continuous section given by the inverse of the canonical isomorphism

$$G(F_\infty)^1 := G(\mathbb{A}_F)^1 \cap G(F_\infty) \xrightarrow{\sim} A_G \backslash G(F_\infty).$$

Moreover the image of the top horizontal arrow is closed by Lemma 17.4.3. We deduce that the bottom horizontal map is proper.

As in the proof of Theorem 17.5.3 the properness of the bottom arrow of (18.13) suffices to prove (18.12). $\square$

Since $A = A_{C_G}$ by Lemma 18.2.3 the measure

$$\tau(H_\gamma) := \text{meas}_{dh_\gamma}(AH_\gamma(F) \backslash H_\gamma(\mathbb{A}_F)) \quad (18.14)$$

of (17.61) is finite if γ is relatively elliptic in the sense of Definition 17.8.4. Here we have abused notation and denoted again by dh_γ the $A \backslash H_\gamma(\mathbb{A}_F)$ -invariant Radon measure induced by the Haar measure on $A \backslash H_\gamma(\mathbb{A}_F)$ used in the definition of the relative orbital integral (18.11).

Recall that we are assuming F is a number field and that H° is the direct product of a reductive group H_r and a unipotent group H_u . We moreover assume H_r satisfies (18.4). The following is the general simple relative trace formula:

Theorem 18.2.5 *Let $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ be a function such that $R(f)$ has cuspidal image and such that if the $H(\mathbb{A}_F)$ -orbit of γ intersects the support of f then γ is relatively elliptic, relatively unimodular, and the H_r -orbit of γ is closed. Then*

$$\sum_{\pi} \text{rtr } \pi(f) = \sum_{[\gamma]} \tau(H_\gamma) \text{RO}_\gamma(f)$$

where the sum on the left is over (H, χ) -distinguished cuspidal automorphic representations π and the sum on the right is over relevant classes $[\gamma]$ that are relatively unimodular, relatively elliptic, and such that the H_r -orbit of γ is closed. Both sums are absolutely convergent.

We recall that if the H_r -orbit of γ is closed, then γ is relatively semisimple by Lemma 17.3.7. Following Langlands, the left-hand side of the trace formula is called the **spectral side** and the right-hand side of the trace formula is called the **geometric side**. We call the trace formula of Theorem 18.2.5 simple because we have imposed assumptions on f to eliminate analytic difficulties on the spectral and geometric sides. It is general because it includes (simple versions of) most trace formulae in the literature as special cases.

We now explain why assumption (18.4) is no essential loss of generality. If we replace an arbitrary subgroup $H \leq G \times G$ with $H(Z_G \times Z_G)$ then (18.4) is satisfied. This has the effect of isolating cuspidal automorphic representations that have a particular central character depending on χ . We chose not to assume that $Z_G \times Z_G \leq H$ for aesthetic reasons. In particular, in the settings of §18.4, §18.5 and Theorem 18.9.2 assuming $Z_G \times Z_G$ is a subgroup of H complicates the formula for local orbital integrals at non-Archimedean places.

We require the following analogue of Lemma 17.7.11:

Lemma 18.2.6 *Assume that $H \leq G \times G$ is reductive and (18.4) holds, that is,*

$$H(\mathbb{A}_F) = (H(\mathbb{A}_F) \cap A_{G^2})(H(\mathbb{A}_F) \cap G^2(\mathbb{A}_F)^1).$$

Let $\Omega \subset A_G \backslash G(\mathbb{A}_F)$ be a compact set. The set of relatively semisimple geometric classes in $G(F)$ whose $H(\mathbb{A}_F)$ -orbit intersects Ω is finite.

Proof. Let

$$H(\mathbb{A}_F)' = H(\mathbb{A}_F) \cap G^2(\mathbb{A}_F)^1$$

and $A_{G,H} = H(\mathbb{A}_F) \cap A_{G^2}$ as above. Then

$$A_{G,H}H(\mathbb{A}_F)' = H(\mathbb{A}_F) \quad (18.15)$$

by assumption and $A_{G,H} \cap H(\mathbb{A}_F)' = 1$. Using the fact that $G(\mathbb{A}_F)$ is the direct product of A_G and $G(\mathbb{A}_F)^1$ and $H(\mathbb{A}_F)$ is the direct product of $A_{G,H}$ and $H(\mathbb{A}_F)'$ we deduce that there is a homeomorphism

$$\begin{aligned} A_G/a(1, A_{G,H}) \times G(\mathbb{A}_F)^1/H(\mathbb{A}_F)' &\xrightarrow{\sim} G(\mathbb{A}_F)/H(\mathbb{A}_F) \\ (z, g) &\longmapsto zg \end{aligned} \quad (18.16)$$

where $a(\cdot, \cdot)$ is the action map of (18.2). Here the quotients $G(\mathbb{A}_F)^1/H(\mathbb{A}_F)'$ and $G(\mathbb{A}_F)/H(\mathbb{A}_F)$ are with respect to the action a . If we let A_G act via its canonical left actions on $A_G/a(1, A_{G,H})$ in the domain and on $G(\mathbb{A}_F)$ in the codomain, the homeomorphism (18.16) is A_G -equivariant. Thus (18.16) induces a homeomorphism

$$G(\mathbb{A}_F)^1/H(\mathbb{A}_F)' \longrightarrow A_G \backslash G(\mathbb{A}_F)/H(\mathbb{A}_F). \quad (18.17)$$

We observe that $H(\mathbb{A}_F)' \geq H(\mathbb{A}_F)^1 \geq H(F)$ and $G(\mathbb{A}_F)^1 \geq G(F)$. The considerations above yield a commutative diagram

$$\begin{array}{ccccc} & & G(F)/H(F) & & \\ & \swarrow & \downarrow & \searrow & \\ A_G \backslash G(\mathbb{A}_F)/H(\mathbb{A}_F) & \xleftarrow{\sim} & G(\mathbb{A}_F)^1/H(\mathbb{A}_F)' & \xrightarrow{\sim} & a(1, A_{G,H})G(\mathbb{A}_F)^1/H(\mathbb{A}_F) \end{array} \quad (18.18)$$

where all arrows are the canonical maps and the horizontal arrows are homeomorphisms.

We claim that given a compact subset

$$\Omega \subseteq a(1, A_{G,H})G(\mathbb{A}_F)^1/H(\mathbb{A}_F),$$

there are only finitely many relatively semisimple geometric classes in $G(F)$ whose image in $a(1, A_{G,H})G(\mathbb{A}_F)^1/H(\mathbb{A}_F)$ intersects Ω . The commutativity of the diagram (18.18) shows that the claim implies the lemma. Let

$$p : a(1, A_{G,H})G(\mathbb{A}_F)^1/H(\mathbb{A}_F) \longrightarrow (G/H)(\mathbb{A}_F)$$

denote the restriction of the natural map $G(\mathbb{A}_F)/H(\mathbb{A}_F) \rightarrow (G/H)(\mathbb{A}_F)$. The set $p(\Omega)$ is compact and $(G/H)(F)$ is discrete and closed in $(G/H)(\mathbb{A}_F)$ by Lemma 17.7.3, hence

$$p(\Omega) \cap (G/H)(F)$$

is finite. In view of Proposition 17.3.4 this implies the claim. $\square$

Proof of Theorem 18.2.5: Let

$$K_f(g, h) = \sum_{\gamma \in G(F)} f(g^{-1}\gamma h).$$

Since we have assumed that $R(f)$ has cuspidal image, one has that

$$K_f(g, h) = K_{f, \text{cusp}}(g, h)$$

in the notation of (16.7). By Theorem 16.2.6 we have that

$$\int_{A_{G,H}H(F)\backslash H(\mathbb{A}_F)} K_f(h_1, h_2)\chi^{-1}(h_1, h_2)d(h_1, h_2) = \sum_{\pi} \text{rtr } \pi(f). \quad (18.19)$$

At this point, if $A_{G,H} = A$ then we can invoke Theorem 17.8.7 to finish the proof. In general we can still follow the argument of the proof of Theorem 17.8.7. We now give the details. We will check below that it is permissible to unfold the integral in (18.19). Assuming this for the moment, the integral in (18.19) is equal to

$$\sum_{[\gamma]} \int f(h_1^{-1}\gamma h_2)\chi^{-1}(h_1, h_2) \frac{d(h_1, h_2)}{dh_{\gamma}} \int_{AH_{\gamma}(F)\backslash H_{\gamma}(\mathbb{A}_F)} \chi^{-1}(h_{\gamma})dh_{\gamma},$$

where the left integral is over

$$(A\backslash H_{\gamma}(\mathbb{A}_F))\backslash (A_{G,H}\backslash H(\mathbb{A}_F))$$

and the sum on $[\gamma]$ is over classes that are relatively unimodular, relatively elliptic, and such that the H_r -orbit of γ is closed. Applying Lemma 17.8.5 the sum above is equal to

$$\begin{aligned} & \sum_{[\gamma]} \tau(H_{\gamma}) \int_{(A\backslash H_{\gamma}(\mathbb{A}_F))\backslash (A_{G,H}\backslash H(\mathbb{A}_F))} f(h_1^{-1}\gamma h_2)\chi^{-1}(h_1, h_2) \frac{d(h_1, h_2)}{dh_{\gamma}} \\ &= \sum_{[\gamma]} \tau(H_{\gamma}) \text{RO}_{\gamma}(f), \end{aligned} \quad (18.20)$$

where the sum is over classes $[\gamma]$ as in the statement of the theorem.

To justify unfolding, using Lemma 17.8.5 it suffices to check that

$$\sum_{[\gamma]} \tau(H_{\gamma}) \int |f(h_1^{-1}\gamma h_2)\chi^{-1}(h_1, h_2)| \frac{d(h_1, h_2)}{dh_{\gamma}} < \infty \quad (18.21)$$

where the integral and the sum on $[\gamma]$ are as above (18.20). The integral in (18.21) converges absolutely by Proposition 18.2.4. It therefore suffices to prove that only finitely many classes $[\gamma]$ can contribute a nonzero summand to

(18.21). Consider the contribution of a particular relatively semisimple class $[\gamma_0]$. By Theorem 17.7.10, for any compact set $\Omega \subset (H_{\gamma_0} \backslash H)(\mathbb{A}_F^\infty)$, there are only finitely many $H(F)$ -orbits $xH(F) \subset (H_{\gamma_0} \backslash H)(F)$ such that $xH(\mathbb{A}_F^\infty)$ intersects Ω . Here we are using that F is a number field, so the condition that $H^1(F_v, H_{\gamma_0}/H_{\gamma_0}^\circ)$ is finite is automatic by Theorem 17.4.4. This implies that there are only finitely many relative classes in the geometric class of $[\gamma_0]$ that can contribute to the sum.

Lastly we show that there are only finitely many geometric classes that can contribute to the sum. By a minor variant of the argument used in the proof of Theorem 17.8.7, it suffices to show that for any compact set $\Omega \subset A_G \backslash G(\mathbb{A}_F)$, there are only finitely many geometric classes in $G(F)$ with closed H_r -orbit whose $H_r(\mathbb{A}_F)$ -orbit intersects Ω . This is the content of Lemma 18.2.6.

We now make some comments on the assumptions on f in Theorem 18.2.5. For more information on the assumption on f involving the geometric side of the trace formula, we refer to exercises 18.3, 18.4 and 18.5.

In order to ensure that f has cuspidal image, one can choose a finite place v and assume

$$f(g) = f_v(g_v)f^v(g^v) \quad (18.22)$$

where $f^v \in C_c^\infty(A_G \backslash G(\mathbb{A}_F^v))$ and $f_v \in C_c^\infty(G(F_v))$ is supercuspidal. This suffices to ensure $R(f)$ has cuspidal image by Lemma 16.4.2. Unfortunately, assuming that f_v is supercuspidal limits the set of π that can be studied using the simple relative trace formula:

Lemma 18.2.7 *Let v be a finite place of F . Assume that $f \in C_c^\infty(G(F_v))$ is supercuspidal. If π_v is an irreducible admissible representation of $G(F_v)$ and $\pi_v(f) \neq 0$, then π_v is supercuspidal.*

Proof. To ease notation, we drop the subscript v , writing $F := F_v$, etc. When $Z_G(F)$ is compact the lemma follows from Exercise 16.6. We reduce to this case. This provides an opportunity to illustrate some techniques useful for considering restrictions of representations to derived subgroups. For the facts recalled in the remainder of this paragraph, see [GK82, Lemma 2.1]. The restriction of π to $Z_G(F)G^{\text{der}}(F)$ decomposes as a finite direct sum of irreducible admissible representations, all with the same multiplicity. Since $Z_G(F)$ acts via scalars on the space of π , the restriction of π to $G^{\text{der}}(F)$ also decomposes into a finite direct sum of irreducible representations, all with the same multiplicity:

$$\pi|_{G^{\text{der}}(F)} = \bigoplus_{i=1}^n \pi_i^{\oplus m}.$$

Here $\pi_i \cong \pi_j$ if and only if $i = j$. The group $G(F)$ acts via conjugation on the set of isomorphism classes of the π_i . If we let $G(F)_{\pi_1} \leq G(F)$ be the subgroup fixing the isomorphism class of π_1 , the group $G(F)/G(F)_{\pi_1}$ permutes the isomorphism classes of the π_i 's transitively.

Since supercuspidal representations are representations whose matrix coefficients are compactly supported modulo the center, π is supercuspidal if and

only if π_i is supercuspidal for all i , which is equivalent to π_i being supercuspidal for a single i due to the transitive action of $G(F)/G(F)_{\pi_1}$.

Since $\pi(f) \neq 0$, there is a φ in the space of π such that

$$\int_{G(F)} f(g)\pi(g)\varphi dg \neq 0$$

which implies that there exists an $a \in G(F)$ such that

$$\int_{G^{\text{der}}(F)} f(ga)\pi(ga)\varphi dg \neq 0.$$

Let $f_a(g) := f(ga)$. For some i , the operator $\pi_i(f_a)$ is nonzero. Our assumption that f is supercuspidal implies that f_a is supercuspidal as a function on $G^{\text{der}}(F)$. Since $Z_{G^{\text{der}}}(F)$ is compact we deduce that π_i is supercuspidal. As already mentioned, this implies that π is supercuspidal. $\square$

Thus if we assume that f_v is supercuspidal, we can only study cuspidal automorphic representations π such that π_v is supercuspidal using functions f as in (18.22). Fortunately there are now more flexible methods available to construct functions $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ such that $R(f)$ has cuspidal image. When $F = \mathbb{Q}$ and G is split and adjoint, Lindenstrass and Venkatesh [LV07] introduced one such construction. Their functions are fixed under a maximal compact subgroup at infinity, but this assumption was later removed in [Mai22]. Another construction is contained in [BPLZZ21]. Under mild assumptions on a cuspidal representation π , they prove that there exists f in an enlargement of $C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ such that $R(f)$ is defined and has cuspidal image and $\pi(f) \neq 0$. The assumptions are mild enough that they are satisfied for every cuspidal automorphic representation of $\text{GL}_n(\mathbb{A}_F)$. Interestingly, the method requires test functions that are not compactly supported at infinity, but merely Schwartz in an appropriate sense.

18.3 Products of subgroups

Assume that $H = H_1 \times H_2 \leq G \times G$, where both H_1 and H_2 are subgroups of G , and for each i the neutral component H_i° is either unipotent or reductive. Then

$$G(F)/H(F) = H_1(F) \backslash G(F)/H_2(F). \quad (18.23)$$

Assume moreover that H satisfies (18.4), and that $\chi = \chi_1 \otimes \chi_2$, where χ_i is a quasi-character of $H_i(\mathbb{A}_F)$. In this case, the formula in Theorem 18.2.5 simplifies somewhat. Let K_∞ be a maximal compact subgroup of $G(F_\infty)$.

Corollary 18.3.1 *Let $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ be a K_∞ -finite function. Assume that $R(f)$ has cuspidal image and that if the $H(\mathbb{A}_F)$ -orbit of γ intersects the support of f then γ is relatively elliptic and the H_γ -orbit of γ is closed. Then*

$$\sum_{\pi} \sum_{\varphi \in \mathcal{B}(\pi)} \mathcal{P}_{\chi_1}(\pi(f)\varphi) \mathcal{P}_{\chi_2}(\bar{\varphi}) = \sum_{[\gamma] \in H_1(F) \backslash G(F) / H_2(F)} \tau(H_\gamma) \text{RO}_\gamma(f)$$

where $\mathcal{B}(\pi)$ is an orthonormal basis of $L_{\text{cusp}}^2(\pi)$ consisting of K_∞ -finite (and hence smooth) vectors. Here the sum on the right is over relevant $[\gamma]$ that are relatively elliptic and such that the H_γ -orbit of γ is closed.

Here we are using the notation (14.4) for the period integrals occurring in the corollary. It is important to point out that only π that are both (H_1, χ_1) and (H_2, χ_2) -distinguished contribute a nonzero summand to the identity in Corollary 18.3.1. Moreover the assumption on relative unimodularity in Theorem 18.2.5 can be (and is) dropped in the case at hand. This sort of relative trace formula first appeared in work of Jacquet [Jac86] and has been applied many times in the intervening years.

Proof. Let π be a cuspidal automorphic representation of $A_G \backslash G(\mathbb{A}_F)$. By Proposition 4.4.7, K_∞ -finite vectors are dense in $L_{\text{cusp}}^2(\pi)$. Thus we can choose an orthonormal basis $\mathcal{B}(\pi)$ of this space consisting of K_∞ -finite vectors. Since f is K_∞ -finite we deduce that in the expansion

$$K_{\pi(f)}(x, y) = \sum_{\varphi \in \mathcal{B}(\pi)} \pi(f)\varphi(x) \bar{\varphi}(y) \quad (18.24)$$

of (16.13) only finitely many φ can contribute a nonzero summand. Since K_∞ -finite vectors are smooth by Proposition 4.4.7, for each $\varphi \in \mathcal{B}(\pi)$ the functions $\pi(f)\varphi \otimes \bar{\varphi}$ on $[G]^2$ are rapidly decreasing by Corollary 9.6.5. Hence Proposition 14.3.3 implies that

$$\text{rtr } \pi(f) = \sum_{\varphi \in \mathcal{B}(\pi)} \mathcal{P}_{\chi_1}(\pi(f)\varphi) \mathcal{P}_{\chi_2}(\bar{\varphi}).$$

To complete the proof, we show that a relatively semisimple element of $G(F)$ is automatically relatively unimodular. If H_1° and H_2° are both reductive and γ is relatively semisimple then H_γ has reductive neutral component by Theorem 17.2.9. Thus by Lemma 3.6.4, $H_\gamma(\mathbb{A}_F)$ is unimodular.

Now assume that H_1° is unipotent. For F -algebras R , one has that

$$H_\gamma(R) = \{(h, \gamma^{-1}h\gamma) \in H_1(R) \times H_2(R)\}.$$

The neutral component of this group is isomorphic to a subgroup of H_1° . Hence it is unipotent, which implies that $H_\gamma(\mathbb{A}_F)$ is unimodular by Exercise 18.9. $\square$

18.4 The simple trace formula

For $\gamma \in G(F)$ let

$$C_\gamma(R) := \{g \in G(R) : g\gamma g^{-1} = \gamma\} \quad (18.25)$$

be the centralizer of γ and let

$$O_\gamma(f) = \int_{C_\gamma(\mathbb{A}_F) \backslash G(\mathbb{A}_F)} f(g^{-1}\gamma g) \frac{dg}{dg_\gamma} \quad (18.26)$$

whenever this integral is well-defined. This is the usual **orbital integral** of f along the conjugacy class of γ . We say an element $\gamma \in G(F)$ is **elliptic** if the measure of $A_G C_\gamma(F) \backslash C_\gamma(\mathbb{A}_F)$ is finite with respect to an $A_G \backslash C_\gamma(\mathbb{A}_F)$ -invariant Radon measure.

The simple trace formula of Deligne and Kazhdan [DKV84, Rog83] is essentially the following special case of the general simple relative trace formula of Theorem 18.2.5:

Corollary 18.4.1 *Let $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$. Assume that $R(f)$ has cuspidal image and that any element of $G(F)$ that is $G(\mathbb{A}_F)$ -conjugate to the support of f is elliptic and semisimple. Then*

$$\sum_{\pi} m(\pi) \operatorname{tr} \pi(f) = \sum_{[\gamma]} \tau(C_\gamma) O_\gamma(f), \quad (18.27)$$

where the sum on the left is over isomorphism classes of cuspidal automorphic representations of $A_G \backslash G(\mathbb{A}_F)$, $m(\pi)$ is the multiplicity of π in the cuspidal spectrum of $L^2([G])$ and the sum on the right is over elliptic semisimple conjugacy classes in $G(F)$.

As in the statement of Theorem 18.2.5, some care must be taken with measures in order for the identity above to be correct. We choose a Haar measure dg' on $A_G \backslash G(\mathbb{A}_F)$ and a Haar measure dg'_γ on $A_G \backslash C_\gamma(\mathbb{A}_F)$ for each semisimple γ . Then we use dg' to define the operator $\pi(f)$ (and hence its trace) and use dg'_γ to define $\tau(C_\gamma)$. Finally, we normalize the Radon measure $\frac{dg}{dg_\gamma}$ on $C_\gamma(\mathbb{A}_F) \backslash G(\mathbb{A}_F)$ used to define $O_\gamma(f)$ so that it corresponds to $\frac{dg'}{dg'_\gamma}$ under the canonical isomorphism

$$C_\gamma(\mathbb{A}_F) \backslash G(\mathbb{A}_F) \xrightarrow{\sim} (A_G \backslash C_\gamma(\mathbb{A}_F)) \backslash (A_G \backslash G(\mathbb{A}_F)). \quad (18.28)$$

Proof. In Theorem 18.2.5, we take χ to be trivial and take H to be the diagonal copy of G inside $G \times G$. Then (18.4) is satisfied, $A = A_{G,H} = \Delta(A_G)$, and a class is just a conjugacy class in $G(F)$. Thus $\operatorname{RO}_\gamma(f) = O_\gamma(f)$.

All classes are trivially relevant, and $\gamma \in G(F)$ is semisimple in the usual sense if and only if it is relatively semisimple by [Ste65, Corollary 6.13]. Thus C_γ° is reductive for all semisimple γ by Theorem 17.2.9. This implies

all semisimple classes are relatively unimodular by Lemma 3.6.4. Finally, it is clear that γ is relatively elliptic if and only if it is elliptic. Thus the geometric side of the trace formula of Theorem 18.2.5 is equal to the right hand of (18.27).

As for the left-hand side,

$$\mathrm{rtr} \pi(f) := \int_{[G]} K_{\pi(f)}(g, g) dg = m(\pi) \mathrm{tr} \pi(f)$$

by the discussion around (16.10). Thus the spectral side of the trace formula of Theorem 18.2.5 is equal to the left-hand side of (18.27). $\square$

To understand Corollary 18.4.1 it is helpful to have some qualitative understanding of orbital integrals. Since orbital integrals factor as explained in Lemma 17.8.3, it suffices to consider local orbital integrals. Thus let v be a non-Archimedean place of F which we omit from notation, writing $F := F_v$. Let $K \leq G(F)$ be a compact open subgroup and let $\mathbb{1}_K$ be its characteristic function. Then for semisimple γ one has that

$$\begin{aligned} \mathrm{O}_\gamma(\mathbb{1}_K) &:= \int_{C_\gamma(F) \backslash G(F)} \mathbb{1}_K(g^{-1}\gamma g) \frac{dg}{dg_\gamma} \\ &= \sum_{\substack{C_\gamma(F)gK \in C_\gamma(F) \backslash G(F)/K \\ g^{-1}\gamma g \in K}} \mathrm{meas}_{\frac{dg}{dg_\gamma}}(C_\gamma(F) \backslash C_\gamma(F)gK) \\ &= \sum_{\substack{C_\gamma(F)gK \in C_\gamma(F) \backslash G(F)/K \\ g^{-1}\gamma g \in K}} \frac{\mathrm{meas}_{dg}(K)}{\mathrm{meas}_{dg_\gamma}(C_\gamma(F) \cap gKg^{-1})}. \end{aligned} \tag{18.29}$$

For general linear groups when K is hyperspecial this can be made much more explicit. We refer to [Kot05, §5] for the computation and many similar computations when $G = \mathrm{GL}_2$ and [Yun13] for the case when $G = \mathrm{GL}_n$. In these cases orbital integrals can be evaluated in fairly elementary terms. However, the situation is wild when one ventures beyond the general linear group [Hal94].

18.5 The simple twisted trace formula

Let θ be a finite order automorphism of a reductive group G over number field F . One then has a right action of G on itself by θ -conjugation, given on points in an F -algebra R by

$$\begin{aligned} G(R) \times G(R) &\longrightarrow G(R) \\ (x, g) &\longmapsto g^{-1}x\theta(g). \end{aligned} \tag{18.30}$$

When θ is the identity, this is simply conjugation. Following the lead of §18.4, we let

$$C_\gamma^\theta(R) := \{g \in G(R) : g^{-1}\gamma\theta(g) = \gamma\} \quad (18.31)$$

be the θ -**centralizer** of γ . Write

$$A_G^\theta := \{a \in A_G : \theta(a) = a\}.$$

We define the **twisted orbital integral**

$$\mathrm{TO}_\gamma(f) := \int_{(A_G^\theta \backslash C_\gamma^\theta(\mathbb{A}_F)) \backslash (A_G \backslash G(\mathbb{A}_F))} f(g^{-1}\gamma\theta(g)) \frac{dg}{dg_\gamma}. \quad (18.32)$$

We say γ is θ -**semisimple** if its G -orbit under θ -conjugation is closed and θ -**elliptic** if

$$\tau(C_\gamma^\theta) := \mathrm{meas}_{dg_\gamma} (A_G^\theta C_\gamma^\theta(F) \backslash C_\gamma^\theta(\mathbb{A}_F))$$

is finite.

We have an operator

$$\begin{aligned} I_\theta : L^2([G]) &\longrightarrow L^2([G]) \\ \varphi &\longmapsto (g \mapsto \varphi(\theta^{-1}(g))) \end{aligned} \quad (18.33)$$

that preserves $L_{\mathrm{cusp}}^2([G])$. For each $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ the composite

$$R(f) \circ I_\theta : L^2([G]) \longrightarrow L^2([G])$$

has kernel function $K_f(g, \theta(h))$. Since $R_{\mathrm{cusp}}(f) := R(f)|_{L_{\mathrm{cusp}}^2([G])}$ is of trace class by Theorem 9.1.2, the same is true of $R_{\mathrm{cusp}}(f) \circ I_\theta$.

For each cuspidal automorphic representation π , the operator $R(f) \circ I_\theta$ restricts to induce an operator

$$\pi(f) \circ I_\theta : L_{\mathrm{cusp}}^2(\pi^\theta) \longrightarrow L_{\mathrm{cusp}}^2(\pi), \quad (18.34)$$

where

$$\pi^\theta(g) := \pi(\theta(g)).$$

The spectral counterpart of a twisted orbital integral is the **twisted trace** of $\pi(f)$ with respect to θ :

$$\mathrm{tr}(\pi(f) \circ I_\theta) := \int_{[G]} K_{\pi(f)}(g, \theta(g)) dg. \quad (18.35)$$

We observe that if π occurs with multiplicity greater than 1 then this is incorporated into the twisted trace. In contrast, we separated the trace and the multiplicity in the statement of Corollary 18.4.1. The twisted trace $\mathrm{tr}(\pi(f) \circ I_\theta)$ vanishes unless $\pi \cong \pi^\theta$ by Exercise 14.2. If $\pi \cong \pi^\theta$ then $\pi(f) \circ I_\theta$

is an endomorphism of $L^2_{\text{cusp}}(\pi)$ and $\int_{[G]} K_{\pi(f)}(g, \theta(g)) dg$ is the trace of the trace class operator (18.34) by [Bri91, Corollary 3.2].

The following is a simple version of the twisted trace formula:

Corollary 18.5.1 *Let $f \in C_c^\infty(A_G \backslash G(\mathbb{A}_F))$. Assume that $R(f)$ has cuspidal image and that any element of $G(F)$ that is θ -conjugate under $G(\mathbb{A}_F)$ to the support of f is θ -elliptic and θ -semisimple. Then*

$$\sum_{\pi} \text{tr}(\pi(f) \circ I_{\theta}) = \sum_{[\gamma]} \tau(C_{\gamma}^{\theta}) \text{TO}_{\gamma}(f), \quad (18.36)$$

where the sum over π is over isomorphism classes of cuspidal automorphic representations of $A_G \backslash G(\mathbb{A}_F)$ such that $\pi \cong \pi^{\theta}$ and the sum on the right is over θ -conjugacy classes in $G(F)$ that are θ -elliptic and θ -semisimple.

Proof. In Theorem 18.2.5, we take χ to be trivial and take H to be the twisted diagonal copy of G in $G \times G$; that is, for F -algebras R ,

$$H(R) = \{(g, \theta(g)) : g \in G(R)\}.$$

In this setting, classes are just θ -conjugacy classes,

$$A_{G,H} = \{(a, \theta(a)) : a \in A_G\}, \quad \text{and} \quad A = \{(a, a) : a = \theta(a), a \in A_G\}.$$

In particular (18.4) is valid and the relative orbital integral $\text{RO}_{\gamma}(f)$ is $\text{TO}_{\gamma}(f)$.

All classes are trivially relevant, all θ -semisimple (resp. θ -elliptic) elements are relatively semisimple (resp. relatively elliptic), and all θ -semisimple elements are relatively unimodular by Theorem 17.2.9. $\square$

The fact that the twisted trace formula isolates representations satisfying $\pi \cong \pi^{\theta}$ was the primary motivation for its development. We explain why this fact is useful in §19.4 below. The twisted trace formula was first introduced by Saito [Sai75] and Shintani [Shi79]. For a completely general twisted trace formula, we refer to [LW13]. A relative version of the simple twisted trace formula (which is another special case of Theorem 18.2.5) was introduced in [Hah09].

18.6 A variant

It has proven to be beneficial to work with a variant of the simple relative trace formula of Theorem 18.2.5. In this section we discuss the variant in impressionistic terms to aid the reader when it is encountered in the literature.

Let X be a smooth affine scheme of finite type over the global field F equipped with a (right) action of the reductive F -group G . Since X is smooth, $C_c^\infty(X(\mathbb{A}_F))$ is defined (see §17.8). We assume for simplicity that $A_G = 1$.

Let $H \leq G$ be a subgroup such that H° is the direct product of a reductive group and a unipotent group and let $\chi : H(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ be a character trivial on $H(F)$. For each $x \in X(F)$ with closed orbit under G , the stabilizer G_x has reductive neutral component by Theorem 17.2.9. For x with closed orbit under G consider the subgroup

$$H_x := G_x \times H \leq G \times G. \quad (18.37)$$

The character χ extends to a character

$$\begin{aligned} G_x(\mathbb{A}_F) \times H(\mathbb{A}_F) &\longrightarrow \mathbb{C}^\times \\ (g, h) &\longmapsto \chi(h) \end{aligned}$$

which we will continue to denote by χ . We are now in the setting of §18.2 and we can form the relative trace

$$\mathrm{rtr}_{H_x, \chi} \pi(f)$$

for any $f \in C_c^\infty(G(\mathbb{A}_F))$.

Now let $\phi \in C_c^\infty(X(\mathbb{A}_F))$. Assume that if $x \in X(F)$ is in the $H(\mathbb{A}_F)$ -orbit of the support of ϕ then x has closed G -orbit. We assume moreover that

$$\int_{H(F) \backslash H(\mathbb{A}_F)} \sum_{x \in X(F)} |\phi(xh) \chi^{-1}(h)| dh < \infty.$$

For conditions ensuring convergence, see Theorem 17.8.7. Thus the integral

$$\int_{H(F) \backslash H(\mathbb{A}_F)} \sum_{x \in X(F)} \phi(xh) \chi^{-1}(h) dh \quad (18.38)$$

is well-defined.

Assume that for each $x \in X(F)$, we can choose $f_x \in C_c^\infty(G(\mathbb{A}_F))$ such that

$$\phi(xy) = \int_{G_x(\mathbb{A}_F)} f_x(g^{-1}y) dg \quad (18.39)$$

for $y \in G(\mathbb{A}_F)$ (this is not always possible). Then

$$\begin{aligned} &\int_{H(F) \backslash H(\mathbb{A}_F)} \sum_{x \in X(F)} \phi(xh) \chi^{-1}(h) dh \\ &= \sum_{x \in X(F)/G(F)} \int_{H_x(F) \backslash H_x(\mathbb{A}_F)} \sum_{\gamma \in G(F)} f_x(h_1^{-1} \gamma h_2) \chi^{-1}(h_2) dh_1 dh_2. \end{aligned} \quad (18.40)$$

If $R(f_x) : L^2([G]) \rightarrow L^2([G])$ has cuspidal image for all $x \in X(F)/G(F)$ then (18.40) is

$$\sum_{x \in X(F)/G(F)} \sum_{\pi} \text{rtr}_{H_x, \chi}(f_x) \quad (18.41)$$

by Theorem 16.2.6. Here the sum is over cuspidal automorphic representations of $G(\mathbb{A}_F)$. In particular studying (18.38) amounts to studying cuspidal automorphic representations π of $G(\mathbb{A}_F)$ such that $\pi \times \pi^\vee$ is (H_x, χ) -distinguished as x varies.

This method of packaging distinction problems together has proven useful in practice. Typically one assumes that X is a spherical variety for G due to the connection with the conjectures of Sakellaridis and Venkatesh (see §14.4). Let us discuss the case of symmetric varieties in the sense of §14.5. Let $\sigma : G \rightarrow G$ be an automorphism of order 2, and for F -algebras R let

$$X(R) := \{g \in G(R) : g^\sigma = g^{-1}\}.$$

Then we have an action

$$\begin{aligned} X(R) \times G(R) &\longrightarrow X(R) \\ (q, g) &\longmapsto g^{-1}q\sigma(g). \end{aligned}$$

The stabilizer of the identity $I \in X(R)$ is just the subgroup $G^\sigma \leq G$ fixed by σ , so this construction places the study of distinction by a given symmetric subgroup of G into a family in a natural manner. The invariant theory of X was studied in [Ric82] and this construction was first studied from the point of view of automorphic representation theory by Jacquet (see [Jac97] for a survey).

We point out that even when it is not possible to choose f_x as in (18.39), it is still possible to expand the function

$$\sum_{x \in X(F)} \phi(xg) \in C^\infty([G])$$

spectrally. More or less it will give something like (18.41), but the f_x involved will not necessarily be compactly supported.

In fact in practice it is more useful to consider cases in which one cannot choose f_x as in (18.39). To explain this we discuss a special case. Let E/F be a quadratic extension, let σ be the generator of $\text{Gal}(E/F)$, and let $G = \text{Res}_{E/F} \text{GL}_n$. Of course $A_G \neq 1$, but the discussion above can be modified to treat this setting. Temporarily let

$$X(R) := \{x \in M_n(E \otimes_F R) : \sigma(x) = x^t\}.$$

It is the space of Hermitian matrices with respect to E/F . Then there is a right action of G on X given on points in an F -algebra R by

$$\begin{aligned} a : X(R) \times G(R) &\longrightarrow X(R) \\ (x, g) &\longmapsto g^t x \sigma(g). \end{aligned} \tag{18.42}$$

Let

$$X^\circ = X \cap \text{Res}_{E/F} \text{GL}_n.$$

The stabilizer of any element of $X^\circ(F)$ is a unitary group. Suppose that we are given $x \in X^\circ(F)$. Let $\phi \in C_c^\infty(X^\circ(\mathbb{A}_F))$. The orbit of x under G is all of X° . By Theorem 17.7.1 the orbit map

$$a(x, \cdot) : G_x(\mathbb{A}_F) \backslash G(\mathbb{A}_F) \longrightarrow X^\circ(\mathbb{A}_F)$$

is proper and hence we have a well-defined pull-back map

$$a(x, \cdot)^* : C_c^\infty(X^\circ(\mathbb{A}_F)) \longrightarrow C_c^\infty(G_x(\mathbb{A}_F) \backslash G(\mathbb{A}_F)).$$

For a given $\phi \in C_c^\infty(X^\circ(\mathbb{A}_F))$ we can then take f_x to be any element of $C_c^\infty(G(\mathbb{A}_F))$ such that

$$\int_{G_x(\mathbb{A}_F)} f(g^{-1}h) dg = (a(x, \cdot)^* \phi)(h).$$

On the other hand, if $\phi \in C_c^\infty(X(\mathbb{A}_F))$, then it is not in general possible to choose f_x so that (18.39) holds (see Exercise 18.11). In this setting Jacquet took H to be the unipotent radical of a Borel subgroup of G and χ to be a generic character and then studied (18.38) in order to prove Theorem 14.5.2 [Jac05b, Jac10] (see also [Ngô99a, Ngô99b]). He worked with functions in $C_c^\infty(X(\mathbb{A}_F))$, not $C_c^\infty(X^\circ(\mathbb{A}_F))$. The fact that X is a linear space, and hence admits a Fourier transform, turns out to be crucial. From the point of view of the theory of spherical varieties, replacing X° by X amounts to replacing a homogeneous spherical variety with a spherical embedding in the sense of Luna and Vust [LV83, Kno91].

Thus studying smooth affine varieties X equipped with actions of reductive groups G that are not homogeneous leads to new insights. One might ask if it is sensible to also consider nonsmooth X . At least if X is spherical this is indeed sensible. Moreover pursuing this line of inquiry leads to exciting new avenues towards Langlands functoriality that are the subject of ongoing research. We refer the reader to [BK00, BK02, BNS16, Get20, GH20, GHL21, GL19, Gu21, Hsu21, JLZ20, Ngô14, Ngô20, Sak12, SW22].

18.7 The Petersson-Bruggeman-Kuznetsov formula

Given two nonzero integers m and n and a positive integer c , the classical **Kloosterman sum** is defined as

$$K(m, n; c) := \sum_{\substack{1 \leq a \leq c \\ (a, c) = 1}} e^{2\pi i(am + \bar{a}n)/c},$$

where $a\bar{a} \equiv 1 \pmod{c}$. Petersson [Pet32] expressed a certain sum of Fourier coefficients of holomorphic weight k cusp forms in terms of the Bessel function J_{k-1} and Kloosterman sums. Historically, this was the first relative trace formula, although cosmetically Petersson's proof appears different. It involved constructing a Poincaré series and computing its coefficients in two different manners rather than integrating an automorphic kernel function. Bruggeman [Bru78] and Kuznetsov [Kuz80] later showed how to incorporate Maass forms into the formula. Traditionally, the formula is called the **Kuznetsov formula**.

Using the formula, Kuznetsov was able to prove estimates for the sum

$$\sum_{c=1}^{\infty} \frac{K(m, n; c)}{c} \phi\left(\frac{2\pi\sqrt{mn}}{c}\right)$$

for suitable $\phi \in C^\infty(\mathbb{R}_{>0})$. We refer to [IK04, Chapter 16] for more details. There are many other analytic number theory papers that use the formula, for example, [Iwa02, GS83]. It has also played a role in work on Langlands' beyond endoscopy proposal [Her11a, Her12, Her16, Sak22, Sar, Ven04].

Following Jacquet, we will derive the Kuznetsov formula as a relative trace formula (see [KL08, KL13] for a nice exposition for GL_2 and [Ye00] for a slightly different take on the higher rank case). The beautiful fact about this trace formula is that the general simple relative trace formula (Theorem 18.2.5) together with a small additional argument for the spectral side can be used to derive it in complete generality, with no additional assumptions on the test functions involved. We state this precisely in Theorem 18.9.2 below. In this sense, it is analytically the simplest trace formula. The Petersson-Kuznetsov formula remains an important tool in analytic number theory and has not been fully exploited in higher rank (see [BB20] for example for recent results for GL_3). We have therefore structured our treatment of the formula with applications to analytic number theory in mind.

We now set notation. Let $N \leq \mathrm{GL}_r$ be the unipotent radical of the Borel B of upper triangular matrices and let $T \leq B$ be the maximal torus of diagonal matrices. For this trace formula, we take

$$H := N \times N \leq \mathrm{GL}_r \times \mathrm{GL}_r.$$

Thus we have a right action

$$\mathrm{GL}_r \times N \times N \longrightarrow \mathrm{GL}_r$$

given on points by $(n_1, n_2) \cdot \gamma = n_1^{-1} \gamma n_2$. For a global or local field F , we denote the stabilizer of $\gamma \in \mathrm{GL}_r(F)$ under this action by

$$(N \times N)_\gamma := H_\gamma \leq N \times N.$$

Thus for F -algebras R , one has that

$$(N \times N)_\gamma(R) := \{(n, \gamma^{-1} n \gamma) : n \in N(R) \cap \gamma N(R) \gamma^{-1}\}. \quad (18.43)$$

Lemma 18.7.1 *Every element $\gamma \in \mathrm{GL}_r(F)$ is relatively unimodular, relatively elliptic, and relatively semisimple.*

Proof. The stabilizer $(N \times N)_\gamma$ is unipotent, hence γ is relatively unimodular by Exercise 18.9. Moreover the fact that $(N \times N)_\gamma$ is unipotent implies that $[(N \times N)_\gamma]$ is compact by Theorem 2.6.3; we deduce that γ is relatively elliptic. Finally Theorem 17.2.10 implies that γ is relatively semisimple. $\square$

The partial determinants

$$\det_i : \mathrm{GL}_r \longrightarrow \mathbb{G}_a$$

given on points by

$$\det_i \left(\begin{smallmatrix} * & * \\ A_i & * \end{smallmatrix} \right) = \det(A_i)$$

are invariant under the action of $N \times N$ for $1 \leq i \leq r$. Here A_i is an $i \times i$ matrix.

For every $w \in \mathrm{GL}_r(F)$ we define a morphism (of schemes, not group schemes)

$$\Upsilon_w : \mathbb{G}_m^r \longrightarrow \mathrm{GL}_r \quad (18.44)$$

by

$$\Upsilon_w(c_1, \dots, c_r) = \begin{pmatrix} c_r/c_{r-1} & & & \\ & c_{r-1}/c_{r-2} & & \\ & & \ddots & \\ & & & c_2/c_1 \\ & & & & c_1 \end{pmatrix} w. \quad (18.45)$$

By the Bruhat decomposition (10.12), one has

$$\mathrm{GL}_r(F) = \coprod_{w \in W(\mathrm{GL}_r, T)(F)} N(F)T(F)wN(F)$$

and thus every element of $N(F) \backslash \mathrm{GL}_r(F) / N(F)$ has a representative of the form $\Upsilon_w(c)$ for some representative w of an element of $W(\mathrm{GL}_r, T)(F)$ and

some $c \in (F^\times)^r$. The action of $N \times N$ on GL_r extends in the obvious manner to an action of $B \times B$ on GL_r . The $B \times B$ -orbit of a given representative w for an element of $W(\mathrm{GL}_r, T)(F)$ is called a **Bruhat cell**. Let

$$w_0 = \begin{pmatrix} & & 1 \\ & \ddots & \\ 1 & & \end{pmatrix}$$

be (the standard representative of) the long Weyl element. The $B \times B$ -orbit of w_0 is the unique open Bruhat cell [Mil17, Theorem 21.73].

Lemma 18.7.2 *The morphism Υ_{w_0} induces a bijection*

$$\Upsilon_{w_0} : (F^\times)^r \longrightarrow N(F) \backslash B(F)w_0B(F)/N(F).$$

Moreover, the stabilizer $(N \times N)_\gamma$ of any $\gamma \in B(F)w_0B(F)$ is trivial.

This lemma implies in particular that all elements of the open Bruhat cell are relevant with respect to any quasi-character in both the local and the global setting.

Proof. It is clear that any element of $N(F) \backslash B(F)w_0B(F)/N(F)$ is in the image of the map Υ_{w_0} . On the other hand, the partial determinants $\det_i$ are all $N \times N$ -invariant and

$$\det_i(\Upsilon_{w_0}(c)) = \pm c_i,$$

where c_i is the i th entry of c and the sign $\pm$ depends only on i and r . Thus Υ_{w_0} is injective and induces a bijection as claimed.

For the claim on the triviality of stabilizers, we can assume $\gamma = tw_0$ for $t \in T(F)$ by what we have already proven. Then if $(n_1, n_2) \in (N \times N)_\gamma(F)$, one has that

$$w_0^{-1}t^{-1}n_1tw_0 = n_2. \tag{18.46}$$

The left-hand side is in the unipotent radical of the Borel opposite to B , so both sides must be equal to I_r . $\square$

18.8 Kloosterman integrals

The geometric side of the Kuznetsov formula is a sum of Kloosterman integrals. We define Kloosterman integrals in this section and discuss their basic properties. The reader unfamiliar with integrals over totally disconnected groups may want to consult Appendix C and complete the exercises of that appendix before reading this section.

We define a morphism

$$\nu : \mathbb{G}_m^{r-1} \longrightarrow T \quad (18.47)$$

via

$$\nu(m) := \begin{pmatrix} \prod_{i=1}^{r-1} m_i & & & \\ & \ddots & & \\ & & m_{r-2}m_{r-1} & \\ & & & m_{r-1} \\ & & & & 1 \end{pmatrix}. \quad (18.48)$$

We note that

$$\nu(m) \begin{pmatrix} 1 & x_1 & & * \\ & \ddots & \ddots & \\ & & 1 & x_{r-1} \\ & & & 1 \end{pmatrix} \nu(m)^{-1} = \begin{pmatrix} 1 & m_1 x_1 & & * \\ & \ddots & \ddots & \\ & & 1 & m_{r-1} x_{r-1} \\ & & & 1 \end{pmatrix}. \quad (18.49)$$

Let F be a global field and fix a nontrivial character

$$\psi : F \backslash \mathbb{A}_F \longrightarrow \mathbb{C}^\times.$$

For $m = (m_i) \in F^{r-1}$, we set

$$\psi_m \begin{pmatrix} 1 & x_1 & & * \\ & \ddots & \ddots & \\ & & 1 & x_{r-1} \\ & & & 1 \end{pmatrix} := \psi \left(\sum_{i=1}^{r-1} m_i x_i \right). \quad (18.50)$$

We also use the obvious local analogue of this notation.

We thus obtain a character

$$\psi_{m_1} \otimes \bar{\psi}_{m_2} : N(\mathbb{A}_F) \times N(\mathbb{A}_F) \longrightarrow \mathbb{C}^\times \quad (18.51)$$

trivial on $N(F) \times N(F)$. We now discuss which elements of $N(F) \backslash \mathrm{GL}_r(F) / N(F)$ are $\psi_{m_1} \otimes \bar{\psi}_{m_2}$ -relevant. One has that

$$S_r \xrightarrow{\sim} W(\mathrm{GL}_r, T)(F).$$

The map sends a permutation $\sigma \in S_r$ to the class of the permutation matrix w_σ such that w_σ has nonzero entries in the $(i, \sigma(i))$ positions.

The following lemma was attributed to Piatetski-Shapiro in [Fri87, p. 175].

Lemma 18.8.1 *Assume $m_1, m_2 \in (F^\times)^{r-1}$. A class in $N(F) \backslash \mathrm{GL}_r(F) / N(F)$ is $\psi_{m_1} \otimes \bar{\psi}_{m_2}$ -relevant only if it is a subset of $N(F)T(F)w_\sigma N(F)$ where $\sigma^{-1}(i) < \sigma^{-1}(i+1)$ implies $\sigma^{-1}(i+1) - \sigma^{-1}(i) = 1$ for $1 \leq i < r$. In other*

words, a class contained in $N(F)T(F)w_\sigma N(F)$ is relevant only if

$$w_\sigma = \begin{pmatrix} & & I_{r_1} \\ & I_{r_2} & \\ & \ddots & \\ I_{r_k} & & \end{pmatrix}$$

where $\sum_{i=1}^k r_i = r$.

Proof. Consider $u := I + \sum_{i=1}^{r-1} x_i E_{i,i+1}$ where $x_i \in \mathbb{A}_F$ and $E_{i,j}$ is the elementary matrix with a 1 in the (i, j) position and zeros elsewhere. One has that

$$w_\sigma u w_\sigma^{-1} = I + \sum_{i=1}^{r-1} x_i E_{\sigma^{-1}(i), \sigma^{-1}(i+1)}.$$

Suppose that $\sigma^{-1}(i) < \sigma^{-1}(i+1)$. Then

$$\{(I + x_i E_{\sigma^{-1}(i), \sigma^{-1}(i+1)}, I + x_i E_{i,i+1}) : x_i \in \mathbb{A}_F\} \leq (N \times N)_{w_\sigma}(\mathbb{A}_F).$$

In order for $\psi_{m_1} \otimes \bar{\psi}_{m_2}$ to be trivial on this subgroup of $(N \times N)_{w_\sigma}(\mathbb{A}_F)$, it is necessary that $\sigma^{-1}(i+1) - \sigma^{-1}(i) = 1$. This proves that $N(F)w_\sigma N(F)$ is $\psi_{m_1} \otimes \bar{\psi}_{m_2}$ -relevant only if $\sigma^{-1}(i) < \sigma^{-1}(i+1)$ implies $\sigma^{-1}(i+1) - \sigma^{-1}(i) = 1$ for $1 \leq i < r$. On the other hand, for $c \in (F^\times)^{r-1}$ and $a \in F^\times$ the element $av(c)w_\sigma$ is $\psi_{m_1} \otimes \bar{\psi}_{m_2}$ -relevant only if w_σ is $\psi_{m_1 c} \otimes \bar{\psi}_{m_2}$ -relevant by a change of variables. Here

$$m_1 c := (m_{11}c_1, \dots, m_{1(r-1)}c_{r-1}),$$

where $m_1 = (m_{11}, \dots, m_{1(r-1)})$ and $c = (c_1, \dots, c_{r-1})$. Thus we deduce that a general element of $N(F)T(F)w_\sigma N(F)$ is $\psi_{m_1} \otimes \bar{\psi}_{m_2}$ -relevant only if $\sigma^{-1}(i) < \sigma^{-1}(i+1)$ implies $\sigma^{-1}(i+1) - \sigma^{-1}(i) = 1$ for $1 \leq i < r$. $\square$

For $f \in C_c^\infty(A_{\mathrm{GL}_r} \backslash \mathrm{GL}_r(\mathbb{A}_F))$ and $\psi_{m_1} \otimes \bar{\psi}_{m_2}$ -relevant $\gamma \in \mathrm{GL}_r(F)$, we define the orbital integral

$$\mathrm{RO}_\gamma(f, m_1, m_2) := \int f(n_1^{-1} \gamma n_2) \psi_{m_1}(n_1) \bar{\psi}_{m_2}(n_2) dn_1 dn_2, \quad (18.52)$$

where the integral is over

$$(N \times N)_\gamma(\mathbb{A}_F) \backslash N(\mathbb{A}_F) \times N(\mathbb{A}_F).$$

By Proposition 18.2.4 and Lemma 18.7.1, the integral in (18.52) is absolutely convergent. It is sometimes called a **Kloosterman integral**. We use the analogous notation and terminology in the local setting.

Proposition 18.8.2 *One has that*

$$\begin{aligned} & \int_{[N]^2} K_f(n_1, n_2) \psi_{m_1}(n_1) \overline{\psi}_{m_2}(n_2) dn_1 dn_2 \\ &= \sum_{[\gamma] \in N(F) \backslash \mathrm{GL}_r(F) / N(F)} \tau((N \times N)_\gamma) \mathrm{RO}_\gamma(f, m_1, m_2) \end{aligned}$$

where the sum is over $\psi_{m_1} \otimes \overline{\psi}_{m_2}$ -relevant classes. The sum is absolutely convergent.

Proof. By Lemma 18.7.1, every $\gamma \in \mathrm{GL}_r(F)$ is relatively unimodular and relatively elliptic. Moreover the H_r -orbit of any $\gamma \in \mathrm{GL}_r(F)$ is obviously closed. Thus the proposition follows from the proof of Theorem 18.2.5. $\square$

In the remainder of this section we study non-Archimedean local relative orbital integrals that are the local analogues of the global relative orbital integrals discussed above. Thus we fix a non-Archimedean place v of F which we omit from notation, writing $F := F_v$, $\psi := \psi_v$, etc. Let $\mathcal{O}_F$ be the ring of integers of F . We assume that ψ is unramified in the sense that $\psi|_{\mathcal{O}_F^\times}$ is trivial and $\psi|_{\varpi^{-1}\mathcal{O}_F}$ is nontrivial. Here ϖ is a choice of uniformizer.

We begin with some easy observations (compare [Fri87, Ste87]):

Lemma 18.8.3 *The Kloosterman integral $\mathrm{RO}_\gamma(\mathbb{1}_{\mathrm{GL}_r(\mathcal{O}_F)}, m_1, m_2)$ is nonzero only if $\det \gamma \in \mathcal{O}_F^\times$ and $\det_i(\gamma) \in \mathcal{O}_F$ for $1 \leq i < r$.*

Proof. This follows from the fact that the partial determinants $\det_i$ are invariant under the action of $N \times N$ on GL_r . $\square$

Lemma 18.8.4 *Let $f^\vee(g) := f(g^{-1})$. Then one has that*

$$\mathrm{RO}_{\Upsilon_w(c)}(f, m_1, m_2) = \mathrm{RO}_{\Upsilon_w(c)^{-1}}(f^\vee, -m_2, -m_1).$$

Proof. We compute that

$$\begin{aligned} & \mathrm{RO}_{\Upsilon_w(c)}(f, m_1, m_2) \\ &= \int f(n_1^{-1} \Upsilon_w(c) n_2) \psi_{m_1}(n_1) \overline{\psi}_{m_2}(n_2) dn_1 dn_2 \\ &= \int f^\vee((n_1^{-1} \Upsilon_w(c) n_2)^{-1}) \psi_{m_1}(n_1) \overline{\psi}_{m_2}(n_2) dn_1 dn_2 \\ &= \int f^\vee(n_2^{-1} \Upsilon_w(c)^{-1} n_1) \psi_{m_1}(n_1) \overline{\psi}_{m_2}(n_2) dn_1 dn_2, \end{aligned} \tag{18.53}$$

where the integrals are over $(N \times N)_{\Upsilon_w(c)}(F) \backslash N(F) \times N(F)$. We have a bijection

$$\begin{aligned} (N \times N)_{\Upsilon_w(c)}(F) \backslash N(F) \times N(F) &\longrightarrow (N \times N)_{\Upsilon_w(c)^{-1}}(F) \backslash N(F) \times N(F) \\ (n_1, n_2) &\longmapsto (n_2, n_1). \end{aligned}$$

Changing variables using this bijection we see that (18.53) is equal to

$$\begin{aligned} & \int f^\vee(n_1^{-1}\Upsilon_w(c)^{-1}n_2)\overline{\psi}_{m_2}(n_1)\psi_{m_1}(n_2)dn_1dn_2 \\ & = \text{RO}_{\Upsilon_w(c)^{-1}}(f^\vee, -m_2, -m_1) \end{aligned}$$

where the integral is now over $(N \times N)_{\Upsilon_w(c)^{-1}}(F) \backslash N(F) \times N(F)$. $\square$

Under suitable assumptions Kloosterman integrals on GL_r can be written in terms of Kloosterman integrals on GL_{r-1} .

Lemma 18.8.5 *Assume that $c = (c_1, c_2, \dots, c_r) \in (\mathcal{O}_F \cap F^\times)^r$ and that $c_1, c_r \in \mathcal{O}_F^\times$. Let*

$$\begin{aligned} \tilde{c} &:= (c_2/c_1, c_3/c_1, \dots, c_r/c_1) \in (\mathcal{O}_F \cap F^\times)^{r-1}, \\ m_1 &:= (m_{11}, m_{12}, \dots, m_{1(r-1)}) \in F^{r-1}, \\ m_2 &:= (m_{21}, m_{22}, \dots, m_{2(r-1)}) \in F^{r-1}, \\ \tilde{m}_1 &:= (m_{11}, m_{12}, \dots, m_{1(r-2)}) \in F^{r-2}, \\ \tilde{m}_2 &:= (m_{22}, m_{23}, \dots, m_{2(r-1)}) \in F^{r-2}. \end{aligned}$$

Then one has that

$$\begin{aligned} & \text{RO}_{\Upsilon_{w_0}(c)}(\mathbb{1}_{\text{GL}_r(\mathcal{O}_F)}, m_1, m_2) \\ & = \mathbb{1}_{\mathcal{O}_F}(m_{1(r-1)}) \mathbb{1}_{\mathcal{O}_F}(m_{21}) \text{RO}_{\Upsilon_{\tilde{w}_0}(\tilde{c})}(\mathbb{1}_{\text{GL}_{r-1}(\mathcal{O}_F)}, \tilde{m}_1, \tilde{m}_2), \end{aligned}$$

where $\tilde{w}_0 \in \text{GL}_{r-1}(F)$ is the standard representative for the long Weyl element.

Proof. To ease notation, let C be the $(r-1) \times (r-1)$ matrix given by

$$C := \begin{pmatrix} & & c_r/c_{r-1} \\ & \ddots & \\ c_2/c_1 & & \end{pmatrix}.$$

For $n_1, n_2 \in N_{r-1}(F)$, $\alpha_1, \alpha_2 \in F^{r-1}$ (the latter viewed as column vectors) we compute

$$\begin{pmatrix} n_1 & \alpha_1 \\ & 1 \end{pmatrix} \begin{pmatrix} & C \\ c_1 & \end{pmatrix} \begin{pmatrix} 1 & \alpha_2^t \\ & n_2 \end{pmatrix} = \begin{pmatrix} c_1 \alpha_1 & n_1 C \\ c_1 & \end{pmatrix} \begin{pmatrix} 1 & \alpha_2^t \\ & n_2 \end{pmatrix} = \begin{pmatrix} c_1 \alpha_1 & \tilde{C} \\ c_1 & c_1 \alpha_2^t \end{pmatrix}. \quad (18.54)$$

Here

$$\tilde{C} = c_1 \alpha_2 \alpha_1^t + n_1 C n_2$$

is an $(r-1) \times (r-1)$ matrix, $c_1 \alpha_1$ is a column vector, and $c_1 \alpha_2^t$ is a row vector. Since we assumed that $c_1 \in \mathcal{O}_F^\times$ and $c_r \in \mathcal{O}_F^\times$, (18.54) is in $\text{GL}_r(\mathcal{O}_F)$ if and only if $\alpha_1, \alpha_2 \in \mathcal{O}_F^{r-1}$ and $n_1 C n_2 \in \text{GL}_{r-1}(\mathcal{O}_F)$. The lemma follows. $\square$

Proposition 18.8.6 *Assume that $c = (c_1, \dots, c_r) \in (\mathcal{O}_F \cap F^\times)^r$ where $c_r \in \mathcal{O}_F^\times$, $c_j \notin \mathcal{O}_F^\times$, and $c_i \in \mathcal{O}_F^\times$ for all $i \neq j$. Then*

$$\mathrm{RO}_{\mathcal{R}_{w_0}(c)}(\mathbb{1}_{\mathrm{GL}_r(\mathcal{O}_F)}, m_1, m_2) = 0$$

unless $m_1, m_2 \in \mathcal{O}_F^{r-1}$, in which case it is equal to

$$\sum_{x \in (\mathcal{O}_F/c_j)^\times} \psi \left(\frac{-m_{2j}x + m_{1(r-j)}c_{j+1}c_{j-1}\bar{x}}{c_j} \right). \quad (18.55)$$

Here $\bar{x} \in \mathcal{O}_F$ is chosen so that $x\bar{x} \equiv 1 \pmod{c}$ and we set $c_{j-1} = 1$ if $j = 1$.

When $F = \mathbb{Q}_p$, (18.55) is a classical Kloosterman sum. This explains why the relative orbital integral $\mathrm{RO}_{\mathcal{R}_{w_0}(c)}(\mathbb{1}_{\mathrm{GL}_r(\mathcal{O}_F)}, m_1, m_2)$ is called a Kloosterman integral.

Proof. By Lemma 18.8.5 iterated $j - 1$ times, we have that

$$\mathrm{RO}_{\mathcal{R}_{w_0}(c)}(\mathbb{1}_{\mathrm{GL}_r(\mathcal{O}_F)}, m_1, m_2) = 0$$

unless $m_{1(r-1)}, \dots, m_{1(r-j+1)}, m_{21}, \dots, m_{2(j-1)} \in \mathcal{O}_F$. Here m_{1i} and m_{2i} are the i th entry of m_1 and m_2 , respectively. Assuming this is the case, the same lemma implies that

$$\mathrm{RO}_{\mathcal{R}_{w_0}(c)}(\mathbb{1}_{\mathrm{GL}_r(\mathcal{O}_F)}, m_1, m_2) = \mathrm{RO}_{\mathcal{R}_{\tilde{w}_0}(\tilde{c})}(\mathbb{1}_{\mathrm{GL}_{r-(j-1)}(\mathcal{O}_F)}, \tilde{m}_1, \tilde{m}_2).$$

Here $\tilde{w}_0 \in \mathrm{GL}_{r-(j-1)}(F)$ is the standard representative for the long Weyl element and

$$\begin{aligned} \tilde{c} &:= (c_j/c_{j-1}, c_{j+1}/c_{j-1}, \dots, c_r/c_{j-1}), \\ \tilde{m}_1 &:= (m_{11}, \dots, m_{1(r-j)}), \\ \tilde{m}_2 &:= (m_{2j}, \dots, m_{2(r-1)}). \end{aligned}$$

We now apply Lemma 18.8.4 to write this as

$$\begin{aligned} &\mathrm{RO}_{\mathcal{R}_{\tilde{w}_0}(\tilde{c})^{-1}}(\mathbb{1}_{\mathrm{GL}_{r-(j-1)}(\mathcal{O}_F)}, -\tilde{m}_2, -\tilde{m}_1) \\ &= \mathrm{RO}_{\mathcal{R}_{\tilde{w}_0}(c_{r-1}/c_r, c_{r-2}/c_r, \dots, c_{j-1}/c_r)}(\mathbb{1}_{\mathrm{GL}_{r-(j-1)}(\mathcal{O}_F)}, -\tilde{m}_2, -\tilde{m}_1). \end{aligned}$$

Applying Lemma 18.8.5 again, this vanishes unless

$$m_{11}, \dots, m_{1(r-j-1)}, m_{2(j+1)}, \dots, m_{2(r-1)} \in \mathcal{O}_F$$

in which case it is

$$\mathrm{RO}_{\mathcal{R}_{\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}}(c_j/c_{j+1}, c_{j-1}/c_{j+1})}(\mathbb{1}_{\mathrm{GL}_2(\mathcal{O}_F)}, -m_{2j}, -m_{1(r-j)}).$$

We compute this directly. It is equal to

$$\begin{aligned}
& \int \mathbb{1}_{\mathrm{GL}_2(\mathcal{O}_F)} \left(\begin{pmatrix} 1-x \\ 1 \end{pmatrix} \begin{pmatrix} c_{j-1}/c_j \\ c_j/c_{j+1} \end{pmatrix} \begin{pmatrix} 1 & y \\ & 1 \end{pmatrix} \right) \bar{\psi}(m_{2j}x) \psi(m_{1(r-j)}y) dx dy \\
&= \int \mathbb{1}_{\mathrm{GL}_2(\mathcal{O}_F)} \begin{pmatrix} -c_j x/c_{j+1} & -c_j x y/c_{j+1} + c_{j-1}/c_j \\ c_j/c_{j-1} & c_j y/c_{j+1} \end{pmatrix} \bar{\psi}(m_{2j}x) \psi(m_{1(r-j)}y) dx dy \\
&= \frac{1}{|c_j|^2} \int \mathbb{1}_{\mathrm{GL}_2(\mathcal{O}_F)} \begin{pmatrix} -x/c_{j+1} & \frac{-x y/c_{j+1} + c_{j-1}}{c_j} \\ c_j/c_{j-1} & y/c_{j+1} \end{pmatrix} \psi \left(\frac{-m_{2j}x + m_{1(r-j)}y}{c_j} \right) dx dy.
\end{aligned}$$

This vanishes unless $m_{1(r-j)}, m_{2j} \in \mathcal{O}_F$, in which case it is equal to

$$\sum_{x \in (\mathcal{O}_F/c_j)^\times} \psi \left(\frac{-m_{2j}x + m_{1(r-j)}c_{j+1}c_{j-1}\bar{x}}{c_j} \right),$$

which completes the proof. $\square$

18.9 Sums of Whittaker coefficients

The spectral side of the Kuznetsov formula is a sum of Whittaker coefficients. We make this precise in the current section and then prove the Kuznetsov formula in Theorem 18.9.2 below.

Let $f \in C_c^\infty(A_{\mathrm{GL}_r} \backslash \mathrm{GL}_r(\mathbb{A}_F))$. If π is a cuspidal automorphic representation of $A_{\mathrm{GL}_r} \backslash \mathrm{GL}_r(\mathbb{A}_F)$ then we can form the relative trace

$$\mathrm{rtr}(\pi(f), m_1, m_2) := \int_{[N]^2} K_{\pi(f)}(n_1, n_2) \psi_{m_1}(n_1) \bar{\psi}_{m_2}(n_2) dn_1 dn_2. \quad (18.56)$$

Since $[N]$ is compact, this is absolutely convergent. We then have the following theorem:

Theorem 18.9.1 *Assume that $R(f)$ has cuspidal image. Then one has an equality of absolutely convergent sums*

$$\begin{aligned}
& \sum_{\pi} \mathrm{rtr}(\pi(f), m_1, m_2) \\
&= \sum_{[\gamma] \in N(F) \backslash \mathrm{GL}_r(F) / N(F)} \tau((N \times N)_\gamma) \mathrm{RO}_\gamma(f, m_1, m_2)
\end{aligned} \quad (18.57)$$

where the sum on the left is over isomorphism classes of cuspidal automorphic representations of $A_{\mathrm{GL}_r} \backslash \mathrm{GL}_r(\mathbb{A}_F)$ and the sum on the right is over $\psi_{m_1} \otimes \bar{\psi}_{m_2}$ -relevant classes.

Proof. This is immediate from Theorem 16.2.6 and Proposition 18.8.2. $\square$

We now use Langlands' spectral expansion of the automorphic kernel function $K_f(x, y)$ from §16.3 to give a general version of Theorem 18.9.1 that

requires no assumption on f . For unexplained notation below we refer to Chapter 10.

Let $B \leq \mathrm{GL}_r$ be the Borel subgroup of upper triangular matrices. Call a parabolic subgroup $P \leq \mathrm{GL}_r$ standard if it contains B . We let $M \leq P$ be the unique Levi subgroup containing the maximal torus of diagonal matrices. Then Langlands' spectral decomposition of $L^2([\mathrm{GL}_r])$ implies that

$$K_f(x, y) = \sum_P n_P^{-1} \sum_{\sigma} \int_{i(\mathfrak{a}_M^{\mathrm{GL}_r})^*} K_{I(\sigma, \lambda)(f)}(x, y) d\lambda \quad (18.58)$$

where the sum on P is over standard parabolic subgroups of GL_r , n_P is the order of the association class of P and the sum on σ is over isomorphism classes of irreducible $A_M \backslash M(\mathbb{A}_F)$ -subrepresentations of $L_{\mathrm{disc}}^2([M])$. Here

$$K_{I(\sigma, \lambda)(f)}(x, y) := \sum_{\varphi \in \mathcal{B}(\sigma)} E(x, I(\sigma, \lambda)(f)\varphi, \lambda) \overline{E(y, \varphi, \lambda)}, \quad (18.59)$$

where $\mathcal{B}(\sigma)$ is an orthonormal basis of

$$\mathrm{Ind}_P^{\mathrm{GL}_r}(L_{\mathrm{disc}}^2([M])(\sigma))$$

consisting of vectors in

$$\mathrm{Ind}_P^{\mathrm{GL}_r}(L_{\mathrm{disc}}^2([M])(\sigma))^0$$

defined as in (10.8). The expression (18.59) is merely an L^2 -expansion of the kernel as a function of y for each fixed x . If f is finite under a maximal compact subgroup of $\mathrm{GL}_r(F_{\infty})$ then the sum can be taken to be finite and hence the identity (18.59) is valid pointwise. Even if this is not the case, the sum is still represented by a smooth function of x and y . We refer to [Art78, §4] for proofs of these assertions. In fact they can be proven by an easy analogue of the argument proving Theorem 16.2.3.

We set

$$\mathrm{rtr}(I(\sigma, \lambda)(f), m_1, m_2) := \int_{[N]^2} K_{I(\sigma, \lambda)(f)}(n_1, n_2) \psi_{m_1}(n_1) \overline{\psi}_{m_2}(n_2) dn_1 dn_2.$$

Since $[N]$ is compact, this is absolutely convergent.

The following is the full version of the Petersson-Bruggemann-Kuznetsov formula:

Theorem 18.9.2 *One has that*

$$\sum_P n_P^{-1} \sum_{\sigma} \int_{i(\mathfrak{a}_M^{\mathrm{GL}_r})^*} |\mathrm{rtr}(I(\sigma, \lambda)(f), m_1, m_2)| d\lambda < \infty.$$

Moreover

$$\begin{aligned} & \sum_P n_P^{-1} \sum_{\sigma} \int_{i(\mathfrak{a}_M^{\mathrm{GL}_r})^*} \mathrm{rtr}(I(\sigma, \lambda)(f), m_1, m_2) d\lambda \\ &= \sum_{[\gamma] \in N(F) \backslash \mathrm{GL}_r(F) / N(F)} \tau((N \times N)_{\gamma}) \mathrm{RO}_{\gamma}(f, m_1, m_2), \end{aligned}$$

where the sum on the right is over $\psi_{m_1} \otimes \bar{\psi}_{m_2}$ -relevant classes.

Proof. The absolute convergence statement follows from the fact that $[N]$ is compact together with the estimates on the terms in the spectral expansion (18.58) contained in [Art78, §4]. Thus by (18.58), we conclude that

$$\begin{aligned} & \int_{[N]^2} K_f(n_1, n_2) \psi_{m_1}(n_1) \bar{\psi}_{m_2}(n_2) dn_1 dn_2 \\ &= \sum_P n_P^{-1} \sum_{\sigma} \int_{i(\mathfrak{a}_M^{\mathrm{GL}_r})^*} \mathrm{rtr}(I(\sigma, \lambda)(f), m_1, m_2) d\lambda. \end{aligned}$$

The theorem now follows from Proposition 18.8.2. $\square$

It is important to observe that for $m_1, m_2 \in (F^{\times})^{r-1}$ only generic representations can contribute to the spectral side of Theorem 18.9.2. In particular, for $P = \mathrm{GL}_r$, only cuspidal representations contribute by the following theorem:

Theorem 18.9.3 *If π is a generic discrete automorphic representation of $A_{\mathrm{GL}_r} \backslash \mathrm{GL}_r(\mathbb{A}_F)$, then π is cuspidal.*

Proof. This is a combination of the main theorem of [MW89] and [JL13, Proposition 2.1]. $\square$

In analytic number theory it is useful to understand the dependence of the quantities $\mathrm{rtr}(I(\sigma, \lambda)(f), m_1, m_2)$ on m_1 and m_2 . We now make this explicit. Let

$$\psi := \psi_{(1, \dots, 1)},$$

where ψ_m is defined in (18.50).

Assume $I(\sigma, \lambda)$ is unramified outside a finite set S of places of F including the infinite places. Using the notation of (11.4) for Whittaker models, we have the following lemma:

Lemma 18.9.4 *Let σ be a discrete automorphic representation of $A_M \backslash M(\mathbb{A}_F)$ and let $\lambda \in i(\mathfrak{a}_M^{\mathrm{GL}_r})^*$. If $I(\sigma, \lambda)$ admits a nonzero global ψ -Whittaker functional then there is a unique vector*

$$W^S \in \mathcal{W}(I(\sigma, \lambda)^S, \psi^S)^{\mathrm{GL}_r(\hat{\mathcal{O}}_F^S)} := \prod_{v \notin S} \mathcal{W}(I(\sigma_v, \lambda), \psi_v)^{\mathrm{GL}_r(\mathcal{O}_{F_v})}$$

such that $W^S(I) = 1$.

Proof. For $v \nmid \infty$ the representation $I(\sigma_v, \lambda)$ is irreducible for $\lambda \in i(\mathfrak{a}_M^{\mathrm{GL}_r})^*$ [Ber84]. Since it admits a nonzero ψ_v -Whittaker functional it is generic. Thus the lemma is an immediate corollary of Theorem 11.4.1. $\square$

For $f \in C_c^\infty(A_{\mathrm{GL}_r} \backslash \mathrm{GL}_r(\mathbb{A}_F) // \mathrm{GL}_r(\hat{\mathcal{O}}_F^S))$ and $m_{1S}, m_{2S} \in (F_S^\times)^{r-1}$, let

$$f_{m_{1S}, m_{2S}}(g) := f(\nu(m_{1S})g\nu(m_{2S})^{-1}) \in C_c^\infty(A_{\mathrm{GL}_r} \backslash \mathrm{GL}_r(\mathbb{A}_F)), \quad (18.60)$$

where ν is defined as in (18.48).

Lemma 18.9.5 *Let σ be discrete automorphic representation of $A_M \backslash M(\mathbb{A}_F)$ and let $\lambda \in i(\mathfrak{a}_M^{\mathrm{GL}_r})^*$. Assume that $I(\sigma, \lambda)$ is unramified outside S and that $\mathrm{rtr}(I(\sigma, \lambda)(f), \mathbf{1}, \mathbf{1}) \neq 0$. One has that*

$$\mathrm{rtr}(I(\sigma, \lambda)(f_{m_{1S}, m_{2S}}), m_1, m_2) = W^S(\nu(m_1)) \overline{W^S}(\nu(m_2)) \mathrm{rtr}(I(\sigma, \lambda)(f), \mathbf{1}, \mathbf{1}),$$

where W^S is as in Lemma 18.9.4.

Here $\mathbf{1} = (1, \dots, 1) \in (F^\times)^{r-1}$. The quantity $W^S(\nu(m_1))$ can be computed using the Casselman-Shalika formula (see (11.13)).

Proof. To ease notation in the proof, write

$$\tilde{f} := f_{m_{1S}, m_{2S}}.$$

We change variables

$$(n_1, n_2) \longmapsto (\nu(m_1)^{-1}n_1\nu(m_1), \nu(m_2)^{-1}n_2\nu(m_2))$$

and use (18.49) to see that

$$\begin{aligned} & \mathrm{rtr}(I(\sigma, \lambda)(\tilde{f}), m_1, m_2) \\ &= \int_{[N]^2} K_{I(\sigma, \lambda)(\tilde{f})}(\nu(m_1)^{-1}n_1\nu(m_1), \nu(m_2)^{-1}n_2\nu(m_2)) \psi(n_1) \overline{\psi}(n_2) dn_1 dn_2 \\ &= \int_{[N]^2} K_{I(\sigma, \lambda)(\tilde{f})}(n_1\nu(m_1), n_2\nu(m_2)) \psi(n_1) \overline{\psi}(n_2) dn_1 dn_2. \end{aligned}$$

Here we have used the fact that $K_{I(\sigma, \lambda)(\tilde{f})}(g_1, g_2)$ is left invariant under $\mathrm{GL}_r(F) \times \mathrm{GL}_r(F)$. This in turn is equal to

$$\int_{[N]^2} K_{I(\sigma, \lambda)(f)}(n_1\nu(m_1^S), n_2\nu(m_2^S)) \psi(n_1) \overline{\psi}(n_2) dn_1 dn_2. \quad (18.61)$$

The function

$$\begin{aligned} \mathrm{GL}_r(\mathbb{A}_F^S) \times \mathrm{GL}_r(\mathbb{A}_F^S) &\longrightarrow \mathbb{C} \\ (g_1^S, g_2^S) &\longmapsto \int_{[N]^2} K_{I(\sigma, \lambda)(f)}(n_1 g_1^S, n_2 g^S) \psi(n_1) \bar{\psi}(n_2) dn_1 dn_2 \end{aligned}$$

lies in $\mathcal{W}(I(\sigma, \lambda)^S, \psi^S) \otimes \mathcal{W}(I(\sigma, \lambda)^S, \bar{\psi}^S)$. Here in the integral we have written g_i^S for the element of $\mathrm{GL}_r(\mathbb{A}_F)$ whose projection to $\mathrm{GL}_r(\mathbb{A}_F^S)$ is g_i^S and whose projection to $\mathrm{GL}_r(F_S)$ is the identity. Thus the lemma follows from Lemma 18.9.5. $\square$

Exercises

18.1. Let $H \leq \mathbb{R}^n$ be a closed real Lie subgroup of $\mathbb{R}^n$ with finitely many connected components. Prove that $H \cong \mathbb{R}^m$ for some $m \leq n$.

18.2. Let T be a split torus over a characteristic zero field k . Prove that any connected subgroup of T is a split torus.

For the next three problems we assume that we are in the setting of §18.2, although we allow F to be a function field. Let v be a finite place of the global field F and let

$$f^v \in C_c^\infty(A_G \backslash G(\mathbb{A}_F^v)), \quad f_v \in C_c^\infty(G(F_v)), \quad f := f_v f^v.$$

Let $\gamma \in G(F)$ and let $\Omega \subset A_G \backslash G(\mathbb{A}_F)$ be the $H(\mathbb{A}_F)$ -orbit of γ .

18.3. Prove that if f_v is supported in the set of elements of $G(F_v)$ with closed H_v -orbit and $f(x) \neq 0$ for some $x \in \Omega$ then γ is relatively semisimple.

18.4. Prove that if f_v is supported in

$$\{\gamma' \in G(F_v) : H_{\gamma'} = H_{\gamma'}^\circ \text{ is smooth}\}$$

and $f(x) \neq 0$ for some $x \in \Omega$ then γ is gcf.

18.5. Prove that if f_v is supported in

$$\{\gamma' \in G(F_v) : \mathrm{Hom}((H \cap \Delta(Z_G)) \backslash H_{\gamma'}, \mathbb{G}_m) \text{ is trivial}\}$$

and $f(x) \neq 0$ for some $x \in \Omega$ then γ is relatively elliptic. Here, as before, $\Delta : G \rightarrow G \times G$ is the diagonal embedding.

18.6. Give an example of a pair of reductive groups $H \leq G$ such that

$$H(\mathbb{A}_F) \neq (A_G \cap H(\mathbb{A}_F))(G(\mathbb{A}_F)^1 \cap H(\mathbb{A}_F)).$$

18.7. State and prove an analogue of (18.29) for twisted orbital integrals.

18.8. For $f \in C_c^\infty(A_{\mathrm{GL}_r} \backslash \mathrm{GL}_r(F_\infty))$ and $m_1, m_2 \in (F_\infty^\times)^{r-1}$, let

$$f_{m_1\infty, m_2\infty}(g) := f(\nu(m_1)g\nu(m_2)^{-1}).$$

For $c := (c_1, \dots, c_r) \in (F_\infty^\times)^r$, prove that

$$\mathrm{RO}_{\Upsilon_w(c)}(f_{m_1\infty, m_2\infty}, m_1, m_2) \neq 0$$

only if for all $v|\infty$ one has that

$$|c_r \det(\nu(m_1)\nu(m_2)^{-1})|_v \asymp_f Y$$

for some $Y \in \mathbb{R}_{>0}$ independent of $v|\infty$ and

$$|\det_i(\nu(m_1)\Upsilon_w(c)\nu(m_2)^{-1})|_v \ll_f |c_r \det \nu(m_1)\nu(m_2)^{-1}|_v^{i/r}$$

for all $v|\infty$ and $1 \leq i \leq r$. In particular, if $w = w_0$, then

$$\mathrm{RO}_{\Upsilon_{w_0}(c)}(f_{m_1\infty, m_2\infty}, m_1, m_2) = 0$$

unless

$$|c_i|_v \ll_f |c_r \det \nu(m_1)\nu(m_2)^{-1}|_v^{i/r} |\det_i(\nu(m_1)w_0\nu(m_2)^{-1})|_v^{-1}$$

for all $v|\infty$ and $1 \leq i \leq r$.

18.9. Let H be a unipotent algebraic group over a global field F . Prove that $H(\mathbb{A}_F)$ is unimodular.

18.10. Let X be the space of $n \times n$ matrices. Let $f \in C_c^\infty(X(\mathbb{A}_F))$. Prove that the restriction of f to $\mathrm{GL}_n(\mathbb{A}_F)$ is smooth but not compactly supported.

18.11. Let X and G be as in (18.42). We have an injection

$$a(I_n, \cdot) : G_{I_n}(\mathbb{A}_F) \backslash G(\mathbb{A}_F) \longrightarrow X(\mathbb{A}_F).$$

Thus we have a pull-back map

$$a(I_n, \cdot)^* : C_c^\infty(X(\mathbb{A}_F)) \longrightarrow C^\infty(G_{I_n}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)). \quad (18.62)$$

Prove that the image of this restriction map is not contained in the space of compactly supported functions on $G_{I_n}(\mathbb{A}_F) \backslash G(\mathbb{A}_F)$.

The remaining exercises explain how the simple trace formula of §18.4 is a special case of the relative trace formula of §18.3. Thus for the remaining exercises let F be a global field and let G_0 be a reductive group over F . Let $G = G_0 \times G_0$ and in the notation of §18.3 let $H_1 = H_2$ be the image of the

diagonal embedding $G_0 \rightarrow G_0 \times G_0$. Thus $H = H_1 \times H_2$ in the notation of §18.2.

18.12. Let π_1 and π_2 be cuspidal automorphic representations of $A_{G_0} \backslash G_0(\mathbb{A}_F)$. For $f_1, f_2 \in C_c^\infty(A_{G_0} \backslash G_0(\mathbb{A}_F))$ prove that

$$\text{rtr}_{H,1} \pi_1 \otimes \pi_2(f_1 \otimes f_2) \neq 0$$

if and only if $\pi_1 \cong \pi_2^\vee$. Here 1 denotes the trivial character of $H(\mathbb{A}_F)$.

18.13. Assuming that $\text{rtr}_{H,1} \pi_1 \otimes \pi_2(f_1 \otimes f_2) \neq 0$ prove that

$$\text{rtr}_{H,1} \pi_1 \otimes \pi_2(f_1 \otimes f_2) = \text{tr} \pi_1(f_2^\vee * f_1)$$

for a suitable choice of Haar measures. Here $f_2^\vee(g) := f_2(g^{-1})$.

18.14. Prove that there is a bijection

$$\begin{aligned} H_1(F) \backslash G(F) / H_2(F) &\longrightarrow G_0(F) / \sim \\ (g_1, g_2) &\longmapsto g_2^{-1} g_1 \end{aligned}$$

where the right-hand side is the set of conjugacy classes in $G_0(F)$. Show that the bijection induces an isomorphism

$$H_{(g_1, g_2)} \xrightarrow{\sim} C_{g_2^{-1} g_1}.$$

18.15. Show that $(g_1, g_2) \in G(F)$ is relatively semisimple (resp. relatively elliptic) if and only if $g_2^{-1} g_1$ is semisimple (resp. elliptic).

18.16. If (g_1, g_2) is relatively semisimple prove that

$$\text{RO}_{(g_1, g_2)}(f_1 \otimes f_2) = \text{O}_{g_2^{-1} g_1}(f_2^\vee * f_1)$$

for a suitable choice of measures. If (g_1, g_2) is relatively elliptic prove that

$$\tau(H_{(g_1, g_2)}) = \tau(C_{g_2^{-1} g_1})$$

for a suitable choice of measures.

Chapter 19

Applications of Trace Formulae



What we know is not much.
What we do not know is
immense.

P.-S. de Laplace

In this chapter we discuss applications of relative trace formulae. Our goal is to provide an introduction to the key motifs of existence and comparison underlying these applications.

19.1 Existence and comparison

Selberg's motivation for proving the trace formula for SL_2 over $\mathbb{Q}$ was to prove the existence of cusp forms that are fixed by $SO_2(\mathbb{R})$. In the process, he actually estimated the number of such cusp forms in a suitable family and showed that they exist in abundance. This was the first application of the trace formula, and it can be viewed as a quantitative existence result. We discuss this in §19.2.

Motivated by the Langlands functoriality conjecture, Jacquet and Langlands [JL70] gave a very different type of application of the trace formula. They compared trace formulae on different groups in order to deduce a relation between automorphic representations on the two groups. This idea has been extended to what is known as the theory of twisted endoscopy and continues to play a key role in automorphic representation theory. We discuss this in the second part of the chapter. All sections of the chapter are more or less brief surveys. Our aim is to whet the appetite of the reader and give some guidance on the literature and future directions of research. We assume for this chapter that F is a number field because most of the results we will cite below are proven in this setting.

19.2 The Weyl law

Let M be a smooth compact Riemannian manifold. Associated to M is a Laplace-Beltrami operator Δ . Call a function $\varphi \in C^\infty(M) \cap L^2(M)$ a Laplacian eigenfunction with eigenvalue λ if

$$\Delta\varphi = \lambda\varphi.$$

It is known that each eigenvalue λ is a nonnegative real number, that the set of eigenvalues is discrete, and that $L^2(M)$ is the Hilbert space direct sum of the corresponding eigenspaces $L^2(M)(\lambda)$. Moreover $\dim L^2(M)(\lambda) < \infty$ for each λ . For $x > 0$, consider the counting function

$$N(x) := \sum_{\lambda \leq x} \dim L^2(M)(\lambda).$$

Here the sum is over the finite set of $\lambda \leq x$ such that $L^2(M)(\lambda) \neq 0$.

The following is the Weyl law (see [Wey12] and [MP49]):

Theorem 19.2.1 (The Weyl law) *One has that*

$$N(x) = \frac{\text{vol}(M)}{(4\pi)^{\frac{\dim M}{2}} \Gamma\left(\frac{\dim M}{2} + 1\right)} x^{\dim M/2} + o(x^{\dim M/2}) \quad (19.1)$$

as $x \rightarrow \infty$. □

Consider the Shimura manifolds $\text{Sh}(G, X)^K$ of §15.2. These are finite unions of Riemannian manifolds. When they are compact, the Weyl law implies the existence of an abundance of automorphic forms on them. However they are often noncompact (see Theorem 2.6.3 for a precise statement). Even if one compactifies $\text{Sh}(G, X)^K$, it is still unclear in the noncompact case whether the eigenforms produced by the Weyl law correspond to the cuspidal spectrum. Selberg developed the trace formula for SL_2 over $\mathbb{Q}$ to address precisely this question [Sel56], and he proved an analogue of Theorem 19.2.1 in this setting.

Müller [Mül07] established the analogue of the Weyl law for $G = \text{SL}_n$ over $\mathbb{Q}$. His proof makes use of the full noninvariant trace formula of Arthur, and his results are valid for cusp forms that are not necessarily fixed by $\text{O}_n(\mathbb{R})$. About the same time, Lindenstrauss and Venkatesh [LV07] proved an analogous result for all split semisimple adjoint groups over $\mathbb{Q}$. Notably, their method is soft in that it does not require the full trace formula, only a simple analogue. It is based on a novel construction of test functions with purely cuspidal image. Their method originally only treated cusp forms that are fixed by a maximal compact subgroup at infinity, but this restriction has been removed in [Mai22].

Using the same technique, the authors [GH15] established a relative analogue of the Weyl law under some assumptions via the general simple relative

trace formula (Theorem 18.2.5). This **relative Weyl law** gives an asymptotic formula for the number of Laplacian eigenfunctions on $\mathrm{Sh}(G, X)^K$ weighted by the L^2 -restriction norm over a locally symmetric subspace associated to a subgroup $H \leq G$.

The approach to the Weyl law via the full Arthur-Selberg trace formula is more complicated, but it still has its merits. The results of Lindenstrauss and Venkatesh mentioned above give no control on the error term in the Weyl law. Using the trace formula, a Weyl law for SL_n over $\mathbb{Q}$ with remainder was established by Lapid and Müller in [LM09] using the approach of [DKV79a, DKV79b] combined with the Arthur-Selberg trace formula.

19.3 The comparison strategy

The Langlands functoriality conjecture predicts that for two reductive groups G_1 and G_2 defined over the number field F and an L -map

$$r : {}^L G_1 \longrightarrow {}^L G_2,$$

there should be a transfer from weak automorphic L -packets of G_1 to weak automorphic L -packets of G_2 . In this section we explain a method for proving the existence of transfers that is applicable in certain situations. Briefly, one can sometimes establish a transfer by comparing the geometric sides of suitable trace formulae for G_1 and G_2 .

In more detail, suppose that we are given subgroups $H_i \leq G_i \times G_i$ for $1 \leq i \leq 2$ and characters

$$\chi_i : (H_i(\mathbb{A}_F) \cap (A_{G_i} \times A_{G_i})) \backslash H_i(\mathbb{A}_F) \longrightarrow \mathbb{C}^\times.$$

For F -algebras R , the groups H_i act on G_i via

$$\begin{aligned} G_i(R) \times H_i(R) &\longrightarrow G_i(R) \\ (g, (h_\ell, h_r)) &\longmapsto h_\ell^{-1} g h_r \end{aligned} \tag{19.2}$$

as in (18.2). We assume that the neutral component of H_i is the direct product of a unipotent and a reductive group. We write $H_{i,r}$ for the unique maximal reductive subgroup of H_i° .

Suppose that we can construct a correspondence

$$\{\chi_2\text{-relevant } \delta \in G_2(F)/H_2(F)\} \longleftrightarrow \{\chi_1\text{-relevant } \gamma \in G_1(F)/H_1(F)\}.$$

For example, one might have a bijection, but often the situation is more complicated. We refer to this as a **geometric correspondence** between relevant relative classes. We say that the class of δ **matches** the class of γ if the two classes correspond. Suppose that for matching δ and γ one

can relate $\tau(H_{2\delta})\text{RO}_\delta(f)$ and $\tau(H_{1\gamma})\text{RO}_\gamma(f^{G_1})$ for suitable test functions $f \in C_c^\infty(A_{G_2} \backslash G_2(\mathbb{A}_F))$ and $f^{G_1} \in C_c^\infty(A_{G_1} \backslash G_1(\mathbb{A}_F))$. Then one can hope to prove an identity of the form

$$\sum_{\delta \in G_2(F)/H_2(F)} \tau(H_{2\delta})\text{RO}_\delta(f) = \sum_{\gamma \in G_1(F)/H_1(F)} \tau(H_{1\gamma})\text{RO}_\gamma(f^{G_1}). \quad (19.3)$$

Here the sums over γ and δ are over χ_1 and χ_2 -relevant elements, respectively, that are additionally relatively regular, relatively elliptic, relatively unimodular and such that the H_{1r} -orbit of γ (resp. the H_{2r} -orbit of δ) is closed. We are assuming these things so that the sums are convergent by Theorem 18.2.5. If an identity like (19.3) can be established for a sufficiently rich set of test functions, then by Theorem 18.2.5 one obtains a relationship between cuspidal automorphic representations π of $A_{G_1} \backslash G_1(\mathbb{A}_F)$ such that $\pi \otimes \pi^\vee$ is (H_1, χ_1) -distinguished and cuspidal automorphic representations π' of $A_{G_2} \backslash G_2(\mathbb{A}_F)$ such that $\pi' \otimes \pi'^\vee$ is (H_2, χ_2) -distinguished. This often leads to a proof of an instance of Langlands functoriality (or its relative analogue due to Jacquet, Sakellaridis, and Venkatesh). We will explain this in more detail momentarily. In many cases one has to incorporate families of (G_i, H_i, χ_i) on both sides of the formula. We ignore this complication because we are just speaking in rough terms.

Let us elaborate on what we mean by a relation between $\tau(H_{2\delta})\text{RO}_\delta(f)$ and $\tau(H_{1\gamma})\text{RO}_\gamma(f^{G_1})$. We assume for simplicity that the geometric correspondence is a bijection, though this is rarely the case. One needs to be able to relate the measures $\tau(H_{2\delta})$ and $\tau(H_{1\gamma})$ for matching δ and γ , but this is usually not the key issue. For simplicity we assume that measures can be chosen so that $\tau(H_{2\delta}) = \tau(H_{1\gamma})$ whenever δ matches γ .

For almost all finite places v , there exists a hyperspecial subgroup $K_{iv} \leq G_i(F_v)$ by Proposition 2.4.8. The map

$$r : {}^L G_1 \longrightarrow {}^L G_2$$

and the Satake isomorphism together induce an algebra homomorphism

$$\hat{r} : C_c^\infty(G_2(F_v) // K_{2v}) \longrightarrow C_c^\infty(G_1(F_v) // K_{1v}) \quad (19.4)$$

(see Exercise 19.2). Dual to this algebra homomorphism we have a transfer

$$\pi_v \longmapsto r(\pi_v) \quad (19.5)$$

from the set of equivalence classes of K_{1v} -unramified representations of $G_1(F_v)$ to the set of equivalence classes of K_{2v} -unramified representations of $G_2(F_v)$. Here we are abusing notation and using the same notation for a representation and its equivalence class. The transfer (19.5) is given by stipulating that the Langlands class of $r(\pi_v)$ is the image under r of the Langlands class of π_v . See below Corollary 7.5.2 for the definition of the Langlands class

of an unramified representation. By Exercise 19.2 for $f \in C_c^\infty(G_2(F_v) // K_{2v})$ one has

$$\mathrm{tr} \pi_v(\widehat{r}(f)) = \mathrm{tr} r(\pi_v)(f). \quad (19.6)$$

Here we normalize the Haar measures implicit in the traces so that the K_i are given measure 1.

Globally, assume S is a finite set of places of F including the infinite places and for each i we are given compact open subgroups $K_i^S \leq G_i(\mathbb{A}_F^S)$ with K_{iv} hyperspecial for all $v \notin S$. Assume moreover that π^S is an admissible irreducible representation of $G_1(\mathbb{A}_F^S)$ with a nonzero K_1^S -fixed vector. We then define

$$r(\pi^S) := \otimes'_{v \notin S} r(\pi_v).$$

It is an irreducible admissible representation of $G_2(\mathbb{A}_F^S)$ with a nonzero K_2^S -fixed vector.

One wants to prove that if γ and δ match then for almost all finite places v , then

$$\mathrm{RO}_\delta(f_v) = \mathrm{RO}_\gamma(\widehat{r}(f_v)) \quad (19.7)$$

for all $f_v \in C_c^\infty(G_2(F_v) // K_{2v})$. This is often too naïve. The equality will only hold up to a function of δ and γ called a **transfer factor**, but we suppress this difficulty.

Formulae of the form (19.7) are known as the **fundamental lemma for the Hecke algebra**. The identity (19.7) for $f_v = \mathbb{1}_{K_{2v}}$ is known as the **fundamental lemma**, or the **fundamental lemma for the unit element of the Hecke algebra**. In cases related to twisted endoscopy it is possible to deduce the fundamental lemma for the Hecke algebra from the fundamental lemma [Clo90a, Hal95, Lab90]. One hopes that the same reduction can be made in general although it is not known how to do this.

Finally, one needs to know that for all places v there is a sufficiently large supply of functions $f_v \in C_c^\infty(G_2(F_v))$ and $f_v^{G_1} \in C_c^\infty(G_1(F_v))$ that **match**, meaning that when δ and γ match one has that

$$\mathrm{RO}_\delta(f_v) = \mathrm{RO}_\gamma(f_v^{G_1}). \quad (19.8)$$

Statements of this type are known as **transfer**. For

$$f = \otimes_v f_v \in C_c^\infty(A_{G_2} \backslash G_2(\mathbb{A}_F))$$

with $f_v \in C_c^\infty(G_2(F_v))$ for all v one has $f_v \in C_c^\infty(G_2(F_v) // K_{2v})$ for almost all v . If $f_v \in C_c^\infty(G_2(F_v) // K_{2v})$ then the fundamental lemma for the Hecke algebra implies that we can take $f_v^{G_1} = \widehat{r}(f_v)$. As in the statement of the fundamental lemma, hoping for an equality as simple as (19.8) is often too naïve; it will only hold up to a transfer factor.

Given the fundamental lemma for the Hecke algebra and transfer, Theorem 18.2.5 implies an identity

$$\sum_{\pi'} \text{rtr}_{H_2, \chi_2} \pi'(f) = \sum_{\pi} \text{rtr}_{H_1, \chi_1} \pi(f^{G_1}), \quad (19.9)$$

at least for f and f^{G_1} satisfying the assumptions of Theorem 18.2.5. Here the sum on the left is over isomorphism classes of cuspidal automorphic representations π' of $A_{G_2} \backslash G_2(\mathbb{A}_F)$ such that $\pi' \otimes \pi'^{\vee}$ is (H_2, χ_2) -distinguished and the sum on the right is over isomorphism classes of cuspidal automorphic representations of $A_{G_1} \backslash G_1(\mathbb{A}_F)$ such that $\pi \otimes \pi^{\vee}$ is (H_1, χ_1) -distinguished.

If f (resp. f^{G_1}) is fixed on the left and right under K_2^S (resp. K_1^S) then (19.9) implies that

$$\sum_{\pi'} \text{tr} \pi'^S(f^S) \text{rtr}_{H_2, \chi_2} \pi'(f_S \mathbb{1}_{K_2^S}) = \sum_{\pi} \text{tr} \pi^S(f^{G_1 S}) \text{rtr}_{H_1, \chi_1} \pi(f_S^{G_1} \mathbb{1}_{K_1^S})$$

(see Exercise 19.4). Though the set of representations on each side of this equation is in general infinite, in practice one can use a version of linear independence of characters to refine it to an identity

$$\begin{aligned} & \text{tr} r(\pi_0^S)(f^S) \sum_{\pi': \pi'^S \cong r(\pi_0^S)} \text{rtr}_{H_2, \chi_2} (\pi'(f_S \mathbb{1}_{K_2^S})) \\ &= \text{tr} \pi_0^S(\widehat{r}(f^S)) \sum_{\pi: \pi^S \cong \pi_0^S} \text{rtr}_{H_1, \chi_1} (\pi(f_S^{G_1} \mathbb{1}_{K_1^S})) \end{aligned} \quad (19.10)$$

for each cuspidal automorphic representation π_0 of $A_{G_1} \backslash G_1(\mathbb{A}_F)$. Of course, both sums can be zero, but if one sum is nonzero, the other is also nonzero.

If one can establish (19.10), then one can use it to establish a transfer from certain weak automorphic L -packets of G_1 to weak automorphic L -packets of G_2 and characterize the image of the transfer. More precisely, assuming the identity (19.10) one can hope to establish a transfer from weak automorphic L -packets of G_1 containing a cuspidal automorphic representation π such that $\pi \otimes \pi^{\vee}$ is (H_1, χ_1) -distinguished to weak automorphic L -packets of G_2 containing a cuspidal automorphic representation π' such that $\pi' \times \pi'^{\vee}$ is (H_2, χ_2) -distinguished.

Often one can refine this comparison to the level of L -packets. One can even use it to define L -packets if G_2 is the Weil restriction of scalars of a general linear group. Especially at this level of refinement, in certain circumstances it may be more appropriate to replace $r: {}^L G_1 \rightarrow {}^L G_2$ with an appropriate map of dual groups of spherical varieties (see §14.4).

To recapitulate, we started with a geometric assumption of matching. Assuming this, we further assumed the fundamental lemma for the Hecke algebra and transfer. Given this pile of assumptions, we asserted that one could prove instances of Langlands functoriality. Amazingly, in many cases it

is possible to make this heuristic method rigorous and prove fantastic results using it.

19.4 Jacquet-Langlands transfer and base change

In this section we describe some settings where the general strategy in §19.3 has been successfully applied. They are related to the trace formula and the twisted trace formula of §18.4 and §18.5.

We start by taking G_1 to be an inner form of the quasi-split reductive group G_2 over F . Then there is an isomorphism

$$r : {}^L G_1 \longrightarrow {}^L G_2 \quad (19.11)$$

by Exercise 19.3. We assume that G_2^{der} is simply connected so that we can apply Theorem 19.4.1 below.

For $1 \leq i \leq 2$ we take $H_i \leq G_i \times G_i$ to be the diagonal copy of G_i and we take χ_i to be trivial. Let G_i act on itself by conjugation and denote by $G_i / \sim$ the GIT quotient. Then the quotient G_i / H_i with respect to the action (19.2) is $G_i / \sim$. In view of Proposition 17.3.4 and Example 17.2, $(G_i / H_i)(F)$ is the set of semisimple conjugacy classes in $G_i(F)$ that are Gal_F -invariant.

Using the definition of an inner form, one can construct an isomorphism of affine F -schemes

$$G_1 / H_1 \xrightarrow{\sim} G_2 / H_2 \quad (19.12)$$

such that the base change of (19.12) to an appropriate finite extension E/F is induced by an isomorphism $G_{1E} \xrightarrow{\sim} G_{2E}$. We give the construction in a special case in Lemma 19.4.2 below; the construction in general is essentially the same. This is almost a geometric correspondence between relative classes (which in the current setting are just conjugacy classes). However there is a slight wrinkle. Consider the map

$$\{\text{semisimple } \gamma \in G_i(F)\} \longrightarrow (G_i / H_i)(F) \quad (19.13)$$

sending γ to its geometric class. Because G_1 need not be quasi-split, for $i = 1$, the map (19.13) need not be surjective. However, for $i = 2$, the group G_2 is quasi-split and hence the map (19.13) is surjective by the following theorem: [Kot82]:

Theorem 19.4.1 (Kottwitz and Steinberg) *Let G be a quasi-split reductive group over a characteristic zero field k with simply connected derived group. A semisimple $G(\bar{k})$ -conjugacy class fixed under Gal_k contains an element of $G(k)$.* $\square$

In the current setting, the fundamental lemma for the Hecke algebra is trivial because G_{1F_v} and G_{2F_v} are isomorphic for almost all place v (see Exercise 19.6). Thus proving transfer of automorphic representations is reduced to understanding transfer of functions. Jacquet and Langlands [JL70] were the first to introduce an argument of this type and they used it to give a complete treatment of the special case where G_1 is the unit group of a quaternion algebra and $G_2 = \mathrm{GL}_2$. Because of this, transfers attached to the maps (19.11) are known as **Jacquet-Langlands transfers**. Though the full theory of Jacquet-Langlands transfers is still not known, many results exist in the literature. In particular, the theory for GL_n is fairly complete.

Let D be a simple algebra of dimension d^2 over F . Let M_{nD} and GL_{nD} be the affine F -schemes whose points in an F -algebra R are given by

$$\begin{aligned} M_{nD}(R) &:= M_n(D \otimes_F R), \\ \mathrm{GL}_{nD}(R) &:= M_n(D \otimes_F R)^\times. \end{aligned} \tag{19.14}$$

Thus M_{nD} is an abelian algebraic group under addition and GL_{nD} is a reductive group under multiplication.

Let E/F be a finite Galois extension such that there is an isomorphism of (not necessarily commutative) E -algebras $M_d(E) \xrightarrow{\sim} D \otimes_F E$. It induces an isomorphism

$$\tilde{t} : (M_{nd})_E \xrightarrow{\sim} (M_{nD})_E \tag{19.15}$$

which in turn induces an isomorphism of affine E -schemes

$$t : (M_{nd})_E / \mathrm{GL}_{ndE} \xrightarrow{\sim} (M_{nD})_E / (\mathrm{GL}_{nD})_E, \tag{19.16}$$

where the quotients are the GIT quotients with respect to the conjugation action. Let GL_{nd} act on itself by conjugation and denote by $\mathrm{GL}_{nd}/\sim$ the GIT quotient. We define $\mathrm{GL}_{nD}/\sim$ similarly.

By restriction t induces an isomorphism

$$t : \mathrm{GL}_{ndE}/\sim \xrightarrow{\sim} (\mathrm{GL}_{nD})_E/\sim. \tag{19.17}$$

Lemma 19.4.2 *There is an isomorphism*

$$t : \mathrm{GL}_{nd}/\sim \xrightarrow{\sim} \mathrm{GL}_{nD}/\sim$$

whose base change to E is (19.17).

Proof. We will show that (19.16) is the base change to E of an isomorphism

$$M_{nd}/\mathrm{GL}_{nd} \xrightarrow{\sim} M_{nD}/\mathrm{GL}_{nD}. \tag{19.18}$$

By restricting this isomorphism to $\mathrm{GL}_n/\sim$ we obtain the isomorphism in the lemma.

To check this, by Galois descent it suffices to check that t is $\text{Gal}(E/F)$ -invariant. In more detail, for any affine scheme X of finite type over F , there is an action of $\text{Gal}(E/F)$ (viewed as a constant group scheme over E , say) on $\text{Spec}(E)$ and on X_E such that the structure map $X_E \rightarrow \text{Spec}(E)$ is equivariant. Concretely, at the level of points, for an E -algebra R , the set $X_E(R)$ is a system of solutions in R to a set of polynomials defined over F and so inherits a $\text{Gal}(E/F)$ -action. If Y is another affine F -scheme of finite type, to check that a morphism $\varphi : X_E \rightarrow Y_E$ is the base change of a morphism $X \rightarrow Y$, it suffices to check that φ is $\text{Gal}(E/F)$ -invariant. For a more sophisticated perspective on Galois descent, [Poo17, §4.3-4.4] is a nice place to begin. Now for $\sigma \in \text{Gal}(E/F)$ consider

$$\tilde{t}^{-1} \circ \sigma \circ \tilde{t} : (M_{nd})_E \longrightarrow (M_{nd})_E.$$

This is an automorphism of $(M_{nd})_E$. It follows from the Skolem-Noether theorem that $\tilde{t}^{-1} \circ \sigma \circ \tilde{t}$ is given by conjugation by some $g_\sigma \in \text{GL}_{nd}(E)$. Thus $\tilde{t}^{-1} \circ \sigma \circ \tilde{t}$ induces the identity automorphism of $(M_{nd})_E / (\text{GL}_{nd})_E$ for all σ , which is to say that t is $\text{Gal}(E/F)$ -invariant. $\square$

For every place v , we have a diagram

$$\begin{array}{ccc} \{\text{semisimple } \delta_v \in \text{GL}_{nd}(F_v)\} & & \{\text{semisimple } \gamma_v \in \text{GL}_{nD}(F_v)\} \\ \downarrow p & & \downarrow p_D \\ (\text{GL}_{nd} / \sim)(F_v) & \xrightarrow{t} & (\text{GL}_{nD} / \sim)(F_v) \end{array}$$

where the quotients on the bottom are the GIT quotients of GL_{nd} and GL_{nD} by the conjugation action of GL_{nd} and GL_{nD} , respectively, and the vertical arrows are the canonical maps. The map p is surjective (see Exercise 19.7), but p_D is not. We say that δ_v and γ_v **match** if $t(p(\delta_v)) = p_D(\gamma_v)$.

An irreducible admissible representation π_v of $\text{GL}_{nd}(F_v)$ is D_v -**compatible** if the character $\Theta_{\pi_v}(\gamma_v)$ is nonzero for some regular semisimple $\delta_v \in \text{GL}_{nd}(F_v)$ that matches a $\gamma_v \in \text{GL}_{nD}(F_v)$. See Theorem 8.5.1 for the definition of the character Θ_{π_v} in the non-Archimedean case. The Archimedean case is similar [Kna86, Theorem 10.25]. An automorphic representation π of $A_{\text{GL}_{nd}} \backslash \text{GL}_{nd}(\mathbb{A}_F)$ is D -compatible if π_v is D_v -compatible for all v . Let S be the finite set of places v of F where $D_v \not\cong M_d(F_v)$. Then π is D -compatible if and only if π_v is D_v -compatible for all $v \in S$.

The following theorem is due to Badulescu, Grbac, and Renard [Bad08, BR10] following several previous works [BDKV84, Rog83] going back to [JL70] where the $n = 2$ case was treated.

Theorem 19.4.3 (Jacquet-Langlands correspondence) *There exists a unique injection r from the set of isomorphism classes of discrete automorphic representations of $A_{\text{GL}_{nD}} \backslash \text{GL}_{nD}(\mathbb{A}_F)$ to the set of isomorphism classes of discrete automorphic representations of $A_{\text{GL}_{nd}} \backslash \text{GL}_{nd}(\mathbb{A}_F)$ such that $r(\pi)_v \cong$*

π_v for $v \notin S$. The image consists of discrete automorphic representations that are D -compatible. $\square$

Here we are using the isomorphism $\mathrm{GL}_{nd}(F_v) \cong \mathrm{GL}_{nD}(F_v)$ for $v \notin S$ to make sense of the assertion that $r(\pi)_v \cong \pi_v$.

In the setting above, the groups G_1 and G_2 were not that different from each other. Indeed, they become isomorphic over the algebraic closure. Saito and Shintani [Sai75, Shi79] discovered the first geometric correspondence involving groups that are not isomorphic over the algebraic closure. It was fully developed for GL_2 by Langlands in [Lan80].

To explain this geometric correspondence, let E/F be a cyclic field extension and let θ be a generator of $\mathrm{Gal}(E/F)$. Let $G_1 = G$ be reductive with simply connected derived group. Take $G_2 = \mathrm{Res}_{E/F}G$, and let

$$r : {}^L G \longrightarrow {}^L \mathrm{Res}_{E/F} G$$

be the map given by the diagonal embedding on the neutral component and the identity on the Galois factor. To set up our geometric correspondence, we let χ_i be trivial, let H_1 be the image of G in $G \times G$ under the diagonal embedding, and let

$$H_2(R) := \{(g, \theta(g)) : g \in \mathrm{Res}_{E/F} G(R)\} \quad (19.19)$$

for F -algebras R . Thus $G(F)/H_1(F)$ is the set of conjugacy classes in $G(F)$ and $\mathrm{Res}_{E/F} G(F)/H_2(F)$ is the set of θ -conjugacy classes in $\mathrm{Res}_{E/F} G(F)$ in the sense of §18.5.

The geometric correspondence is defined as follows [Kot82]. Let $\delta \in \mathrm{Res}_{E/F} G(F)$ and let

$$\mathrm{N}(\delta) := \theta^{[E:F]-1}(\delta) \cdot \theta^{[E:F]-2}(\delta) \cdots \delta.$$

View $\mathrm{N}(\delta)$ as an element of $G(E)$. Then

$$\theta(\mathrm{N}(\delta)) = \delta \mathrm{N}(\delta) \delta^{-1}$$

which implies that the $G(E)$ -conjugacy class of $\mathrm{N}(\delta)$ is fixed under $\mathrm{Gal}(E/F)$. Thus there is an element of $G(F)$ in the conjugacy class by Theorem 19.4.1. We call such an element a **norm** of δ . This gives us a map

$$\begin{array}{c} \{\theta\text{-semisimple conjugacy classes in } \mathrm{Res}_{E/F} G(F)\} \\ \downarrow \\ \{\text{semisimple conjugacy classes in } G(F)\}. \end{array} \quad (19.20)$$

We say that a θ -semisimple δ and a semisimple γ **match** if γ is a norm of δ (see §18.5 for the notion of a θ -semisimple class).

This geometric correspondence forms the basis for a comparison of the trace formula of §18.4 and the twisted trace formula of §18.5. We point out that one might anticipate such a comparison by consideration of the spectral sides of the two trace formulae. Indeed, the image of the functorial transfer induced by r is contained in the set of representations π' that satisfy $\pi' \cong \pi'^\theta$, and the spectral side of the twisted trace formula is given in terms of such representations.

Unlike in the Jacquet-Langlands case, the fundamental lemma is nontrivial in this setting. However Kottwitz found a beautiful, direct, and simple argument establishing it [Kot86a]. The fundamental lemma for the whole Hecke algebra was then deduced by Clozel [Clo90a] and Labesse [Lab90]. This was the basis for the work of Arthur and Clozel on cyclic base change for GL_n stated in §13.4.

19.5 Twisted endoscopy

The twisted trace formula in §18.5 was formulated for an arbitrary finite order automorphism θ of an arbitrary reductive group G . Thus one can try to complete a comparison similar to that of the previous section in this level of generality. There are serious new difficulties that appear. The key point is that for essentially every reductive group other than GL_n two semisimple elements of $G(F)$ that are $G(\overline{F})$ -conjugate need not be $G(F)$ -conjugate.

Indeed, in the settings described in §19.4, we did not really relate elements of $G_1(F)/H_1(F)$ and $G_2(F)/H_2(F)$. We actually related elements of $(G_1/H_1)(F)$ and $(G_2/H_2)(F)$. For general G_i and H_i , these are quite different. Because of this subtlety, to understand the quotient $(G_2/H_2)(F)$, it is appropriate to place G_1 into a whole family of so-called **twisted endoscopic groups**. Making this precise is part of the **theory of twisted endoscopy**. It would take at least another book to discuss this fully and fortunately such books are already available. The reader might start with [Kot84, Kot86b] which are based on [Lan79b, Lan83]. More recent references include [CHLN11, KS99, Lab99]. Perhaps the definitive treatment is [MW16a, MW16b].

For arbitrary θ , the geometric comparison was made precise fairly early in the references mentioned above, though it is far more complicated than the geometric comparisons mentioned in §19.4 for the general linear group. Originally, it was expected that the fundamental lemma in this context would be a routine combinatorial exercise as it was in the case of GL_2 , and this is why it was described as a lemma. In fact, proving the fundamental lemma, transfer, and then applying them became a struggle of Homeric proportions involving at least two generations of mathematicians over a span of almost 30 years.

While the fundamental lemma was isolated and made precise by Shelstad and Langlands [LS87], Arthur began developing the trace formula in a form that was designed to apply the lemma. It would take too much space to record his work and accomplishments here; fortunately, there is a freely available archive of his work online. Some of the fruits of his work are described in §13.4 and §13.8. We will also not give an overview of the “long march” (in the words of Ngô, [Ngô10a]) to the fundamental lemma, preferring to leave this to the survey [Ngô10a] written by the person who completed the proof in his Fields medal winning work.

We would be remiss not to point out that despite the amazing success of twisted endoscopy, it has obvious drawbacks. As currently understood, it can only hope to establish cases of functoriality where the image of the functorial lift consists of representations that are isomorphic to their conjugates under a single automorphism. A generalization to two automorphisms is proposed in [Get12, Get20]. These papers are based on interconnected ideas of Braverman, Kazhdan, L. Lafforgue, Langlands, Ngô, and Sakellaridis [BK00, Laf14, Lan04, Ngô14, Sak12].

19.6 The interplay of distinction and twisted endoscopy

The theory of twisted endoscopy discussed in the previous section has reached a degree of maturity. One understands how to compare the twisted trace formulae and the trace formula. Since Jacquet has given us the new tool of the relative trace formula, we now have a much broader arena to apply the experience gained in the theory of twisted endoscopy and prove new results. We mention a few geometric correspondences here and connect them to the results on distinction reviewed in Chapter 14.

We begin by explaining the comparison underlying Theorem 14.5.2, which characterizes the cuspidal automorphic representations of $\mathrm{Res}_{E/F}\mathrm{GL}_n(\mathbb{A}_F)$ that are distinguished by a quasi-split unitary group. Here E/F is a quadratic extension. Let $G_1 = \mathrm{GL}_n$, $G_2 = \mathrm{Res}_{E/F}\mathrm{GL}_n$, and let

$$r : {}^L\mathrm{GL}_n \longrightarrow {}^L\mathrm{Res}_{E/F}\mathrm{GL}_n$$

be the base change map. It is given by the diagonal embedding on the neutral component and the identity on the Galois factor.

In this case, our basic outline from §19.3 must be modified along the lines of §18.6. Thus for F -algebras R , we let

$$X(R) := \{x \in M_n(E \otimes_F R) : \theta(x) = x^t\},$$

where θ is the generator of $\mathrm{Gal}(E/F)$. This admits an action of $\mathrm{Res}_{E/F}\mathrm{GL}_n$:

$$X(R) \times \mathrm{Res}_{E/F}\mathrm{GL}_n(R) \longrightarrow X(R)$$

$$(x, g) \mapsto \theta(g)^t xg.$$

The stabilizer of any $x \in X(F) \cap \text{Res}_{E/F} \text{GL}_n(F)$ is a unitary group. For $f \in C_c^\infty(X(\mathbb{A}_F))$, the function

$$g \mapsto \sum_{x \in X(F)} f(\theta(g)^t xg)$$

is left $G(F)$ -invariant. As in §18.6, it can be expanded spectrally in terms of automorphic representations of $\text{Res}_{E/F} \text{GL}_n(\mathbb{A}_F)$ that are distinguished by a unitary group.

Let N be the unipotent radical of the Borel subgroup of upper triangular matrices in GL_n . It acts on X via restriction of the action of $\text{Res}_{E/F} \text{GL}_n$. Let

$$\begin{aligned} \chi_2 : \text{Res}_{E/F} N(\mathbb{A}_F) &\longrightarrow \mathbb{C}^\times \\ u &\longmapsto \psi(u\theta(u)), \end{aligned}$$

where $\psi : N(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ is a generic character trivial on $N(F)$. We do not need to modify the geometric setting for G_1 . We can simply take

$$H_1(R) := \{(u_1^{-t}, u_2) : u_1, u_2 \in N(R)\}$$

for an F -algebra R and let $\chi_1(u_1, u_2) = \psi(u_1^t u_2)$. Note that this is precisely the geometric setting of the Kuznetsov formula of §18.7, apart from slightly modifying the action of the unipotent groups involved. Let $T \leq \text{GL}_n$ be the maximal torus of diagonal matrices. We have a diagram

$$\begin{array}{ccc} & T(F) & \\ p_2 \swarrow & & \searrow p_1 \\ X(F)/\text{Res}_{E/F} N(F) & & G_1(F)/H_1(F) \end{array}$$

where the two arrows send an element of $T(F)$ to its class. We say that a χ_2 -relevant δ and χ_1 -relevant γ match if there is a $t \in T(F)$ such that $\delta \in p_2(t)$ and $\gamma \in p_1(t)$. In this setting, the fundamental lemma was conjectured by Jacquet and Ye [JY96] and proven in two different manners by Ngô [Ngô99a, Ngô99b] and Jacquet [Jac04, Jac05b].

Jacquet and Rallis [JR11] later proposed a related comparison that eventually led to the proof of the results on the Gan-Gross-Prasad conjecture discussed in §14.7. A useful reference is [Cha19]. Let E/F be a quadratic extension of number fields and let U_n be the quasi-split unitary group of §13.7. We take

$$G_1 = U_{n+1} \times U_n \quad \text{and} \quad G_2 = \text{Res}_{E/F} \text{GL}_{n+1} \times \text{Res}_{E/F} \text{GL}_n$$

and let

$$r : {}^L(\mathrm{U}_{n+1} \times \mathrm{U}_n) \longrightarrow {}^L(\mathrm{Res}_{E/F}\mathrm{GL}_{n+1} \times \mathrm{Res}_{E/F}\mathrm{GL}_n)$$

be induced by L -maps ${}^L\mathrm{U}_{n+1} \rightarrow {}^L\mathrm{Res}_{E/F}\mathrm{GL}_{n+1}$ and ${}^L\mathrm{U}_n \rightarrow \mathrm{Res}_{E/F}\mathrm{GL}_n$ of §13.7. We take $H_i \leq G_i \times G_i$ to be the subgroups with points in an F -algebra R given by

$$\begin{aligned} H_2(R) &:= \{((\begin{smallmatrix} g & \\ & 1 \end{smallmatrix}), g) : g \in \mathrm{Res}_{E/F}\mathrm{GL}_n(R)\} \times \mathrm{GL}_{n+1}(R) \times \mathrm{GL}_n(R), \\ H_1(R) &:= \iota(\mathrm{U}_n)(R) \times \iota(\mathrm{U}_n)(R), \end{aligned}$$

where $\iota : \mathrm{U}_n \rightarrow \mathrm{U}_{n+1} \times \mathrm{U}_n$ is defined by a special case of (14.18). Thus H_2 is isomorphic to $\mathrm{Res}_{E/F}\mathrm{GL}_n \times \mathrm{GL}_{n+1} \times \mathrm{GL}_n$ and H_1 is isomorphic to $\mathrm{U}_n \times \mathrm{U}_n$. Letting

$$\eta : F^\times \backslash \mathbb{A}_F^\times / \mathrm{N}_{E/F}(\mathbb{A}_E^\times) \longrightarrow \{\pm 1\}$$

be the character attached to E/F by class field theory, we set

$$\chi_2(((\begin{smallmatrix} g & \\ & 1 \end{smallmatrix}), g), (g_1, g_2)) = \eta(\det g_1)^{n+1} \eta(\det g_2)^n.$$

We take χ_1 to be the trivial character. In this setting, one has an isomorphism of GIT quotients

$$G_2/H_2 \cong G_1/H_1$$

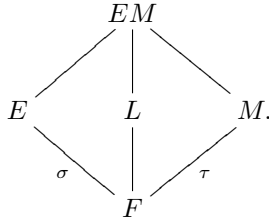
and we say that relatively regular semisimple $\delta \in G_2(F)$ and $\gamma \in G_1(F)$ match if their geometric classes correspond under this isomorphism. It turns out that regular semisimple elements have trivial stabilizers so the question of relevance is irrelevant here.

In this setting, the fundamental lemma was proven by Z. Yun [Yun11]. W. Zhang [Zha14] proved transfer and used it to deduce the global Gan-Gross-Prasad conjecture for unitary groups under certain local restrictions. These were later removed by various authors as explained in §14.7.

Both of the comparisons mentioned above are in some sense much simpler than the comparisons encountered in the theory of twisted endoscopy. The origin of this is easy to pinpoint. It is that stabilizers of relatively regular semisimple elements are trivial. As the reader can see from Chapter 17, the Galois cohomology of centralizers creates a great deal of complication in understanding orbital integrals. There is no reason to expect that one will always be able to avoid nontrivial centralizers, and it is important to develop relative analogues of the theory of twisted endoscopy discussed in §19.5.

As an example where it does not seem possible to avoid centralizers, we discuss the comparison underlying Theorem 14.6.1. As discussed in §14.6, this setup and its variants are fertile ground for future research.

For a quadratic extension E/F , we let $\mathrm{U}_{nE/F} \leq \mathrm{Res}_{E/F}\mathrm{GL}_n$ be the quasi-split unitary group defined in §13.7. We give ourselves a biquadratic extension



Thus σ is a generator for $\text{Gal}(E/F)$ and τ is a generator for $\text{Gal}(M/F)$. Let $G_1 = \text{Res}_{E/F} \text{U}_{nEM/E}$ and $G_2 = \text{Res}_{EM/F} \text{GL}_n$. Theorem 14.6.1 is a result characterizing cuspidal automorphic representations of

$$\text{Res}_{E/F} \text{U}_{nEM/E}(\mathbb{A}_F) = \text{U}_{nEM/E}(\mathbb{A}_E)$$

distinguished by $\text{U}_{nM/F}$ (under certain simplifying assumptions). To prove it one studies the behavior of distinguished representations under the functorial transfer attached to the L -map

$$r : {}^L \text{Res}_{E/F} \text{U}_{nEM/E} \longrightarrow {}^L \text{Res}_{EM/F} \text{GL}_n$$

induced by the L -map ${}^L \text{U}_{nM/F} \rightarrow {}^L \text{Res}_{M/F} \text{GL}_n$ of §13.7.

The automorphism σ induces an automorphism of $\text{Res}_{E/F} \text{U}_{nEM/E}$ with $\text{U}_{nM/F}$ as its fixed point group and an automorphism (still denoted σ) of $\text{Res}_{EM/F} \text{GL}_n$ with $\text{Res}_{M/F} \text{GL}_n$ as its fixed point group. Likewise, the automorphism τ induces an automorphism of $\text{Res}_{EM/F} \text{GL}_n$ with $\text{Res}_{E/F} \text{U}_{nEM/E}$ as its fixed point group. We set $\theta = \sigma \circ \tau$. Its fixed point group is $\text{Res}_{L/F} \text{U}_{nEM/L}$. For F -algebras R , we take

$$\begin{aligned}
 H_2(R) &= \text{Res}_{M/F} \text{GL}_n(R) \times \text{Res}_{L/F} \text{U}_{nEM/L}(R), \\
 H_1(R) &= \text{U}_{nM/F}(R) \times \text{U}_{nM/F}(R).
 \end{aligned}$$

To relate these two quotients we proceed as follows. We have morphisms (of affine F -schemes, not groups)

$$\begin{aligned}
 B_\theta &: \text{Res}_{EM/F} \text{GL}_n \longrightarrow \text{Res}_{EM/F} \text{GL}_n, \\
 B_\sigma &: \text{Res}_{E/F} \text{U}_{nEM/E} \longrightarrow \text{Res}_{E/F} \text{U}_{nEM/E}
 \end{aligned}$$

given on points by $B_\theta(g) := g\theta(g^{-1})$ and $B_\sigma(g) := g\sigma(g^{-1})$. Let Q_2 be the scheme theoretic image of B_θ and Q_1 the scheme theoretic image of B_σ . We then have isomorphisms

$$\begin{aligned}
 G_2/H_2 &\longrightarrow Q_2/\text{Res}_{M/F} \text{GL}_n, \\
 G_1/H_1 &\longrightarrow Q_1/\text{U}_{nM/F},
 \end{aligned}$$

where $\text{Res}_{M/F} \text{GL}_n$ acts via τ -conjugation on Q_2 and $\text{U}_{nM/F}$ acts via conjugation on Q_1 . This is formally similar to the geometric setting underlying

base change discussed in §19.4: one has a set of τ -conjugacy classes and a set of conjugacy classes that one wishes to relate. For a relatively regular and relatively semisimple $\delta \in \text{Res}_{EM/F} \text{GL}_n(F)$, we say that γ is a **norm** of δ if there is an $h \in \text{Res}_{M/F} \text{GL}_n(\bar{F})$ such that

$$h\gamma\sigma(\gamma^{-1})h^{-1} = \delta\theta(\delta^{-1})\tau(\delta\theta(\delta^{-1})).$$

One can show that every δ that is relatively regular and relatively semisimple admits a norm, and we say that

$$\delta \in G_2(F) = \text{Res}_{EM/F} \text{GL}_n(F) = \text{GL}_n(EM)$$

and

$$\gamma \in G_1(F) = \text{Res}_{E/F} \text{U}_{nEM/E}(F) = \text{U}_{nEM/E}(E)$$

match if γ is a norm of δ . In this context, the fundamental lemma can be obtained from the fundamental lemma for base change and transfer for certain test functions was proved in [GW14]. This provides enough information to establish Theorem 14.6.1. However there is a relative analogue of twisted endoscopy lurking here that has yet to be developed. This theory of twisted relative endoscopy will probably be necessary to obtain a version of Theorem 14.6.1 valid for arbitrary cuspidal automorphic representations of $\text{U}_{nEM/E}(\mathbb{A}_E)$.

Exercises

19.1. Relate the Laplace-Beltrami operator on a Shimura manifold $\text{Sh}(G, X)^K$ and the Casimir operator of Exercise 4.8.

19.2. For $i = 1, 2$, let G_i be a reductive group over the non-Archimedean local field F admitting a hyperspecial subgroup $K_i \leq G_i(F)$. If

$$r : {}^L G_1 \longrightarrow {}^L G_2$$

is an L -map, prove that there is a unique algebra homomorphism

$$\hat{r} : C_c^\infty(G_2(F) // K_2) \longrightarrow C_c^\infty(G_1(F) // K_1) \quad (19.14)$$

such that if π is an irreducible admissible K_1 -unramified representation of $G_1(F)$ with K_2 -unramified transfer $r(\pi)$ to $G_2(F)$ then

$$\text{tr } r(\pi)(f) = \text{tr } \pi(\hat{r}(f))$$

for all $f \in C_c^\infty(G_2(F) // K_2)$. Here we normalize the Haar measures implicit in the traces so that the K_i are given measure 1.

19.3. If G_1 and G_2 are two reductive groups over a local or global field F that are inner forms of each other, prove that ${}^L G_1$ is isomorphic to ${}^L G_2$.

19.4. Let G be a reductive group over the global field F and let $H \leq G \times G$ and $\chi : A_{G,H} \backslash H(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ be as in §18.2. Let S be a finite set of places of F including the infinite places and let $K^S < G(\mathbb{A}_F^S)$ be a compact open subgroup such that K_v is hyperspecial for $v \notin S$. Let

$$f = f_S f^S \in C_c^\infty(A_G \backslash G(\mathbb{A}_F) // K^S).$$

Prove that

$$\mathrm{rtr}(\pi(f)) = \mathrm{tr}(f^S) \mathrm{rtr}(\pi(f_S \mathbb{1}_{K^S})).$$

19.5. Construct the isomorphism (19.12).

19.6. Let G_1 and G_2 be reductive groups over a global field F that are inner forms of each other. Prove that $G_{1F_v} \cong G_{2F_v}$ for all but finitely many places v of F .

19.7. Let k be a field of characteristic zero. Prove that every semisimple conjugacy class in $\mathrm{GL}_n(\bar{k})$ that is fixed by Gal_k has an element in $\mathrm{GL}_n(k)$. Prove moreover that two elements $\gamma_1, \gamma_2 \in \mathrm{GL}_n(k)$ that are $\mathrm{GL}_n(\bar{k})$ -conjugate are $\mathrm{GL}_n(k)$ -conjugate.

19.8. Let $T \leq \mathrm{GL}_n$ be a maximal torus over a field k . Prove that $\mathcal{D}(k, T, \mathrm{GL}_n)$, defined as in (17.23), has only one element.

Appendix A

Groups Attached to Involutions of Algebras

We define certain groups attached to involutions of algebras. Many of the structures introduced in Chapter 1 are then described explicitly for these groups.

A.1 Algebras with involution

Our goal in this appendix is to provide interesting examples of the general theory of reductive groups outlined in Chapter 1. We focus on so-called classical groups. For more information about exceptional groups one possible reference is [KMRT98].

Let D be a semisimple algebra over a field k_0 of characteristic not equal to 2. We let k_0^{sep} be a separable closure of k_0 and let $\bar{k}_0 \geq k_0^{\text{sep}}$ be an algebraic closure. We assume that D is equipped with an **involution** $*$: $D \rightarrow D$, that is, an k_0 -linear map $x \mapsto x^*$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$. A **homomorphism** of algebras with involution $\phi : (D, *) \rightarrow (D', \dagger)$ is a homomorphism of k_0 -algebras $D \rightarrow D'$ such that $\phi(x^*) = \phi(x)^\dagger$. An isomorphism of algebras with involution is a bijective homomorphism of algebras with involution.

We assume that one of the following holds:

- (1) The center k of D is k_0 , k_0 is fixed pointwise by $*$, and D is simple.
- (2) The center k of D is a quadratic extension of k_0 , the involution $*$ induces the nontrivial automorphism of k over k_0 , and D is simple.
- (3) The center k of D is isomorphic to $k_0 \oplus k_0$, the involution $*$ induces the nontrivial automorphism of k over k_0 , and $D \cong D_0 \oplus D_0$ where D_0 is a simple k_0 -algebra.

In case (1) (resp. (2)) one refers to $*$ as an involution of the first (resp. second) kind. In case (3) one refers to $*$ as a split involution of the second kind.

Example A.1. Let $D = M_n(k_0)$, the space of $n \times n$ matrices over k_0 . Then for any symmetric or skew-symmetric matrix $J \in \text{GL}_n(k_0)$ the map $x \mapsto Jx^t J^{-1}$ defines an involution of the first kind on D .

Example A.2. Assume k/k_0 is a quadratic extension of fields; we denote the action of the nontrivial element of $\text{Gal}(k/k_0)$ using a bar. Let $D = M_n(k)$. Let $J \in \text{GL}_n(k)$ be a hermitian matrix, i.e. $J^t = \bar{J}$. Then the map $x^* := J\bar{x}^t J^{-1}$ defines an involution of the second kind on D .

We observe that given a field extension k'/k_0 the base extension $D_{k'} := D \otimes_{k_0} k'$ is again a semisimple algebra, now over k' , and $*$ induces a k' -linear involution on $D_{k'}$ that we will continue to denote by $*$. If $*$ is an involution of the first kind or a split involution of the second kind, then for any extension k'/k the induced involution will again be of the first kind or a split involution of the second kind, respectively. Assume we are in case (2). If $k \otimes_{k_0} k'$ is again a field then $*$ is an involution of the second kind on $D_{k'}$. If $k \otimes_{k_0} k'$ is not a field then the center of $D_{k'}$ is isomorphic to $k' \oplus k'$ and $*$ induces a split involution of the second kind on $D_{k'}$. The fact that involutions of the second kind become split involutions after a field extension is our only motivation for discussing split involutions of the second kind.

Let k'/k_0 be a field extension. We say that D **splits over** k' if $D_{k'}$ is isomorphic to $M_n(k \otimes_{k_0} k')$. Here we view the space $M_n(k \otimes_{k_0} k')$ of $n \times n$ matrices over $k \otimes_{k_0} k'$ as an algebra via matrix multiplication. The following is proved in [Sch85, §8.5]:

Theorem A.1.1 *There exists a finite separable extension k'/k_0 over which D splits.* $\square$

Choose a field k'/k_0 over which D splits. We then have an embedding

$$D \longrightarrow D \otimes_{k_0} k' \cong M_n(k \otimes_{k_0} k') \quad (\text{A.1})$$

for some n , where the arrow sends x to $x \otimes 1$. For each $x \in D$ we can take the image of x in $M_n(k \otimes_{k_0} k')$ and then take its determinant and trace. For all k_0 -algebras R this yields the **reduced norm**

$$\text{N} : D \otimes_{k_0} R \longrightarrow k \otimes_{k_0} R \quad (\text{A.2})$$

and the **reduced trace**

$$\text{Tr} : D \otimes_{k_0} R \longrightarrow k \otimes_{k_0} R \quad (\text{A.3})$$

respectively. The definition is independent of the choice of k' and it follows from the Skolem-Noether theorem that it is also independent of the choice of isomorphism in (A.1). If $D = M_n(k)$ where k is a field then the reduced trace is the usual trace and the reduced norm is the determinant.

There are various groups we can attach to these data. For k_0 -algebras R we let

$$\begin{aligned}\mathrm{GL}_D(R) &:= (D \otimes_{k_0} R)^\times, \\ \mathrm{SL}_D(R) &:= \{g \in \mathrm{GL}_D(R) : N(g) = 1\}.\end{aligned}\tag{A.4}$$

If D splits over k' then $\mathrm{GL}_{Dk'} \cong \mathrm{GL}_{nk'}$ via an isomorphism sending $\mathrm{SL}_{Dk'}$ to $\mathrm{SL}_{nk'}$ in case (1) and

$$\mathrm{GL}_{Dk'} \cong \mathrm{GL}_{nk'} \times \mathrm{GL}_{nk'}$$

in case (2) and (3).

Consider the algebraic groups over k_0 whose points in an k_0 -algebra R are given respectively by

$$\begin{aligned}\mathrm{Iso}(D, *) (R) &:= \{g \in (D \otimes_{k_0} R)^\times : gg^* = I_D\}, \\ \mathrm{SIso}(D, *) (R) &:= \{g \in (D \otimes_{k_0} R)^\times : gg^* = I_D, N(g) = 1\}.\end{aligned}\tag{A.5}$$

Here I_D is the identity of D . In case (3), $I_D = (I_{D_0}, I_{D_0})$. These groups are referred to as the **isometry group** and the **special isometry group** of the pair $(D, *)$, respectively. We have

$$\mathrm{SIso}(D, *) \leq \mathrm{Iso}(D, *).\tag{A.6}$$

Lemma A.1.2 *The group GL_D is reductive, the group SL_D is semisimple, and $\mathrm{GL}_D^{\mathrm{der}} = \mathrm{SL}_D$. The groups $\mathrm{Iso}(D, *)^\circ$ and $\mathrm{SIso}(D, *)$ are reductive. When $\mathrm{SIso}(D, *)$ is semisimple then $(\mathrm{Iso}(D, *)^\circ)^{\mathrm{der}} = \mathrm{SIso}(D, *)$.*

In fact it follows from theorems A.3.1, A.3.2, and A.3.3, below that in most cases $\mathrm{SIso}(D, *)$ is almost simple, that is, it is semisimple and its Lie algebra is a simple Lie algebra.

Proof. An algebraic group G is semisimple (resp. reductive) if and only if its base change along any field extension is semisimple (resp. reductive) [Mil17, Proposition 19.5 and 19.13]. Thus GL_D is reductive and SL_D is semisimple. Since the kernel of the reduced norm $N : \mathrm{GL}_D \rightarrow \mathrm{Res}_{k/k_0} \mathbb{G}_m$ is SL_D we have $\mathrm{GL}_D^{\mathrm{der}} \leq \mathrm{SL}_D$. This inclusion is a monomorphism, and since it induces an isomorphism $\mathrm{GL}_D^{\mathrm{der}}(\bar{k}_0) \xrightarrow{\sim} \mathrm{SL}_D(\bar{k}_0)$ it is a quotient map (see the discussion after Corollary 1.3.7). Hence the inclusion is an isomorphism, that is, $\mathrm{GL}_D^{\mathrm{der}} = \mathrm{SL}_D$.

Let G be a smooth affine algebraic group over k_0 and let $\sigma : G \rightarrow G$ be an automorphism of order 2. Let

$$G^\sigma(R) := \{g \in G(R) : g^\sigma = g\}$$

be the fixed points of this automorphism. Since we assumed the residual characteristic of k_0 is not 2, we have that G^σ is a smooth algebraic group [Edi92, Proposition 3.4]. Moreover, if G is simply connected then G^σ is a reductive group by [Ste68, Theorem 8.1]. Setting $G = \mathrm{SL}_D$ and $\sigma(g) = (g^*)^{-1}$

and observing that SL_D is simply connected we deduce that $\mathrm{SIsO}(D, *)$ is reductive.

On the other hand the reduced norm induces an isomorphism between $\mathrm{Iso}(D, *)^\circ / \mathrm{SIsO}(D, *)$ and a subgroup of $\mathrm{Res}_{k/k_0} \mathbb{G}_m$. Since the unipotent radical of $\mathrm{Iso}(D, *)^\circ_{\bar{k}_0}$ has trivial image in $(\mathrm{Res}_{k/k_0} \mathbb{G}_m)_{\bar{k}_0}$ we deduce that the unipotent radical of $\mathrm{Iso}(D, *)^\circ_{\bar{k}_0}$ is contained in the unipotent radical of $\mathrm{SIsom}(D, *)_{\bar{k}_0}$, which is trivial. Thus $\mathrm{Iso}(D, *)^\circ$ is reductive.

The fact that $\mathrm{Iso}(D, *)^\circ / \mathrm{SIsO}(D, *)$ is isomorphic to a subgroup of $\mathrm{Res}_{k/k_0} \mathbb{G}_m$ also implies that $(\mathrm{Iso}(D, *)^\circ)^{\mathrm{der}} \leq \mathrm{SIsO}(D, *)$. When $\mathrm{SIsO}(D, *)$ is semisimple it is its own derived group [Mil17, Summary 21.54]. We deduce the last assertion of the lemma. $\square$

We leave the following lemma to the reader:

Lemma A.1.3 *In case (3), there is an isomorphism*

$$\mathrm{Iso}(D, *) \xrightarrow{\sim} \mathrm{GL}_D \quad (\text{A.7})$$

that sends $\mathrm{SIsO}(D, *)$ to SL_D . $\square$

Because of Lemma A.1.3, we assume that we are in case (1) or (2) for the remainder of this appendix.

Let us describe the Lie algebras of the groups introduced above. The natural maps

$$\begin{array}{ccc} & \mathrm{SL}_D & \\ \nearrow & & \searrow \\ \mathrm{SIsO}(D, *) & & \mathrm{GL}_D \\ \searrow & & \nearrow \\ & \mathrm{Iso}(D, *) & \end{array}$$

are all monomorphisms, or equivalently closed immersions. Using Proposition 1.3.6 and the definition of the Lie functor in (1.16) we see that we have injections of k_0 -vector spaces

$$\begin{array}{ccc} & \mathrm{Lie} \, \mathrm{SL}_D & \\ \nearrow & & \searrow \\ \mathrm{Lie} \, \mathrm{SIsO}(D, *) & & \mathrm{Lie} \, \mathrm{GL}_D \\ \searrow & & \nearrow \\ & \mathrm{Lie} \, \mathrm{Iso}(D, *) & \end{array} \quad (\text{A.8})$$

Thus all of these Lie algebras may be viewed as subalgebras of $\mathrm{Lie} \, \mathrm{GL}_D$, with the bracket given by restriction of the bracket on $\mathrm{Lie} \, \mathrm{GL}_D$.

Lemma A.1.4 *There is a canonical k_0 -linear isomorphism $\text{Lie GL}_D \xrightarrow{\sim} D$ that sends the Lie bracket on Lie GL_D to the commutator $[x, y] := xy - yx$ on D .*

Proof. The canonical k_0 -linear isomorphism is given by

$$\begin{aligned} \ker((D \otimes_{k_0} k_0[t]/(t^2))^\times \rightarrow D^\times) &= \{I_D + xt : x \in D\} \xrightarrow{\sim} D \\ I_D + xt &\mapsto x. \end{aligned} \tag{A.9}$$

Let V be D viewed as a k_0 -vector space. The left multiplication action of D on V induces a faithful representation $\text{GL}_D \rightarrow \text{GL}_V$. This in turn yields a commutative diagram

$$\begin{array}{ccc} \text{Lie GL}_D & \hookrightarrow & \text{Lie GL}_V \\ \downarrow \sim & & \downarrow \sim \\ D & \hookrightarrow & \mathfrak{gl}_V(k_0). \end{array}$$

As discussed in §1.6, taking Lie algebras is a functor and the Lie algebra bracket on $\mathfrak{gl}_V(k_0)$ is given by $[X, Y] = X \circ Y - Y \circ X$. The description of the Lie bracket on Lie GL_D follows. $\square$

Regard D as a k_0 -vector space. Combining (A.8) with Lemma A.1.4 we see that we can identify each of the Lie algebras occurring in (A.8) with a subspace of D . We use this identification without further comment below.

Lemma A.1.5 *One has that*

$$\begin{aligned} \text{Lie SL}_D &= \{x \in D : \text{Tr}(x) = 0\}, \\ \text{Lie Iso}(D, *) &= \{x \in D : x = -x^*\}, \\ \text{Lie SISO}(D, *) &= \{x \in D : x = -x^*, \text{Tr}(x) = 0\}. \end{aligned}$$

Proof. For each $x \in D$ consider the element $I_D + xt$ of $D \otimes_{k_0} k_0[t]/(t^2)$. Using the isomorphism (A.9) we see that

$$\text{Lie SL}_D = \{x \in D : N(I_D + xt) = 1\} = \{x \in D : \text{Tr}(x) = 0\}.$$

Moreover

$$\begin{aligned} \text{Lie Iso}(D, *) &= \{x \in D : (I_D + xt)(I_D + xt)^* = I_D\} \\ &= \{x \in D : x = -x^*\}. \end{aligned}$$

Finally, the last equality follows because

$$\text{Lie SISO}(D, *) = \text{Lie Iso}(D, *) \cap \text{Lie SL}_D.$$

$\square$

A.2 Split simple algebras

Assume D is split over k_0 . Thus there is an isomorphism

$$D \cong \text{End}_{k_0}(V) = M_n(k)$$

where $V = k^n$. Here as above k is the center of D . We use this isomorphism to identify D and $\text{End}_{k_0}(V)$ in this section. We have a perfect pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : V \times V &\longrightarrow k \\ ((v_i), (w_i)) &\longmapsto \langle (v_i), (w_i) \rangle := \sum_{i=1}^n v_i w_i. \end{aligned} \tag{A.10}$$

We denote by t the transpose with respect to this pairing; this is just the standard transpose.

In case (1) and case (2) the involution $*$ of $D = \text{End}_{k_0}(V)$ can be described as follows [KMRT98, Proposition 2.19 and 2.20]:

- (1) There is a $J \in \text{GL}_V(k_0)$ such that for $x \in \text{End}_{k_0}(V)$ one has $x^* = Jx^t J^{-1}$ and one of the following holds:
 - (1a) J is symmetric (i.e. $J^t = J$),
 - (1b) J is skew symmetric (i.e. $J^t = -J$).
- (2) There is a $J \in \text{GL}_V(k)$ that is hermitian (i.e. $\overline{J}^t = J$) such that for $x \in \text{End}_{k_0}(V)$ one has $x^* = J\overline{x}^t J^{-1}$.

In case (1a) we say that the involution is **of orthogonal type**, in case (1b) we say that the involution is **of symplectic type**, and in case (2) we say that the involution is **of unitary type**. More generally, if $(D', *)$ is a simple k_0 -algebra (not necessarily split) with an involution, we say that $(D', *)$ is of orthogonal (resp. symplectic) type if $(D'_{k_0^{\text{sep}}}, *)$ is of orthogonal (resp. symplectic) type. If $*$ defines a split involution of the second type on $D'_{k_0^{\text{sep}}}$ then we say $(D', *)$ is of unitary type. We point out that in case (1b) n must be even.

For $v, w \in V$ let

$$\langle v, w \rangle_J := \langle v, Jw \rangle. \tag{A.11}$$

This is a nondegenerate bilinear form on V . We list its properties depending on the cases mentioned above. Let $v, w \in V$.

- (1) The form $\langle \cdot, \cdot \rangle_J$ is k_0 -bilinear and
 - (1a) symmetric: $\langle v, w \rangle_J = \langle w, v \rangle_J$,
 - (1b) alternating: $\langle v, w \rangle_J = -\langle w, v \rangle_J$.
- (2) The form $\langle \cdot, \cdot \rangle_J$ is k -antilinear, that is, $a\langle v, w \rangle = \langle av, w \rangle = \langle v, \overline{a}w \rangle$ for $a \in k^\times$, and hermitian: $\langle v, w \rangle_J = \overline{\langle w, v \rangle_J}$.

Lemma A.2.1 *Let $(D_1, \dagger_1)$ and $(D_2, \dagger_2)$ be k_0 -algebras with involution such that $\dim_{k_0} D_1 = \dim_{k_0} D_2$. Assume that one of the following holds:*

- $\dagger_1$ and $\dagger_2$ are both involutions of the first kind of orthogonal type,
- $\dagger_1$ and $\dagger_2$ are both involutions of the first kind of symplectic type,
- $\dagger_1$ and $\dagger_2$ are both involutions of the second kind.

Then $(D_{1k_0^{\text{sep}}}, \dagger_1)$ is isomorphic to $(D_{2k_0^{\text{sep}}}, \dagger_2)$.

Proof. Recall that D_1 and D_2 split over a separable extension by Theorem A.1.1. Thus in the third case $\dagger_1$ and $\dagger_2$ are split involutions of the second kind. One checks that there is an integer n and an isomorphism

$$D_{ik_0^{\text{sep}}} \cong M_n(k_0^{\text{sep}}) \oplus M_n(k_0^{\text{sep}})$$

sending $\dagger_i$ to the involution $(x, y) \mapsto (y, x)$.

In the first and second cases by the discussion at the beginning of this section we may assume that $D_{1k_0^{\text{sep}}} = D_{2k_0^{\text{sep}}} = M_n(k_0^{\text{sep}})$ and that $x^{\dagger_i} = J_i x^t J_i^{-1}$. The matrices J_1 and J_2 are both either symmetric or skew-symmetric, and in either case we can choose a $g \in \text{GL}_n(k_0^{\text{sep}})$ such that $gJ_1g^t = J_2$. This implies the desired equivalence. $\square$

For k_0 -algebras R let

$$\begin{aligned} G_J(R) &:= \{g \in \text{GL}_n(k \otimes_{k_0} R) : \langle gv, gw \rangle_J = \langle v, w \rangle_J\}, \\ SG_J(R) &:= G_J(R) \cap \text{SL}_n(R). \end{aligned} \tag{A.12}$$

Then

$$\text{Iso}(D, *) = G_J \quad \text{and} \quad \text{SIso}(D, *) = SG_J. \tag{A.13}$$

There is standard notation for the groups G_J and SG_J and standard names for the groups depending on the type of involution and J . They are given in the following table:

Case	Name	G_J	SG_J
(1a)	orthogonal group of $\langle, \rangle_J$	O_J	SO_J
(1b)	symplectic group of $\langle, \rangle_J$	Sp_J	Sp_J
(2)	unitary group of $\langle, \rangle_J$	U_J	SU_J

Here in case (1a) the group SO_J is known as the special orthogonal group and in case (2) the group SU_J is known as the special unitary group. In case (1b) $SG_J = G_J$.

In order to facilitate computations and simplify certain statements below, let

$$J_n := \begin{pmatrix} & & 1 \\ & \ddots & \\ & & \\ 1 & & \end{pmatrix} \in \mathrm{GL}_n(\mathbb{Z})$$

be the antidiagonal matrix. For even n let

$$J'_n := \begin{pmatrix} & J_{n/2} \\ -J_{n/2} & \end{pmatrix}. \quad (\text{A.14})$$

We set

$$\begin{aligned} \mathrm{O}_n &:= \mathrm{O}_{J_n}, & \mathrm{SO}_n &:= \mathrm{SO}_{J_n}, \\ \mathrm{Sp}_n &:= \mathrm{Sp}_{J'_n}, \\ \mathrm{U}_n &:= \mathrm{U}_{J_n}, & \mathrm{SU}_n &:= \mathrm{SU}_{J_n}. \end{aligned}$$

This agrees with the notation from §13.7.

Lemma A.2.2 *Let n be the dimension of V as above. The rank and dimension of SG_J are given in the following table:*

Case	Rank	Dimension
(1a)	$\lfloor \frac{n}{2} \rfloor$	$\frac{(n-1)n}{2}$
(1b)	$\frac{n}{2}$	$\frac{(n+1)n}{2}$
(2)	$n - 1$	$n^2 - 1$

Proof. The quantities considered in the lemma are unchanged under base change so we may replace SG_J with $(\mathrm{SG}_J)_{\bar{k}_0}$. Therefore by Lemma A.2.1 it suffices to compute the rank and dimension of SO_n , Sp_n , and SU_n . One can compute the rank directly. The dimension of a smooth algebraic group over k_0 is the same as the dimension of its Lie algebra as a k_0 -vector space [Mil17, Proposition 1.37]. The dimension of $\mathrm{Lie SO}_n$, $\mathrm{Lie Sp}_n$ and $\mathrm{Lie SU}_n$ can be computed using Lemma A.1.5. $\square$

A.3 Classification of forms

We now drop the assumption that D is split and discuss the groups $\mathrm{SG}(D, *)$ as $(D, *)$ varies. Forms and inner forms of algebraic groups were defined in §14.7. A form is said to be an **outer form** if it is not an inner form. The following is [Mil17, Theorem 24.62]:

Theorem A.3.1 *The forms of SO_n are the groups $\mathrm{SG}(D, *)$ where D is a simple algebra of dimension n^2 over k_0 and $*$ is an involution of the first kind of orthogonal type. Two such forms $\mathrm{SG}(D, *)$ and $\mathrm{SG}(D', \dagger)$ are isomorphic if and only if $(D, *) \cong (D', \dagger)$. $\square$*

We will not describe when the forms are inner and outer in this case.

The symplectic analogue is [Mil17, Theorem 24.47]:

Theorem A.3.2 *Assume n is even. The forms of Sp_n are all inner. They are the $G(D, *)$ where D is a simple algebra of dimension n^2 over k_0 and $*$ is an involution of the first kind of symplectic type. Two such forms $\mathrm{SG}(D, *)$ and $\mathrm{SG}(D', \dagger)$ are isomorphic if and only if $(D, *) \cong (D', \dagger)$. $\square$*

Finally, the analogue for involutions of the second kind is [Mil17, Theorem 24.43 and 24.44]:

Theorem A.3.3 *The inner forms of SL_n over a field k_0 are given by SL_D where D is a simple algebra of dimension n^2 over k_0 . Two such inner forms SL_D and $\mathrm{SL}_{D'}$ are isomorphic if and only if $D \cong D'$ as k_0 -algebras.*

The outer forms are given by $\mathrm{SG}(D, *)$ where $(D, *)$ is a simple algebra of the second kind with center k , a quadratic field extension of k_0 , and where $\dim_k D = n^2$. Two such outer forms $\mathrm{SG}(D, *)$ and $\mathrm{SG}(D', \dagger)$ are isomorphic if and only if $(D, *) \cong (D', \dagger)$. $\square$

The group SO_n is almost simple for $n \geq 3$, the group Sp_{2n} is almost simple for $n \geq 1$, and the groups SU_n and SL_n are almost simple for $n \geq 2$. This justifies our claim after Lemma A.1.2 above.

A.4 Parabolic subgroups

In this section we collect some information about parabolic subgroups of the groups G_J° and SG_J for the choices of J isolated in §A.2. First it is helpful to state the behavior of parabolic subgroups when one restricts to the derived group [DG74, §XXVI, Proposition 1.19]:

Proposition A.4.1 *Let G be a reductive group. There is a bijection*

$$\begin{aligned} \{\text{parabolic subgroups } P \leq G\} &\xrightarrow{\sim} \{\text{parabolic subgroups } P' \leq G^{\mathrm{der}}\} \\ P &\longmapsto P \cap G^{\mathrm{der}} \\ Z_G^t P' &= N_G(P') \longleftarrow P' \end{aligned}$$

where Z_G^t is the maximal subtorus of Z_G . Moreover, the unipotent radical of P is equal to the unipotent radical of $P \cap G^{\mathrm{der}}$. $\square$

To apply this proposition in the case at hand, it is helpful to point out that

$$Z_{\mathrm{GL}_D}(R) = \{xI_D : x \in (k \otimes_{k_0} R)^\times\}.$$

Because of Proposition A.4.1, to give a description of the parabolic subgroups of G_J° , it suffices to give a description of the parabolic subgroups of SG_J .

To ease notation, let

$$SG_n := \begin{cases} SO_n & \text{in case (1a),} \\ Sp_n & \text{in case (1b),} \\ SU_n & \text{in case (2).} \end{cases} \quad (\text{A.15})$$

Let $T_n \leq GL_n$ be the maximal torus of diagonal matrices, let $B_n \leq GL_n$ be the Borel subgroup of upper triangular matrices, and let N_n be the unipotent radical of B_n . Set

$$T := (\text{Res}_{k/k_0} T_n \cap SG_n)^\circ, \quad N := \text{Res}_{k/k_0} N_n \cap SG_n, \quad B := TN. \quad (\text{A.16})$$

Let $\epsilon \in \{1, -1\}$ be 1 in case (1a) and (2) and -1 in case (1b). Let T_0 be the trivial group over k_0 in case (1) and let

$$T_0(R) := \{a \in (k \otimes_{k_0} R)^\times : \bar{a}^{-1} = a\} \quad (\text{A.17})$$

in case (2). If $n = 2\ell$ is even, for k_0 -algebras R we have that

$$\begin{aligned} T(R) &= \left\{ \begin{pmatrix} t & \\ & J_\ell \bar{t}^{-1} J_\ell^{-1} \end{pmatrix} : t \in T_\ell(k \otimes_{k_0} R), \det(t\bar{t}^{-1}) = 1 \right\}, \\ N(R) &= \left\{ \begin{pmatrix} u & ux \\ & J_\ell (\bar{u}^{-1})^t J_\ell^{-1} \end{pmatrix} : \begin{array}{l} (u, x) \in N_\ell(k \otimes_{k_0} R) \times M_\ell(k \otimes_{k_0} R) \\ J_\ell \bar{x}^t J_\ell^{-1} = -\epsilon x \end{array} \right\}. \end{aligned}$$

If $n = 2\ell + 1$ is odd, we have that

$$\begin{aligned} T(R) &= \left\{ \begin{pmatrix} t & a \\ & J_\ell \bar{t}^{-1} J_\ell^{-1} \end{pmatrix} : (t, a) \in T_\ell(k \otimes_{k_0} R) \times T_0(R), \det(t\bar{t}^{-1}) = a^{-1} \right\}, \\ N(R) &= \left\{ \begin{pmatrix} u & uy & ux \\ & 1 & -\bar{y}^t J_\ell^{-1} \\ & J_\ell \bar{u}^{-t} J_\ell^{-1} & \end{pmatrix} : \begin{array}{l} (u, y) \in N_\ell(k \otimes_{k_0} R) \times (k \otimes_{k_0} R)^\ell \\ x \in M_\ell(k \otimes_{k_0} R), J_\ell \bar{x}^t J_\ell^{-1} + x = -y \otimes (\bar{y}^t J_\ell^{-1}) \end{array} \right\}. \end{aligned}$$

Here in case (1) the extension k/k_0 is trivial and the bars are understood to be the identity automorphism.

In general the intersection of a maximal torus of a reductive group G with a subgroup H need not be a maximal torus of H , and the intersection of a Borel subgroup of G with H need not be a Borel subgroup of H . Despite this, we have the following lemma:

Lemma A.4.2 *The group T is a maximal torus of SG_n and B is a Borel subgroup of SG_n . In particular SG_n is quasi-split.*

Proof. Using Lemma A.2.2 one checks that $\dim T$ is equal to the rank of SG_n which implies that T is a maximal torus. Now B is a connected solvable subgroup of SG_n and hence to prove that it is a Borel subgroup, it suffices to prove that its dimension equal to the dimension of a Borel subgroup B' of $SG_{n\bar{k}_0}$. We have

$$\dim B' = \frac{\dim SG_{n\bar{k}_0}}{2} + \frac{\dim T_{\bar{k}_0}}{2}$$

by the Bruhat decomposition [Mil17, Theorem 21.84], and the quantity on the right may be computed using Lemma A.2.2. Upon computing the dimension of B one deduces the lemma. $\square$

We combine Lemma A.4.2 with the discussion above to give some examples illustrating groups that are not quasi-split and quasi-split but not split. It is helpful to first state the following theorem [Mil17, Corollary 23.53]:

Theorem A.4.3 *Let G be a quasi-split reductive group and let G' be an inner form of G . Then G' is isomorphic to G if and only if G' is quasi-split as well.* $\square$

This theorem, combined with theorems A.3.1, A.3.2, A.3.3 and Lemma A.4.2, implies the following two corollaries:

Corollary A.4.4 *If $SG(D, *)$ is quasi-split then D is split.* $\square$

Corollary A.4.5 *If k/k_0 is a field extension then the group SU_n is quasi-split but not split.* $\square$

We choose B as our minimal parabolic subgroup and describe the standard parabolic subgroups of SG_n . Let $T_s \leq T$ be the maximal split torus. For k_0 -algebras R one has

$$T_s(R) := \left\{ \begin{pmatrix} t & & \\ & J_{n/2} t J_{n/2}^{-1} & \\ & & \ddots \end{pmatrix} : t \in T_{n/2}(R) \right\} \quad (\text{A.18})$$

when n is even and

$$T_s(R) := \left\{ \begin{pmatrix} t & & & \\ & 1 & & \\ & & J_{(n-1)/2} t J_{(n-1)/2}^{-1} & \\ & & & \ddots \end{pmatrix} : t \in T_{(n-1)/2}(R) \right\} \quad (\text{A.19})$$

when n is odd. Thus $T = T_s$ unless we are in case (2).

Let

$$t = \begin{pmatrix} t_1 & & & \\ & \ddots & & \\ & & t_{\lfloor n/2 \rfloor} & \\ & & & \ddots \end{pmatrix}.$$

For $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ define

$$\varepsilon_j \begin{pmatrix} t & & \\ & J_{n/2} t J_{n/2}^{-1} & \\ & & \ddots \end{pmatrix} := t_j$$

when n is even and

$$\varepsilon_j \begin{pmatrix} t & & & \\ & 1 & & \\ & & J_{(n-1)/2} t J_{(n-1)/2}^{-1} & \\ & & & \ddots \end{pmatrix} := t_j$$

when n is odd.

To determine the set of roots $\Phi(SG_n, T_s)$ of T_s in SG_n it suffices to determine the set of positive roots $\Phi(SG_n, T_s)^+$ defined by B , that is, the set of roots in B . Indeed,

$$\Phi(SG_n, T_s) = \Phi(SG_n, T_s)^+ \amalg \Phi(SG_n, T_s)^- \quad (\text{A.20})$$

where $\Phi(SG_n, T_s)^-$ is the set of roots in the opposite parabolic B^{op} , and hence

$$\Phi(SG_n, T_s)^- = \{-\chi : \chi \in \Phi(SG_n, T_s)^+\}.$$

To avoid some degenerate special cases we assume $n \geq 2$. The set $\Phi(SG_n, T_s)^+$ and the type of the root system is given by the following table:

Case	$\Phi(SG_n, T_s)^+$	Type
$\text{SO}_n, n \text{ even}$	$\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq \frac{n}{2}$ $\varepsilon_i + \varepsilon_j : 1 \leq i < j \leq \frac{n}{2}$	$D_{n/2}$
$\text{SO}_n, n \text{ odd}$	$\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq \frac{n-1}{2}$ $\varepsilon_i : 1 \leq i \leq \frac{n-1}{2}$ $\varepsilon_i + \varepsilon_j : 1 \leq i < j \leq \frac{n-1}{2}$	$B_{(n-1)/2}$
$\text{Sp}_n, n \text{ even}$	$\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq \frac{n}{2}$ $\varepsilon_i + \varepsilon_j : 1 \leq i \leq j \leq \frac{n}{2}$	$C_{n/2}$
$\text{SU}_n, n \text{ odd}$	$\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq \frac{n-1}{2}$ $\varepsilon_i : 1 \leq i \leq \frac{n-1}{2}$ $\varepsilon_i + \varepsilon_j : 1 \leq i \leq j \leq \frac{n-1}{2}$	$BC_{(n-1)/2}$
$\text{SU}_n, n \text{ even}$	same as Sp_n	$C_{n/2}$

When n is even define

$$\alpha_j := \begin{cases} \varepsilon_j - \varepsilon_{j+1} & \text{if } 1 \leq j < n/2 \\ \varepsilon_{j-1} + \varepsilon_j & \text{if } j = n/2, \quad \text{SO}_n \text{ case} \\ 2\varepsilon_j & \text{if } j = n/2, \quad \text{Sp}_n \text{ case} \end{cases} \quad (\text{A.21})$$

and when n is odd

$$\alpha_j = \begin{cases} \varepsilon_j - \varepsilon_{j+1} & \text{if } 1 \leq j < (n-1)/2 \\ \varepsilon_j & \text{if } j = (n-1)/2. \end{cases} \quad (\text{A.22})$$

The tuple $\Delta := (\alpha_1, \alpha_2, \dots, \alpha_{[n/2]})$ is the set of simple roots associated to B . To verify that the table above and the computation of the simple roots is correct, the standard reference [Bou02, §VI.4] is helpful. For comparison with loc. cit. we point out that BC_r is the unique irreducible nonreduced root system of rank r (see [Bou02, §VI.4.14]).

The derived groups of Levi subgroups of SG_n are isomorphic to a product of special linear groups together with a group isomorphic to SG_i for some

integer $0 \leq i \leq n$ of the same parity as n . Let us restrict our attention to maximal proper parabolic subgroups and make this precise.

Recall from §1.9 that standard parabolic subgroups (with respect to B) are in bijection with subsets of the set of simple roots Δ . The standard maximal proper parabolic subgroups are the parabolic subgroups of the form $P_{\Delta - \{\alpha_j\}}$ for $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$. The unique Levi subgroup of $P_{\Delta - \{\alpha_j\}}$ containing T is $M_{\Delta - \{\alpha_j\}}$. We leave the proof of the following proposition to the reader:

Proposition A.4.6 *Let R be a k_0 -algebra. In case (1) the group $M_{\Delta - \{\alpha_j\}}(R)$ is*

$$\left\{ \begin{pmatrix} g & & \\ & h & \\ & & J_j g^{-t} J_j^{-1} \end{pmatrix} : (g, h) \in \mathrm{GL}_j(R) \times \mathrm{SG}_{n-2j}(R) \right\}.$$

In case (2) the group $M_{\Delta - \{\alpha_j\}}(R)$ is

$$\left\{ \begin{pmatrix} g & & \\ & h & \\ & & J_j \bar{g}^{-t} J_j^{-1} \end{pmatrix} : (g, h) \in \mathrm{GL}_j(R) \times \mathrm{U}_{n-2j}(R), \det(g\bar{g}^{-1}) \det h = 1 \right\}.$$

□

We end this section by describing the root data of SG_n in case (1). Recall that this is the tuple

$$(X^*(T), X_*(T), \Phi(\mathrm{SG}_n, T), \Phi^\vee(\mathrm{SG}_n, T))$$

defined in §1.8. We freely use the results of that section.

We have already described $\Phi(\mathrm{SG}_n, T)$. We have also described the base $\Delta \subset \Phi(\mathrm{SG}_n, T)$ corresponding to the Borel subgroup B above. As recalled in §7.3 above, the set $\Delta^\vee := \{\alpha^\vee : \alpha \in \Delta\}$ is a base for the root system $\Phi^\vee(\mathrm{SG}_n, T)$. It hence determines $\Phi^\vee(\mathrm{SG}_n, T)$ [Mil17, Proposition C.52]. For $x \in R^\times$ and $1 \leq j \leq \lfloor \frac{n}{2} \rfloor$ let

$$t_j(x) := \begin{pmatrix} I_{j-1} & & \\ & x & \\ & & I_{\lfloor n/2 \rfloor - j} \end{pmatrix}.$$

When n is even let

$$\mu_j(x) := \begin{pmatrix} t_j(x) & & \\ & J_{n/2} t_j(x) J_{n/2}^{-1} & \end{pmatrix}$$

and when n is odd let

$$\mu_j(x) := \begin{pmatrix} t_j(x) & & \\ & 1 & \\ & & J_{n-1/2} t_j(x) J_{(n-1)/2}^{-1} \end{pmatrix}.$$

Proposition A.4.7 *Assume we are in case (1). When n is even we have*

$$\alpha_j^\vee := \begin{cases} \mu_j - \mu_{j+1} & \text{if } 1 \leq j < n/2 \\ \mu_{j-1} + \mu_j & \text{if } j = n/2, \text{ SO}_n \text{ case} \\ \mu_j & \text{if } j = n/2, \text{ Sp}_n \text{ case} \end{cases}$$

and when n is odd we have

$$\alpha_j^\vee = \begin{cases} \mu_j - \mu_{j+1} & \text{if } 1 \leq j < (n-1)/2 \\ 2\mu_j & \text{if } j = (n-1)/2. \end{cases}$$

Proof. Recall the reflection s_{α_j} of (1.30). As explained in (1.33), we must show

$$s_{\alpha_j}(\chi) = \chi - \langle \chi, \alpha_j^\vee \rangle \alpha_j \quad (\text{A.23})$$

for all $\chi \in X^*(T)$. If $1 \leq j < \lfloor \frac{n}{2} \rfloor$ then the reflection s_α is represented by

$$\left(\begin{pmatrix} I_{j-1} & & \\ & 1 & \\ & & I_{n/2-j-1} \end{pmatrix} \quad J_{n/2} \begin{pmatrix} I_{j-1} & & \\ & 1 & \\ & & I_{n/2-j-1} \end{pmatrix} J_{n/2}^{-1} \right)$$

if n is even and

$$\left(\begin{pmatrix} I_{j-1} & & \\ & 1 & \\ & & I_{(n-1)/2-j-1} \end{pmatrix} \quad 1 \quad J_{n/2} \begin{pmatrix} I_{j-1} & & \\ & 1 & \\ & & I_{(n-1)/2-j-1} \end{pmatrix} J_{(n-1)/2}^{-1} \right)$$

if n is odd. Thus for $1 \leq j < \lfloor \frac{n}{2} \rfloor$ we have $s_{\alpha_j}(\varepsilon_j) = \varepsilon_{j+1}$ and $s_{\alpha_j}(\varepsilon_i) = \varepsilon_i$ if $i \notin \{j, j+1\}$. This implies the formula for $\alpha_j^\vee$ stated in the proposition when $1 \leq j < \lfloor n/2 \rfloor$.

Assume n is even and we are in the SO_n case. Then $s_{\alpha_{n/2}}$ is represented by

$$\left(\begin{pmatrix} I_{n/2-2} & & \\ & \begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix} & \\ & & I_{n/2-2} \end{pmatrix} \right).$$

Thus for $1 \leq j < \frac{n}{2} - 1$ we have $s_{\alpha_{n/2}}(\varepsilon_j) = \varepsilon_j$ and $s_{\alpha_{n/2}}(\varepsilon_{n/2-1}) = -\varepsilon_{n/2}$, $s_{\alpha_{n/2}}(\varepsilon_{n/2}) = -\varepsilon_{n/2-1}$. We deduce that $\alpha_{n/2}^\vee = \mu_{n/2-1} + \mu_{n/2}$.

If n is odd and we are in the SO_n case then $s_{\alpha_{(n-1)/2}}$ is represented by

$$\begin{pmatrix} I_{(n-1)/2-1} & & \\ & \begin{smallmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{smallmatrix} & \\ & & I_{(n-1)/2-1} \end{pmatrix}.$$

Thus for $1 \leq j < \lfloor \frac{n}{2} \rfloor$ we have $s_{\alpha_{n/2}}(\varepsilon_j) = \varepsilon_j$ and $s_{\alpha_{n/2}}(\varepsilon_{n/2}) = -\varepsilon_{n/2}$. We obtain $\alpha_{(n-1)/2}^\vee = 2\mu_{(n-1)/2}$.

Finally if n is even and we are in the Sp_{2n} case then $s_{\alpha_{n/2}}$ is represented by

$$\begin{pmatrix} I_{n/2-1} & & \\ & \begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} & \\ & & I_{n/2-1} \end{pmatrix}.$$

Thus for $1 \leq j < \lfloor \frac{n}{2} \rfloor$ we have $s_{\alpha_j}(\varepsilon_j) = \varepsilon_j$ and $s_{\alpha_{n/2}}(\varepsilon_{n/2}) = -\varepsilon_{n/2}$. This implies $\alpha_{n/2}^\vee = \mu_{n/2}$. $\square$

Comparing Proposition A.4.7 with the description of the simple roots of various root systems in (A.21) and (A.22) we obtain the following table:

Case	Type of $\Phi(\mathrm{SG}_n, T_s)$	Type of $\Phi^\vee(\mathrm{SG}_n, T_s)$
SO_n , n even	$D_{n/2}$	$D_{n/2}$
SO_n , n odd	$B_{(n-1)/2}$	$C_{(n-1)/2}$
Sp_n , n even	$C_{n/2}$	$D_{n/2}$

Appendix B

The Iwasawa Decomposition

We prove the existence of the Iwasawa decomposition using theorems of Bruhat and Tits to treat the ramified case. We also discuss the compatibility of the Iwasawa decomposition and the Levi decomposition of parabolic subgroups.

B.1 Introduction

Let F be a local or global field and let P be a parabolic subgroup of a reductive F -group G . Fix a Levi subgroup $M \leq P$ and let $N \leq P$ be the unipotent radical. Thus $P = MN$. In the local case, we say that a maximal compact subgroup $K \leq G(F)$ is **in good position with respect to (P, M)** if the Iwasawa decomposition

$$G(F) = P(F)K \tag{B.1}$$

holds and

$$P(F) \cap K = (M(F) \cap K)(N(F) \cap K). \tag{B.2}$$

Similarly, if F is a global field we say that a maximal compact subgroup $K = \prod_v K_v \leq G(\mathbb{A}_F)$ is **in good position with respect to (P, M)** if K_v is good position with respect to (P_{F_v}, M_{F_v}) for all places v of F .

Our main goal in this appendix is to prove the following theorem:

Theorem B.1.1 *Let G be a reductive group over a global or local field F and let $P \leq G$ be a parabolic subgroup with Levi subgroup M . There exists a maximal compact subgroup K of $G(\mathbb{A}_F)$ or $G(F)$, respectively, in good position with respect to (P, M) . We can even assume that $M(\mathbb{A}_F) \cap K$ and $M(F) \cap K$ are maximal compact subgroups of $M(\mathbb{A}_F)$ and $M(F)$, respectively.*

The proof of this theorem, in general, invokes the formidable work of Bruhat-Tits. However, at least outside of a finite set S of places of F including the infinite places that in practice one can control, one can obtain the decomposition using much less. Our aim in this appendix is to explain this to readers who have some background in algebraic geometry. The proof will also explain precisely how one chooses K , at least outside of finite set S of places of F . The reason for including this material is twofold. First, in most applications one needs at least a basic understanding of the relationship between the K and $P(\mathbb{A}_F)$ occurring in the Iwasawa decomposition. Second, the proof gives us the opportunity to introduce concepts and techniques that are useful in a broad context.

At the end of the appendix in §B.5 we record some complementary information in the local unramified case. We include a statement of the Cartan decomposition in this setting.

B.2 Some group schemes

In order to explain how one can choose K given P , it is useful to discuss notions of reductive and parabolic subgroups over the base ring $\mathcal{O}_F$. Our primary reference for this is [Con14]. Before beginning, we recall some terminology that is ubiquitous in algebraic geometry (specialized to the case of schemes over an affine base).

Let k be a commutative ring with identity. A **geometric point** of the affine scheme $\mathrm{Spec}(k)$ is the spectrum $\bar{s} := \mathrm{Spec}(\bar{L})$ of an algebraic closure of a field L equipped with a morphism

$$\bar{s} \longrightarrow \mathrm{Spec}(k).$$

Recall from §1.2 that to give such a morphism is equivalent to giving a morphism of commutative rings with identity $k \rightarrow \bar{L}$. A **geometric fiber** of a k -scheme X is the base change $X_{\bar{s}}$ where $\bar{s}$ is a geometric point of $\mathrm{Spec}(k)$.

Definition B.2.1. A **reductive group scheme** $\mathcal{G}$ over k is an affine smooth group scheme such that its geometric fibers are reductive groups.

This (trivially) recovers our original definition of reductive group when k is a field. Now suppose $\mathcal{G}$ is a reductive group scheme over k .

Definition B.2.2. A **parabolic subgroup scheme** $\mathcal{P}$ of $\mathcal{G}$ is a smooth affine k -group scheme equipped with a monic homomorphism $\mathcal{P} \rightarrow \mathcal{G}$ of group schemes such that for all geometric points $\bar{s} \rightarrow \mathrm{Spec}(k)$ the fiber $\mathcal{P}_{\bar{s}}$ is a parabolic subgroup of $\mathcal{G}_{\bar{s}}$. A **Borel subgroup scheme** is a parabolic subgroup scheme $\mathcal{B}$ such that $\mathcal{B}_{\bar{s}}$ is a Borel subgroup of $\mathcal{G}_{\bar{s}}$ for all geometric points $\bar{s} \rightarrow \mathrm{Spec}(k)$.

If $\mathcal{P}$ is a parabolic subgroup of $\mathcal{G}$ then the map $\mathcal{P} \rightarrow \mathcal{G}$ is a closed immersion by [Con14, Proposition 5.2.3].

Assume we are given a reductive group scheme $\mathcal{G}$ and a parabolic subgroup scheme $\mathcal{P} \leq \mathcal{G}$. Consider the functor which assigns to any k -algebra R the set of closed subgroup schemes of $\mathcal{G}$ that are étale locally conjugate to $\mathcal{P}$. Then this functor is representable by a scheme called $\mathcal{G}/\mathcal{P}$ over k . This scheme is not affine, but it is smooth and proper and equipped with a canonical k -ample line bundle (and hence admits a canonical embedding into a projective space over k) [Con14, Corollary 5.2.8]. We note that there is a natural transformation of functors $\mathcal{G} \rightarrow \mathcal{G}/\mathcal{P}$ given on points in a k -algebra R by

$$\begin{aligned} \mathcal{G}(R) &\longrightarrow (\mathcal{G}/\mathcal{P})(R) \\ g &\longmapsto g\mathcal{P}(R)g^{-1}. \end{aligned} \tag{B.3}$$

If R is a field the map (B.3) is surjective:

Lemma B.2.3 *If k is a field then the map*

$$\mathcal{G}(k) \longrightarrow (\mathcal{G}/\mathcal{P})(k)$$

is surjective and induces a bijection $\mathcal{G}(k)/\mathcal{P}(k) \xrightarrow{\sim} (\mathcal{G}/\mathcal{P})(k)$.

Proof. Let $\bar{k}$ be an algebraic closure of k . By the generalities on parabolic subgroups over fields recalled after Theorem 1.9.3, every parabolic subgroup of $\mathcal{G}(k)$ that is $\mathcal{G}(\bar{k})$ -conjugate to $\mathcal{P}$ is in fact $\mathcal{G}(k)$ -conjugate to $\mathcal{P}$. $\square$

The following lemma is extremely useful:

Lemma B.2.4 *Let $\mathcal{H}$ be a smooth affine group scheme of finite type over a Henselian local ring $\mathcal{O}$ with finite residue field κ and let $\mathcal{X}$ be a $\mathcal{H}$ -torsor. If the special fiber $\mathcal{H}_\kappa$ is connected then*

$$\mathcal{X}(\mathcal{O}) \neq \emptyset.$$

Proof. Since $\mathcal{G}$ is smooth, so is $\mathcal{X}$ (see §17.6). Therefore, by Hensel's lemma (Lemma 17.7.5), the map $\mathcal{X}(\mathcal{O}) \rightarrow \mathcal{X}(\kappa)$ is surjective. It therefore suffices to show that $\mathcal{X}(\kappa) \neq \emptyset$. But $\mathcal{X}_\kappa$ is a torsor under the connected affine algebraic group $\mathcal{H}_\kappa$, hence $\mathcal{X}(\kappa) \neq \emptyset$ by a theorem of Lang [Lan56]. $\square$

We can now prove a version of the Iwasawa decomposition if we assume the existence of the parabolic subgroup scheme $\mathcal{P}$:

Theorem B.2.5 *Suppose that S is a finite set of places of a global field F containing the infinite places, that $\mathcal{G}$ is a reductive group scheme over $\mathcal{O}_F^S$ and that $\mathcal{P} \leq \mathcal{G}$ is a parabolic subgroup scheme over $\mathcal{O}_F^S$. Then*

$$\mathcal{G}(\mathbb{A}_F^S) = \mathcal{P}(\mathbb{A}_F^S)\mathcal{G}(\hat{\mathcal{O}}_F^S).$$

Proof. Suppose that $g \in \mathcal{G}(\mathbb{A}_F^S)$. Then $g \in \mathcal{G}(\mathcal{O}_{F_v})$ for all but finitely many places v , so it suffices to show that for every $v \notin S$ one has that

$$\mathcal{G}(F_v) = \mathcal{P}(F_v)\mathcal{G}(\mathcal{O}_{F_v}).$$

Fix $v \notin S$ and let $\varpi \in \mathcal{O}_{F_v}$ be a uniformizer.

One has a commutative diagram

$$\begin{array}{ccc} \mathcal{G}(\mathcal{O}_{F_v}) & \longrightarrow & \mathcal{G}(F_v) \\ a \downarrow & & \downarrow \\ (\mathcal{G}/\mathcal{P})(\mathcal{O}_{F_v}) & \xrightarrow{b} & (\mathcal{G}/\mathcal{P})(F_v). \end{array}$$

The scheme $\mathcal{G}/\mathcal{P}$ is projective over $\mathcal{O}_{F_v}$, hence proper, and thus the arrow b is a bijection by the valuative criterion of properness. In view of Lemma B.2.3 it therefore suffices to show that the arrow a is surjective. The fiber of a over any point of $(\mathcal{G}/\mathcal{P})(\mathcal{O}_{F_v})$ is a $\mathcal{P}$ -torsor, and hence has a point by Lemma B.2.4. $\square$

B.3 The dynamic method

In the statement of Theorem B.1.1, there is no mention of the group scheme $\mathcal{G}$ or $\mathcal{P}$; one has only the reductive group G and its parabolic subgroup P . To apply Theorem B.2.5 in the setting of Theorem B.1.1 we must construct $\mathcal{G}$ and $\mathcal{P}$ given G and P . One starts by choosing a faithful representation $G \rightarrow \mathrm{GL}_n$. For a sufficiently large finite set S of places of F containing the infinite ones, the schematic closure $\mathcal{G}$ of G in $\mathrm{GL}_{n\mathcal{O}_F^S}$ is then reductive by Proposition 2.4.8. Thus we can construct $\mathcal{G}$. We now construct $\mathcal{P}$.

Proposition B.3.1 *Let S be a finite set of places of a global field F containing the infinite places and let $\mathcal{G}$ be a reductive group scheme over $\mathcal{O}_F^S$. Let P be a parabolic subgroup of $\mathcal{G}_F$ with Levi decomposition $P = MN$. After possibly adding a finite set of places to S , the schematic closure $\mathcal{P}$ of P in $\mathcal{G}$ is a parabolic subgroup scheme of $\mathcal{G}$ admitting a decomposition*

$$\mathcal{P} = \mathcal{M} \ltimes \mathcal{N}, \tag{B.4}$$

where $\mathcal{M}$ is a reductive group scheme, $\mathcal{N}$ is a connected smooth group scheme with unipotent fibers, $\mathcal{M}_F = M$, and $\mathcal{N}_F = N$.

In the setting of the proposition, we have $\mathcal{P}(\mathcal{O}_{F_v}) = \mathcal{M}(\mathcal{O}_{F_v}) \ltimes \mathcal{N}(\mathcal{O}_{F_v})$ for $v \notin S$. Moreover $\mathcal{M}$ and $\mathcal{N}$ are the schematic closures of their generic fibers in $\mathcal{G}$.

Applying Lemma 2.4.6 this implies that

$$P(F_v) \cap \mathcal{G}(\mathcal{O}_{F_v}) = (M(F_v) \cap \mathcal{G}(\mathcal{O}_{F_v}))(N(F_v) \cap \mathcal{G}(\mathcal{O}_{F_v})),$$

the condition required in (B.2). We also note that $\mathcal{M}(\mathcal{O}_{F_v}) \leq M(F_v)$ is a hyperspecial subgroup (and in particular is maximal, see Theorem 2.4.3).

To prove Proposition B.3.1 we use the so-called **dynamic** description of parabolic subgroups. Suppose that we are given an affine group scheme $\mathcal{G}$ of finite presentation over a ring k and a cocharacter, i.e. a morphism of group schemes

$$\lambda : \mathbb{G}_m \longrightarrow \mathcal{G}.$$

This defines a left action of $\mathbb{G}_m$ on $\mathcal{G}$

$$\lambda : \mathbb{G}_m \times \mathcal{G} \longrightarrow \mathcal{G}$$

given on points by

$$\lambda(t, g) := \lambda(t)g\lambda(t)^{-1}.$$

We say that $\lim_{t \rightarrow 0} \lambda(t, g)$ **exists** if the morphism λ extends to a morphism

$$\lambda : \mathbb{G}_a \times \mathcal{G} \longrightarrow \mathcal{G}.$$

Example B.1. Let $\lambda : \mathbb{G}_m \rightarrow \mathrm{GL}_3$ be the cocharacter given as

$$\lambda(t) = \begin{pmatrix} t & & \\ & t^{-1} & \\ & & t^{-2} \end{pmatrix}.$$

Then

$$\lambda(t) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \lambda(t)^{-1} = \begin{pmatrix} a_{11} & t^2 a_{12} & t^3 a_{13} \\ t^{-2} a_{21} & a_{22} & t a_{23} \\ t^{-3} a_{31} & t^{-1} a_{32} & a_{33} \end{pmatrix}.$$

Thus $\lim_{t \rightarrow 0} \lambda(t, g)$ exists if and only if g is a point of the Borel subgroup of upper triangular matrices. Moreover $\lim_{t \rightarrow 0} \lambda(t, g) = 1$, the identity in GL_3 , if and only if g lies in the group of strictly upper triangular matrices, and g commutes with the action of λ if and only if g is a diagonal matrix.

Let $P(\lambda)$ be the group functor on k -algebras given on a k -algebra R by

$$P(\lambda)(R) := \left\{ g \in \mathcal{G}(R) : \lim_{t \rightarrow 0} \lambda(t, g) \text{ exists} \right\}.$$

We also define

$$N(\lambda)(R) = \left\{ g \in \mathcal{G}(R) : \lim_{t \rightarrow 0} \lambda(t, g) = 1 \right\}$$

and let $Z(\lambda)(R) \leq \mathcal{G}(R)$ be the subgroup that commutes with the action λ . We then have

$$P(\lambda)(R) = Z(\lambda)(R) \ltimes N(\lambda)(R). \quad (\text{B.5})$$

In the example above, $P(\lambda)$ turned out to be a parabolic subgroup of the ambient group scheme. Moreover, the geometric fibers of $N(\lambda)$ (resp. $Z(\lambda)$) turned out to be the unipotent radical (resp. Levi subgroup) of the corresponding fiber of $P(\lambda)$. This is a general phenomenon, and is the basis of the **dynamic** description of parabolic subgroups, which is summarized in Theorem B.3.2 and Theorem B.3.3 below. For the proof of Theorem B.3.2 see [Con14, Theorem 4.1.7, Example 5.2.2]:

Theorem B.3.2 *Assume that $\mathcal{G}$ is smooth. Then the functor $P(\lambda)$ is represented by a closed smooth subgroup of $\mathcal{G}$. If $\mathcal{G}$ has connected fibers then so does $P(\lambda)$. If $\mathcal{G}$ is connected and reductive, $\mathcal{T} \leq \mathcal{G}$ is a maximal torus, and λ has image in $\mathcal{T}$ then $P(\lambda)$ is a parabolic subgroup scheme of $\mathcal{G}$. $\square$*

Here a torus $\mathcal{T}$ over $\text{Spec}(k)$ is a group scheme over k that is fppf locally isomorphic to $\mathbb{G}_m^n$ for some $n \geq 0$. In other words, for each point $x \in \text{Spec}(k)$ there exists a neighborhood $U \subseteq \text{Spec}(k)$ of x and a faithfully flat morphism $U' \rightarrow U$ of finite presentation such that $\mathcal{T}_{U'} \cong \mathbb{G}_{mU'}^n$. A torus $\mathcal{T}$ is a maximal torus of $\mathcal{G}$ if it is a closed subgroup scheme of $\mathcal{G}$ such that for all geometric points $\bar{s}$ of $\text{Spec}(k)$, the fiber $\mathcal{T}_{\bar{s}}$ is a maximal torus of $\mathcal{G}_{\bar{s}}$.

Theorem B.3.3 *Assume that k is a field and G is a reductive group over k . Let $P \leq G$ be a parabolic subgroup. It admits a Levi decomposition. Given a Levi decomposition $P = MN$, there is a cocharacter $\lambda : \mathbb{G}_m \rightarrow G$ such that $P = P(\lambda)$, $M = Z(\lambda)$, and $N = N(\lambda)$.*

Proof. By [Mil17, Theorem 25.1] there is a cocharacter $\lambda : \mathbb{G}_m \rightarrow G$ such that $P = P(\lambda)$. Thus P admits the Levi decomposition (B.5). Suppose we are given a Levi decomposition $P = MN$. Then $N = N(\lambda)$ since both groups are the unipotent radical of P . All Levi subgroups of P are conjugate under $N(k)$ by [Spr09, Proposition 16.1.1], so upon conjugating λ by an element of $N(k)$ we may assume that $M = Z(\lambda)$. $\square$

Using these theorems, we can now prove Proposition B.3.1:

Proof of Proposition B.3.1: Let P be a parabolic subgroup of G . Then there is a cocharacter $\lambda : \mathbb{G}_m \rightarrow G$ such that $P = P(\lambda)$ by Theorem B.3.3. Let T be a maximal torus of P containing the image of λ ; then T is also a maximal torus of G . Choose a faithful representation $G \rightarrow \text{GL}_n$. Upon choosing a sufficiently large finite set S of places of F , we can assume that the schematic closure $\mathcal{G}$ of G in $\text{GL}_{n\mathcal{O}_F^S}$ is reductive (see Proposition 2.4.8), that the schematic closure $\mathcal{T}$ of T in $\mathcal{G}$ is a maximal torus of $\mathcal{G}$ [Con14, Corollary 3.1.8 and Proposition 3.1.9], and that the cocharacter $\lambda : \mathbb{G}_m \rightarrow T$ extends to a cocharacter $\Lambda : \mathbb{G}_m \rightarrow \mathcal{T}$. Here we are using some basic facts on spreading out that are summarized in [Pool17, §3.2]. Let

$$\mathcal{P} := P(\Lambda).$$

Then by Theorem B.3.2, $\mathcal{P}$ is a parabolic subgroup of $\mathcal{G}$. Since $\mathcal{P}$ is the schematic closure of its generic fiber we deduce the proposition. $\square$

B.4 Proof of Theorem B.1.1

We now complete the proof of Theorem B.1.1. In view of Theorem B.2.5 and Proposition B.3.1, it suffices to prove Theorem B.1.1 in the local case. Thus we assume for the remainder of the section that F is a local field.

We will actually prove something more precise than what is required by Theorem B.1.1. Fix a minimal parabolic subgroup $P_0 \leq G$ and a Levi subgroup $M_0 \leq P_0$. We write N_0 for the unipotent radical of P_0 . We call a pair (P, M) consisting of a parabolic subgroup $P \leq G$ and a Levi subgroup $M \leq P$ a **standard parabolic pair** if $P_0 \leq P$ and $M_0 \leq M$.

Theorem B.4.1 *Assume F is Archimedean. There is a maximal compact subgroup $K \leq G(F)$ such that for any standard parabolic pair (P, M) , one has that $G(F) = P(F)K$, $M(F) \cap K$ is a maximal compact subgroup of $M(F)$, and $P(F) \cap K = M(F) \cap K$.*

Proof. By [Kna02, Proposition 7.31], the Iwasawa decomposition $G(F) = P(F)K$ holds for some maximal compact subgroup $K \leq G(F)$. Let $G(F)^+$ be the neutral component of the real Lie group $G(F)$. All maximal compact subgroups of $G(F)$ are $G(F)^+$ -conjugate by Theorem 4.4.2. Thus $G(F) = P(F)K$ holds for all maximal compact subgroups K of $G(F)$.

By Theorem 4.4.1, any maximal compact subgroup of $G(F)$ intersects all components of the real Lie group $G(F)$. Combining this statement with the fact that all maximal compact subgroups of $G(F)$ are $G(F)^+$ -conjugate one can prove the following assertion: A subgroup $K \leq G(F)$ is a maximal compact subgroup if and only if $K \cap G(F)^+$ is a maximal compact subgroup of $G(F)^+$ and K intersects all components of $G(F)$. In view of this, the assertion that we can choose K so that $M(F) \cap K$ is a maximal compact subgroup of $M(F)$ is part of [HC75, Lemma 8].

Consider the image of K in $P(F)/N(F) = M(F)$. It is a compact subgroup containing $M(F) \cap K$, hence equal to $M(F) \cap K$. If $mn \in K$ with $(m, n) \in M(F) \times N(F)$, we deduce that $m \in M(F) \cap K$, and hence $n \in N(F) \cap K = I$. This completes the proof of the last assertion. $\square$

We now prepare to prove the non-Archimedean analogue of Theorem B.4.1. We require some results and terminology from Bruhat-Tits theory. The book [KP23] is a wonderful resource for more details on the constructions that follow.

Let $BT(G)$ denote the Bruhat-Tits building of G over F . It is a topological space equipped with a continuous action of $G(F)$ and is a union of subsets called **apartments** $A(G, T)$ indexed by the maximal split tori T of G . We

will require the notion of special and hyperspecial points of $BT(G)$; we refer to [KP23, Definitions 1.3.39 and 7.11.1] for definitions. When consulting references to [KP23] below, it is helpful to know that special points exist and a special point is a vertex in the terminology of loc. cit. (see [KP23, Proposition 1.3.43]). Let

$$G(F)^1 := \bigcap_{\chi \in X^*(G)} \ker(|\cdot| \circ \chi : G(F) \longrightarrow \mathbb{R}_{>0}). \quad (\text{B.6})$$

Proposition B.4.2 *If $x \in BT(G)$ is a special point, then the stabilizer $G(F)_x^1$ of x in $G(F)^1$ is a maximal compact subgroup of $G(F)$.*

Proof. By [KP23, Theorem 4.2.15(1)], $G(F)_x^1$ is a maximal bounded open subgroup of $G(F)$. The group $G(F)_x^1$ is clearly closed in $G(F)$, hence compact. On the other hand, any maximal compact subgroup of $G(F)$ is bounded, and moreover is open by Theorem 2.4.5. The proposition follows. $\square$

Theorem B.4.3 *Assume F is non-Archimedean. There is a maximal compact subgroup $K \leq G(F)$ such that for any standard parabolic pair (P, M) , one has that $G(F) = P(F)K$, $M(F) \cap K$ is a maximal compact subgroup of $M(F)$, and $P(F) \cap K = (M(F) \cap K)(N(F) \cap K)$.*

Proof. Assume that $T \leq M_0$ is a maximal split torus. Let $x \in A(G, T)$ be a special point. Then $K := G(F)_x^1$ is a maximal compact subgroup of $G(F)$ by Proposition B.4.2. Moreover $G(F)_x^1$ contains the parahoric subgroup $\mathcal{P}_x = G(F)_x^0$ attached to x (see [KP23, §7.4] for the definition of a parahoric subgroup). Thus by [KP23, Theorem 5.3.4] one has $P_0(F)G(F)_x^1 = G(F)$. It follows immediately that $P(F)K = G(F)$ for all standard parabolic subgroups P of G .

We now claim that $M(F) \cap K$ is a maximal compact subgroup of $M(F)$. Assuming this, the rest of the theorem follows as in the proof of Theorem B.4.1. The inclusion $M \leq G$ induces a projection

$$p : A(G, T) \longrightarrow A(M, T).$$

The image of a special point is again a special point [KP23, §6.7.2]. On the other hand $M(F) \cap K = M(F)_{p(x)}^1$ by [KP23, Corollary 9.7.3]. Thus $M(F) \cap K$ is maximal by Proposition B.4.2. $\square$

B.5 Addenda in the hyperspecial case

Let F be a non-Archimedean local field with ring of integers $\mathcal{O}_F$. Let ϖ be a uniformizer. Suppose that G is a reductive group over F that is unramified (quasi-split and split over an unramified extension). Thus there is a smooth

model $\mathcal{G}$ of G over $\mathcal{O}_F$ with reductive fibers by Theorem 2.4.3. The group $\mathcal{G}(\mathcal{O}_F)$ is a hyperspecial subgroup of $G(F)$.

Lemma B.5.1 *There exists a Borel subgroup scheme $\mathcal{B}$ of $\mathcal{G}$ and a maximal torus $\mathcal{T}$ of $\mathcal{G}$ satisfying $\mathcal{T} \leq \mathcal{B}$.*

Proof. We first check that a Borel subgroup exists. The functor assigning to an $\mathcal{O}_F$ -scheme S the set of Borel subgroups of $\mathcal{G}_S$ is representable by a smooth proper $\mathcal{O}_F$ -scheme, say $\mathcal{X}$ [DG74, XXII, Corollaire 5.8.3(i)]. We have $\mathcal{X}(F) = \mathcal{X}(\mathcal{O}_F)$ by the valuative criterion of properness, so there is a Borel subgroup $\mathcal{B} \leq \mathcal{G}$. The existence of $\mathcal{T}$ is [DG74, XXII, Corollaire 5.9.7]. $\square$

Lemma B.5.2 *Let $\mathcal{B}$ be a Borel subgroup scheme of $\mathcal{G}$ and let $\mathcal{T} \leq \mathcal{B}$ be a maximal torus. The subgroup $\mathcal{T}(\mathcal{O}_F) = \mathcal{T}(F) \cap \mathcal{G}(\mathcal{O}_F)$ is a maximal compact subgroup of $\mathcal{T}(F)$ and $\mathcal{G}(\mathcal{O}_F)$ is in good position with respect to $(\mathcal{B}_F, \mathcal{T}_F)$.*

Proof. The assertion that $\mathcal{T}(\mathcal{O}_F) \leq \mathcal{T}(F)$ is a maximal compact subgroup is [Mac17, Proposition 4.6]. One has a decomposition

$$\mathcal{B} = \mathcal{T} \ltimes \mathcal{N}$$

where $\mathcal{N}$ is a connected smooth group scheme with unipotent fibers [DG74, XXVI, Proposition 1.6]. In particular, $\mathcal{N}_F$ is the unipotent radical of $\mathcal{B}_F$. This implies that

$$\mathcal{B}(F) \cap \mathcal{G}(\mathcal{O}_F) = \mathcal{B}(\mathcal{O}_F) = \mathcal{T}(\mathcal{O}_F) \mathcal{N}(\mathcal{O}_F) = (\mathcal{T}(F) \cap \mathcal{G}(\mathcal{O}_F))(\mathcal{N}(F) \cap \mathcal{G}(\mathcal{O}_F)).$$

Moreover, $\mathcal{B}(F) \mathcal{G}(\mathcal{O}_F) = G(F)$ by the argument proving Theorem B.2.5 $\square$

Let $\mathcal{H}$ be a smooth affine group scheme of finite type over $\mathcal{O}_F$. We say that a torus $\mathcal{T} \leq \mathcal{H}$ is a **maximal split torus** if its generic (resp. special) fiber is a maximal split torus of the generic (resp. special) fiber of $\mathcal{H}$.

Lemma B.5.3 *Assume the special fiber of $\mathcal{H}$ is connected. Let $\mathcal{T}$ and $\mathcal{T}'$ be maximal split tori of $\mathcal{H}$. Then $\mathcal{T}$ and $\mathcal{T}'$ are $\mathcal{H}(\mathcal{O}_F)$ -conjugate.*

Proof. For $\mathcal{O}_F$ -algebras R consider the transporter

$$\text{Transp}(\mathcal{T}, \mathcal{T}')(R) := \{g \in \mathcal{H}(R) : g\mathcal{T}g^{-1} = \mathcal{T}'\}.$$

It is represented by a smooth closed subscheme of $\mathcal{H}$ [Con14, Proposition 2.1.2]. We are to show that $\text{Transp}(\mathcal{T}, \mathcal{T}')(\mathcal{O}_F) \neq \emptyset$. By Hensel's lemma (Lemma 17.7.5) it suffices to check that $\text{Transp}(\mathcal{T}, \mathcal{T}')(\mathcal{O}_F / \varpi) \neq \emptyset$. This follows from Theorem 1.7.6. $\square$

We now state the Cartan decomposition, which was introduced in a less precise form in Theorem 5.5.3:

Theorem B.5.4 (The Cartan decomposition) *Let $\mathcal{T}$ be a maximal split torus of $\mathcal{G}$ and let $B \geq \mathcal{T}_F$ be a Borel subgroup of G . Then*

$$G(F) = \coprod_{\lambda} \mathcal{G}(\mathcal{O}_F)\lambda(\varpi)\mathcal{G}(\mathcal{O}_F)$$

where the disjoint union is over all $\lambda \in X_*(\mathcal{T}_F)$ that are dominant with respect to B .

Proof. By [Her11b, §3] there exists a reductive group scheme $\mathcal{G}'$ with Borel subgroup $\mathcal{B}'$, and a maximal split torus $\mathcal{T}' \leq \mathcal{B}'$ such that

$$G(F) = \coprod_{\lambda} \mathcal{G}'(\mathcal{O}_F)\lambda(\varpi)\mathcal{G}'(\mathcal{O}_F) \quad (\text{B.7})$$

where the disjoint union is over all $\lambda \in X_*(\mathcal{T}'_F)$ that are antidominant with respect to $\mathcal{B}'_F$. More specifically, in the notation of [KP23], $\mathcal{G}'$ is the parahoric group scheme $\mathcal{G}_x^0$ attached to a hyperspecial point $x \in BT(G)$. We point out that in [Her11b, §3] the author assumes that F is of characteristic zero, but this is not used in the argument.

We claim that (B.7) remains valid if we instead take the disjoint union over λ that are antidominant with respect to any Borel subgroup $B'' \leq G$ containing $\mathcal{T}'_F$. If B'' contains $\mathcal{T}'_F$ then it contains $C_G(\mathcal{T}'_F)$. All Borel subgroups containing $C_G(\mathcal{T}'_F)$ are conjugate under $N_G(\mathcal{T}'_F)(F)$ by [Bor91, §21.3]. On the other hand, every element of $N_G(\mathcal{T}'_F)(F)$ lies in $N_{\mathcal{G}'}(\mathcal{T}')(\mathcal{O}_F)C_G(\mathcal{T}'_F)(F)$ by [KP23, Proposition 9.9.1]. The claim follows. By replacing B'' by its opposite we see that (B.7) remains valid if we instead take the disjoint union over λ that are dominant with respect to any Borel subgroup $B'' \leq \mathcal{G}_F$ containing $\mathcal{T}'_F$.

By [KP23, Proposition 9.9.3], we can choose the hyperspecial point x so that the identity on the generic fibers extends to an isomorphism

$$\iota : \mathcal{G} \longrightarrow \mathcal{G}'.$$

In particular ι induces an identification $\mathcal{G}(\mathcal{O}_F) = \mathcal{G}'(\mathcal{O}_F)$. By Lemma B.5.3, $\iota(\mathcal{T})$ and $\mathcal{T}'$ are $\mathcal{G}'(\mathcal{O}_F)$ -conjugate, and the theorem follows. $\square$

Appendix C

Poisson Summation

We define the adelic version of the Fourier transform on a vector space and state the corresponding version of Poisson summation.

C.1 The standard additive characters

All of the material in this appendix is either implicit or explicit in Tate's thesis [Tat67]. A leisurely account is given in [RV99, Chapter 7].

The goal of this section is to fix a standard additive character on every local field and an additive character of $F \backslash \mathbb{A}_F$ for every global field. We first consider the case of completions of $\mathbb{Q}$ and $\mathbb{F}_p(t)$ for primes p . We let

$$\begin{aligned}\psi_{\mathbb{R}} : \mathbb{R} &\longrightarrow \mathbb{C}^{\times} \\ a &\longmapsto e^{-2\pi i a}\end{aligned}$$

and

$$\begin{aligned}\psi_{\mathbb{Q}_p} : \mathbb{Q}_p &\longrightarrow \mathbb{C}^{\times} \\ a &\longmapsto e^{2\pi i \operatorname{pr}(a)}.\end{aligned}$$

Here the principal part $\operatorname{pr}(a) \in \mathbb{Z}[p^{-1}]$ is any element such that

$$a \equiv \operatorname{pr}(a) \pmod{\mathbb{Z}_p}.$$

If F is a characteristic zero local field then it is an extension of a completion F_0 of $\mathbb{Q}$, and we let

$$\psi_F := \psi \circ \operatorname{tr}_{F/F_0} : F \longrightarrow \mathbb{C}^{\times}.$$

We now treat the characteristic p case. We define the additive character

$$\begin{aligned}\psi_{\mathbb{F}_p((t^{-1}))} : \mathbb{F}_p((t^{-1})) &\longrightarrow \mathbb{C}^\times \\ \sum_{n=-\infty}^r a_n t^n &\longmapsto e^{-2\pi i a_1/p}.\end{aligned}$$

Here the a_i are representatives in $\mathbb{Z}$ for $a_i \in \mathbb{F}_p$. Let ϖ be an irreducible polynomial in $\mathbb{F}_p[t]$ and let $\mathbb{F}_p(t)_\varpi$ is the completion of $\mathbb{F}_p(t)$ at the corresponding place. If $k = \mathbb{F}_p[t]/\varpi$ then we define

$$\begin{aligned}\psi_{\mathbb{F}_p(t)_\varpi} : \mathbb{F}_p(t)_\varpi &\longrightarrow \mathbb{C}^\times \\ \sum_{n=r}^{\infty} a_n \varpi^n &\longrightarrow e^{2\pi i \text{tr}_{k/\mathbb{F}_p}(a_{-1})/p}.\end{aligned}$$

Here the a_i are in k (or rather a fixed set of representatives for $k = \mathbb{F}_p[t]/\varpi$ in $\mathbb{F}_p[t]$). On the right, we are implicitly identifying $\text{tr}_{k/\mathbb{F}_p}(a_{-1})$ with an element of $\mathbb{Z}$ mapping to it under the quotient map $\mathbb{Z} \rightarrow \mathbb{F}_p$. If F is a characteristic p local field then it is an extension of a completion F_0 of $\mathbb{F}_p(t)$ and we let

$$\psi_F := \psi \circ \text{tr}_{F/F_0} : F \longrightarrow \mathbb{C}^\times.$$

One checks that all of the ψ_F we have defined are indeed nontrivial characters.

For an abelian topological group $\mathcal{A}$ let $\widehat{\mathcal{A}}$ be the unitary dual of $\mathcal{A}$. It is the set of all continuous homomorphisms $\mathcal{A} \rightarrow \mathbb{C}^\times$. For the proof of the following lemma see [Wei95, §II.5]:

Lemma C.1.1 *If F is a local field and $\psi : F \rightarrow \mathbb{C}^\times$ is any nontrivial character then there is a bijection*

$$\begin{aligned}F &\longrightarrow \widehat{F} \\ \alpha &\longmapsto (x \mapsto \psi(\alpha x)).\end{aligned}$$

□

If F is a global field we let

$$\psi_F := \prod_v \psi_{F_v}. \tag{C.1}$$

This defines a nontrivial character of $\mathbb{A}_F$ that is trivial on F . The global analogue of Lemma C.1.1 is the following lemma [Wei95, §IV.2]:

Lemma C.1.2 *If F is a global field and $\psi : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times$ is any nontrivial character then there is a bijection*

$$\begin{aligned}F &\longrightarrow \widehat{F \backslash \mathbb{A}_F} \\ \alpha &\longmapsto (x \mapsto \psi(\alpha x)).\end{aligned}$$

□

C.2 Local Schwartz spaces and Fourier transforms

Let F be a local field. If F is non-Archimedean we define $\mathcal{S}(F) = C_c^\infty(F)$, and if F is Archimedean we let $\mathcal{S}(F)$ denote the usual Schwartz space. We refer to $\mathcal{S}(F)$ as the Schwartz space of F .

If W is a finite dimensional F -vector space, we define $\mathcal{S}(W)$ analogously. Let

$$\langle \cdot, \cdot \rangle : W \times W \longrightarrow F \quad (\text{C.2})$$

be a perfect pairing, let $\psi : F \rightarrow \mathbb{C}^\times$ be a character, and let dy be a Haar measure on W . In the case where $W = F$, the pairing is usually taken to be $\langle x, y \rangle := xy$. For $f \in \mathcal{S}(W)$, one has a **Fourier transform**

$$\widehat{f}(x) := \int_W f(y) \psi(\langle x, y \rangle) dy$$

which is also an element of $\mathcal{S}(W)$. In the Archimedean case the assertion that $\widehat{f} \in \mathcal{S}(W)$ is standard, and in the non-Archimedean case it is Exercise C.3.

The Fourier transform depends on ψ , the pairing $\langle \cdot, \cdot \rangle$ and the Haar measure dy . Given ψ and $\langle \cdot, \cdot \rangle$, there is a unique way to normalize dy so that for all $f \in \mathcal{S}(W)$ the Fourier inversion formula holds:

$$f(x) = \widehat{\widehat{f}}(-x). \quad (\text{C.3})$$

This Haar measure dy is known as the **self-dual Haar measure** (with respect to the pairing and additive character).

It follows from the Fourier inversion formula that the Fourier transform defines a $\mathbb{C}$ -linear isomorphism

$$\widehat{\cdot} : \mathcal{S}(W) \xrightarrow{\sim} \mathcal{S}(W). \quad (\text{C.4})$$

C.3 Global Schwartz spaces and Poisson summation

Let F be a global field and let W be a finite dimensional F -vector space. For an F -algebra R , let $W(R) := W \otimes_F R$. We define

$$\mathcal{S}(W(\mathbb{A}_F)) := \mathcal{S}(W(F_\infty)) \otimes_{\mathbb{C}} C_c^\infty(W(\mathbb{A}_F^\infty)).$$

When F is a number field, we view $W(F_\infty)$ as an $\mathbb{R}$ -vector space to make sense of $\mathcal{S}(W(F_\infty))$.

Let

$$\langle \cdot, \cdot \rangle : W \times W \longrightarrow F$$

be a perfect pairing, let $\psi : F \backslash \mathbb{A}_F \rightarrow \mathbb{C}^\times$ be a nontrivial character, and let dy be a Haar measure on $W(\mathbb{A}_F)$. For $f \in \mathcal{S}(W(\mathbb{A}_F))$, we then have an **adelic Fourier transform**

$$\widehat{f}(x) := \int_{W(\mathbb{A}_F)} f(y) \psi(\langle x, y \rangle) dy. \quad (\text{C.5})$$

Lemma C.3.1 *One has that $\widehat{f} \in \mathcal{S}(W(\mathbb{A}_F))$.*

Proof. It suffices to verify the lemma in the special case where $W = F^r$ and $\langle, \rangle$ is the standard product

$$\langle (x_1, \dots, x_r), (y_1, \dots, y_r) \rangle \mapsto \sum_{i=1}^r x_i y_i.$$

Let S be a finite set of places of F including the infinite places and all places where F_v is ramified. If S is large enough then

$$f = f_S \mathbb{1}_{(\widehat{\mathcal{O}}_F^S)^r}.$$

It follows from Lemma C.1.2 that $\psi = \psi_S \psi^S$ where $\psi^S = \prod_{v \notin S} \psi_{F_v}$. Finally we can assume that $dy = dy_S dy^S$ where $dy^S = \prod_{v \notin S} dy_v$ and $dy_v(\mathcal{O}_{F_v}^r) = 1$ for $v \notin S$. Thus

$$\widehat{f} = \widehat{f}_S \widehat{\mathbb{1}}_{(\widehat{\mathcal{O}}_F^S)^r}.$$

We already know that $\widehat{f}_S \in \mathcal{S}(W(F_\infty)) \otimes \mathcal{S}(W(F_S^\infty))$, so it suffices to check that $\widehat{\mathbb{1}}_{(\widehat{\mathcal{O}}_F^S)^r} = \mathbb{1}_{(\widehat{\mathcal{O}}_F^S)^r}$. This is the content of Exercise C.5. $\square$

Given $\langle, \rangle$ and ψ , there is a unique choice of Haar measure dy on $W(\mathbb{A}_F)$ such that the Fourier inversion formula holds:

$$f(x) = \widehat{\widehat{f}}(-x). \quad (\text{C.6})$$

This is the **self-dual Haar measure** (with respect to $\langle, \rangle$ and ψ).

As in the local setting, the Fourier inversion formula implies that the Fourier transform induces an isomorphism

$$\widehat{\cdot} : \mathcal{S}(W(\mathbb{A}_F)) \xrightarrow{\sim} \mathcal{S}(W(\mathbb{A}_F)). \quad (\text{C.7})$$

We can now state an adelic version of Poisson summation [Wei95, §VII.2]:

Theorem C.3.2 (Poisson summation) *For $f \in \mathcal{S}(W(\mathbb{A}_F))$ one has that*

$$\sum_{\gamma \in W(F)} f(\gamma) = \sum_{\gamma \in W(F)} \widehat{f}(\gamma).$$

$\square$

Exercises

C.1. Let F be a local field. Prove that the maps ψ_F defined in §C.1 are characters.

C.2. Let F be a global field. Prove that ψ_F is a character of $\mathbb{A}_F$ trivial on F .

C.3. Let F be a non-Archimedean local field and let $f \in C_c^\infty(F)$. Prove that $\widehat{f} \in C_c^\infty(F)$. Here $\widehat{f}$ can be defined with respect to any Haar measure on F and nontrivial character $\psi : F \rightarrow \mathbb{C}^\times$.

C.4. Prove that the Fourier inversion formula (C.3) holds when F is a non-Archimedean local field.

C.5. Let F be a non-Archimedean local field that is unramified over its prime field. Show that

$$\widehat{\mathbb{1}_{\mathcal{O}_F}} = \mathbb{1}_{\mathcal{O}_F}$$

where the Fourier transform is defined with respect to the standard additive character ψ_F and the Haar measure on F such that $dy(\mathcal{O}_F) = 1$. Conclude that this Haar measure is self-dual with respect to ψ_F and the pairing $(x, y) \mapsto xy$.

Appendix D

Alternate Conventions Related to Adelic Quotients

Let G be a reductive group over a global field F . When there is a nontrivial split torus in the center of G , the quotient $G(F)\backslash G(\mathbb{A}_F)$ is not of finite measure. We discuss the relationship between several methods of coming to grips with this fact.

D.1 The quotients

Recall the groups $G(\mathbb{A}_F)^1$ and A_G defined in §2.6. Choices of Haar measures on $G(\mathbb{A}_F)$ and A_G induce $G(\mathbb{A}_F)$ -invariant Radon measures on $G(F)\backslash G(\mathbb{A}_F)$ and on $A_G G(F)\backslash G(\mathbb{A}_F)$ as explained in §3.6. The quotient

$$[G] := A_G G(F)\backslash G(\mathbb{A}_F)$$

is always of finite measure by Theorem 2.6.3, but the quotient

$$G(F)\backslash G(\mathbb{A}_F)$$

has finite measure if and only if A_G is trivial. This is why we have worked with the quotient $A_G G(F)\backslash G(\mathbb{A}_F)$ throughout this book. In this section we explain how this is no essential loss of generality. At the same time we indicate the relationship between our conventions and some other common conventions related to adelic quotients. To avoid some distracting complications we will assume throughout this appendix that F is a number field.

Recall the homomorphism

$$H_G : G(\mathbb{A}_F) \longrightarrow \mathfrak{a}_G$$

defined in (10.1). For each $\lambda \in \mathfrak{a}_{G\mathbb{C}}^*$ let

$$L^2(G(F)\backslash G(\mathbb{A}_F), \lambda) \tag{D.1}$$

be the space of measurable functions $\varphi : G(F) \backslash G(\mathbb{A}_F) \rightarrow \mathbb{C}$ such that

$$\varphi(ag) = e^{\langle H_G(a), \lambda \rangle} \varphi(g) \quad (\text{D.2})$$

for all $a \in A_G$, $g \in G(\mathbb{A}_F)$ and

$$\int_{G(F) \backslash G(\mathbb{A}_F)^1} |\varphi(g)|^2 dg < \infty.$$

This is a Hilbert space with inner product

$$(\varphi_1, \varphi_2) := \int_{G(F) \backslash G(\mathbb{A}_F)^1} \varphi_1(g) \overline{\varphi_2(g)} dg.$$

The right action of $G(\mathbb{A}_F)$ on $G(F) \backslash G(\mathbb{A}_F)$ induces a left action of $G(\mathbb{A}_F)$ on $L^2(G(F) \backslash G(\mathbb{A}_F), \lambda)$. In this manner $L^2(G(F) \backslash G(\mathbb{A}_F), \lambda)$ becomes a Hilbert space representation of $G(\mathbb{A}_F)$. It is unitary if and only if $\lambda \in i\mathfrak{a}_G^*$.

Similarly $L^2(G(F) \backslash G(\mathbb{A}_F)^1)$ is a unitary representation of $G(\mathbb{A}_F)^1$. For each $\lambda \in \mathfrak{a}_G^*$ let $\mathbb{C}(e^{\langle H_G(\cdot), \lambda \rangle})$ be a 1 dimensional complex vector space on which $G(\mathbb{A}_F)$ acts via $e^{\langle H_G(\cdot), \lambda \rangle}$. Since F is a number field $G(\mathbb{A}_F)$ is the internal direct product of A_G and $G(\mathbb{A}_F)^1$. Thus for each $\lambda \in i\mathfrak{a}_G^*$ we have a representation

$$\mathbb{C}(e^{\langle H_G(\cdot), \lambda \rangle}) \otimes L^2(G(F) \backslash G(\mathbb{A}_F)^1)$$

of $A_G \times G(\mathbb{A}_F)^1 = G(\mathbb{A}_F)$. There is an equivalence of representations of $G(\mathbb{A}_F)$

$$\mathbb{C}(e^{\langle H_G(\cdot), \lambda \rangle}) \otimes L^2(G(F) \backslash G(\mathbb{A}_F)^1) \xrightarrow{\sim} L^2(G(F) \backslash G(\mathbb{A}_F), \lambda) \quad (\text{D.3})$$

induced by the map sending $\varphi \in L^2(G(F) \backslash G(\mathbb{A}_F)^1)$ to the function

$$\begin{aligned} A_G \times G(\mathbb{A}_F)^1 &\longrightarrow \mathbb{C} \\ (a, g) &\longmapsto e^{\langle H_G(a), \lambda \rangle} \varphi(g). \end{aligned}$$

When $\lambda \in i\mathfrak{a}_G$ these equivalences are in fact unitary equivalences of representations.

Upon restriction to $G(\mathbb{A}_F)^1$ the representations $L^2(G(F) \backslash G(\mathbb{A}_F), \lambda)$ all become isomorphic, and as representations of $G(\mathbb{A}_F)$ they only differ by characters of A_G . Thus the spaces $L^2(G(F) \backslash G(\mathbb{A}_F), \lambda)$ all encode essentially the same information.

Lemma D.1.1 *One has unitary equivalences of $G(\mathbb{A}_F)$ -representations*

$$\begin{aligned} L^2(G(F) \backslash G(\mathbb{A}_F)) &\cong \int_{i\mathfrak{a}_G}^{\oplus} \mathbb{C}(e^{\langle H_G(\cdot), \lambda \rangle}) \otimes L^2(G(F) \backslash G(\mathbb{A}_F)^1) d\lambda \\ &\cong \int_{i\mathfrak{a}_G}^{\oplus} L^2(G(F) \backslash G(\mathbb{A}_F), \lambda) d\lambda. \end{aligned}$$

Here the direct integrals are defined as in §3.8 and $d\lambda$ is a Haar measure on $i\mathfrak{a}_G$.

Proof. Given the equivalence (D.3) it suffices to prove the first isomorphism. As mentioned above, since we have assumed F is a number field, the group $G(\mathbb{A}_F)$ is the internal direct product of A_G and $G(\mathbb{A}_F)^1$. Hence the isomorphism $H_G : A_G \xrightarrow{\sim} \mathfrak{a}_G$ induces an isomorphism $G(\mathbb{A}_F) \xrightarrow{\sim} \mathfrak{a}_G \times G(\mathbb{A}_F)^1$, which in turn induces an isomorphism

$$G(F) \backslash G(\mathbb{A}_F) \xrightarrow{\sim} \mathfrak{a}_G \times G(F) \backslash G(\mathbb{A}_F)^1.$$

This isomorphism is $G(\mathbb{A}_F)$ -equivariant if we let $G(\mathbb{A}_F)$ act on the right via $(x, g) \cdot (a'g') = (x + H_G(a'), gg')$ for $(a', g') \in A_G \times G(\mathbb{A}_F)^1$. Thus we have a $G(\mathbb{A}_F)$ -equivariant isomorphism

$$L^2(G(F) \backslash G(\mathbb{A}_F)) \xrightarrow{\sim} L^2(\mathfrak{a}_G) \hat{\otimes} L^2(G(F) \backslash G(\mathbb{A}_F)^1)$$

where the hat denotes the completed tensor product. By Fourier analysis

$$L^2(\mathfrak{a}_G) \cong \int_{i\mathfrak{a}_G^*}^{\oplus} \mathbb{C}(e^{\langle H_G(\cdot), \lambda \rangle}) d\lambda.$$

The lemma follows. $\square$

The moral of the discussion above is that to study $L^2(G(F) \backslash G(\mathbb{A}_F))$ it suffices to study $L^2(A_G G(F) \backslash G(\mathbb{A}_F))$ or equivalently $L^2(G(F) \backslash G(\mathbb{A}_F)^1)$. It is more or less a matter of taste as to whether one works with $G(\mathbb{A}_F)^1$ or $A_G \backslash G(\mathbb{A}_F)$. However, when A_G is nontrivial these spaces are not equal. The reason that we chose to work with $A_G \backslash G(\mathbb{A}_F)$ in this book is that elements of $C_c^\infty(A_G \backslash G(\mathbb{A}_F))$ can all be written as finite sums of restricted direct products

$$f_\infty \otimes \otimes'_{v \nmid \infty} f_v$$

where $f_\infty \in C_c^\infty(A_G \backslash G(F_\infty))$ and $f_v \in C_c^\infty(G(F_v))$ for all $v \nmid \infty$. This is not true for elements of $G(\mathbb{A}_F)^1$; they are merely restrictions to $G(\mathbb{A}_F)^1$ of a finite sum of restricted direct products

$$f'_\infty \otimes \otimes'_{v \nmid \infty} f'_v$$

where $f'_\infty \in C_c^\infty(G(F_\infty))$ and $f'_v \in C_c^\infty(G(F_v))$ for all $v \nmid \infty$. It can be slightly awkward to work with these restrictions in certain situations.

D.2 Separating by central character

Let $\chi : Z_G(F) \backslash Z_G(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ be a quasi-character. We define

$$L^2(G(F)\backslash G(\mathbb{A}_F), \chi) \quad (\text{D.4})$$

be the space of measurable functions $\varphi : G(F)\backslash G(\mathbb{A}_F) \rightarrow \mathbb{C}$ such that

$$\varphi(zg) = \chi(z)\varphi(g) \quad (\text{D.5})$$

for all $z \in Z_G(\mathbb{A}_F)$, $g \in G(\mathbb{A}_F)$ and

$$\int_{G(F)\backslash G(\mathbb{A}_F)^1} |\varphi(g)|^2 dg < \infty.$$

This is a Hilbert space with inner product

$$(\varphi_1, \varphi_2) := \int_{G(F)\backslash G(\mathbb{A}_F)^1} \varphi_1(g) \overline{\varphi_2(g)} dg.$$

The right action of $G(\mathbb{A}_F)$ on $G(F)\backslash G(\mathbb{A}_F)$ induces a left action of $G(\mathbb{A}_F)$ on $L^2(G(F)\backslash G(\mathbb{A}_F), \chi)$ and this gives $L^2(G(F)\backslash G(\mathbb{A}_F), \chi)$ the structure of a Hilbert space representation. It is unitary if and only if χ is a character.

For every quasi-character $\chi : Z_G(F)\backslash Z_G(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ there is a unique $\lambda \in \mathfrak{a}_{G\mathbb{C}}^*$ such that $\chi e^{\langle H_G(\cdot), \lambda \rangle}|_{Z_G(\mathbb{A}_F)}$ is trivial on A_G . On the other hand for any $\lambda \in \mathfrak{a}_{G\mathbb{C}}^*$ we have an isomorphism of Hilbert spaces

$$\begin{aligned} L^2(G(F)\backslash G(\mathbb{A}_F), \chi) &\xrightarrow{\sim} L^2\left(G(F)\backslash G(\mathbb{A}_F), \chi e^{\langle H_G(\cdot), \lambda \rangle}|_{Z_G(\mathbb{A}_F)}\right) \\ \varphi &\longmapsto \varphi e^{\langle H_G(\cdot), \lambda \rangle} \end{aligned} \quad (\text{D.6})$$

that intertwines the actions of $G(\mathbb{A}_F)^1$. Therefore there is no essential loss of generality in assuming that χ is trivial on A_G .

Let $\chi : A_G Z_G(F)\backslash Z_G(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ be a character. We may view

$$L^2(G(F)\backslash G(\mathbb{A}_F), \chi)$$

as a unitary representation of $G(\mathbb{A}_F)$, $A_G\backslash G(\mathbb{A}_F)$ (since A_G acts trivially) or $G(\mathbb{A}_F)^1$ (by restriction). In the previous section we explained that as λ varies over $\mathfrak{a}_{G\mathbb{C}}^*$ the spaces $L^2(G(F)\backslash G(\mathbb{A}_F), \lambda)$ encode equivalent representations of $G(\mathbb{A}_F)^1$. However, if $\chi, \chi' : A_G Z_G(F)\backslash Z_G(\mathbb{A}_F) \rightarrow \mathbb{C}^\times$ are two distinct characters, then no irreducible subrepresentation of

$$L^2(G(F)\backslash G(\mathbb{A}_F), \chi)$$

is equivalent to an irreducible subrepresentation of $L^2(G(F)\backslash G(\mathbb{A}_F), \chi')$. Here we may view the two spaces as representations of $G(\mathbb{A}_F)$, $A_G\backslash G(\mathbb{A}_F)$, or $G(\mathbb{A}_F)^1$.

More precisely one has the following:

Lemma D.2.1 *One has a unitary equivalence of $A_G\backslash G(\mathbb{A}_F)$ -representations*

$$L^2([G]) \xrightarrow{\sim} \widehat{\bigoplus_{\chi} L^2(G(F) \backslash G(\mathbb{A}_F), \chi)}$$

where the Hilbert space direct sum is over characters

$$\chi : A_G Z_G(F) \backslash Z_G(\mathbb{A}_F) \longrightarrow \mathbb{C}^{\times}.$$

A discrete automorphic representation of $A_G \backslash G(\mathbb{A}_F)$ occurs in

$$L^2(G(F) \backslash G(\mathbb{A}_F), \chi)$$

if and only if it has central character χ .

Proof. The isomorphism in the lemma is simply the decomposition of $L^2([G])$ into isotypical components indexed by the representations of the compact group $A_G Z_G(F) \backslash Z_G(\mathbb{A}_F)$ [DE09, Theorem 7.3.2]. The second assertion is a direct consequence of the isomorphism. $\square$

Thus working with $L^2(G(F) \backslash G(\mathbb{A}_F), \chi)$ instead of $L^2([G])$ isolates automorphic representations with central character χ . In the literature when authors work with $L^2(G(F) \backslash G(\mathbb{A}_F), \chi)$ they often modify the Hecke algebra as well, replacing

$$C_c^{\infty}(A_G \backslash G(\mathbb{A}_F))$$

with the related space

$$C_c^{\infty}(G(\mathbb{A}_F), \chi^{-1}).$$

This is the space of smooth functions $f : G(\mathbb{A}_F) \rightarrow \mathbb{C}$ that are compactly supported modulo $Z_G(\mathbb{A}_F)$ and satisfy $f(zg) = \chi^{-1}(z)f(g)$ for $(z, g) \in Z_G(\mathbb{A}_F) \times G(\mathbb{A}_F)$. In this book we did not use this convention because it adds an extra layer of notation.

Hints to selected exercises

Chapter 1

1.12 Use the fact that parabolic subgroups would have unipotent elements. This would imply that D has zero divisors, contradicting the assumption that it is a division algebra.

Chapter 2

2.7 One approach uses the alternate definition of group schemes described in Exercise 1.7.

2.8 See [CF86, Chapter II.6].

2.11 Note that $\mathrm{GL}_n(F_S)$ is an open subset of $\mathfrak{gl}_n(F_S)$ for any finite set S of places of F and use weak approximation for $\mathbb{A}_F$ (see Theorem 2.1.9).

2.15 In the Archimedean case, use the Gram-Schmidt process. In the non-Archimedean case, one can proceed in an elementary manner using induction on n . See [Bum97, Proposition 4.5.2].

2.16 Let $H_G : G(\mathbb{A}_F) \rightarrow \mathrm{Hom}(X^*(G), \mathbb{R})$ be defined by

$$\langle H_G(g), \lambda \rangle = \log |\lambda(g)|$$

for $\lambda \in X^*(G)$. Then $\ker H_G = G(\mathbb{A}_F)^1$. Using (1.14) show that H_G is injective when restricted to A_G . Show moreover that the subgroup index $[H_G(A_G) : H_G(G(\mathbb{A}_F))]$ is finite and that when F is a number field $H_G(A_G) = H_G(G(\mathbb{A}_F))$.

2.20 Using the description of $\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}}) / \mathrm{GL}_2(\widehat{\mathbb{Z}})$ given at the end of §2.6, the exercise can be deduced from the following well-known fact: Every

element of the complex upper half-plane is $\mathrm{SL}_2(\mathbb{Z})$ -equivalent (under Möbius transformations) to an element z satisfying $-\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2}$ and $|z| > 1$.

Chapter 3

3.1 See [Fol95, Proposition 2.6].

3.2 Average a left Haar measure.

3.3 See [Fol95, §2.4], bearing in mind that $\delta_G(g) = \Delta(g)^{-1}$ in the notation of loc. cit.

3.15 Prove first that the complex vector space admits a G -invariant inner product.

Chapter 4

4.3 See [Bum97, Theorem 2.4.1].

4.11 The proof is an extension of the well-known construction of all the finite dimensional irreducible representations of the Lie algebra $\mathfrak{sl}_2$.

Chapter 5

5.6 Show that V has a countable basis and invoke Lemma 4.5.2.

5.11 See [Car79].

5.12 Mimic the proof of Proposition 4.2.7.

Chapter 7

7.3 Let $\mathbb{C}[M]$ be the group algebra of a free abelian group M . We have a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \mathbb{C}[X_*(T_{sF^{\mathrm{sep}}})] & \xrightarrow{a} & \mathbb{C}[X_*(T_{F^{\mathrm{sep}}})] & \xrightarrow{b} & \mathbb{C}[X_*(T_{F^{\mathrm{sep}}})] \\
 & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
 0 & \longrightarrow & \mathcal{O}[\widehat{T}_s] & \xrightarrow{\nu^*} & \mathcal{O}[\widehat{T}] & \xrightarrow{c} & \mathcal{O}(\widehat{T})
 \end{array}$$

where the vertical isomorphisms are given by Exercise 7.2, a is induced by the inclusion $T_s \rightarrow T$ and $b(\lambda) := \lambda^{-1}\text{Fr} \circ \lambda$. Here the action of Fr is well-defined because T splits over an unramified extension. The map c is induced by

$$\begin{aligned}\widehat{T}(R) &\longrightarrow \widehat{T}(R) \\ t &\longmapsto t^{-1}\text{Fr}(t)\end{aligned}$$

for $\mathbb{C}$ -algebras R . The fact that T splits over an unramified extension implies that the top row of the diagram is exact, hence so is the bottom. The lemma is a consequence of the exactness of the bottom row of the diagram.

7.4 See [Her11b, Lemma 2.1].

7.9 See [Lau96, Lemma D.1.5].

Chapter 8

8.5 Right exactness is the only point that is not obvious. Let (σ, V) and (σ', W) be smooth representations of $G(F)$ and let $a : V \rightarrow W$ be an intertwining map. Let $\varphi : G(F) \rightarrow W$ be an element of $\text{Ind}_P^G(W)$, and choose a compact open subgroup $K \leq G(F)$ fixing φ . By Lemma 8.2.1 for all $g \in G(F)$, one has

$$\varphi(g) \in V^{M(F) \cap gKg^{-1}}.$$

Since $V^{M(F) \cap gKg^{-1}} \rightarrow W^{M(F) \cap gKg^{-1}}$ is surjective by Exercise 7.9 there is a $v_g \in V^{M(F) \cap gKg^{-1}}$ such that $a(v_g) = \varphi(g)$. Fix a set of representatives X for $P(F) \backslash G(F) / K$. We then define

$$\begin{aligned}\widetilde{\varphi} : G &\longrightarrow V \\ mn x k &\longmapsto \delta_P^{1/2}(m) \sigma(m) v_x\end{aligned}$$

for $(n, m, x, k) \in N(F) \times M(F) \times X \times K$. The function $\widetilde{\varphi}$ is fixed by K and is therefore smooth, and by construction $a(\widetilde{\varphi}) = \varphi$. Thus the map $\text{Ind}_P^G(V) \rightarrow \text{Ind}_P^G(W)$ is surjective.

8.6 Additivity is clear. Assume that

$$0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow 0$$

is exact. It is easy to verify that $V_{1N} \rightarrow V_{2N} \rightarrow V_{3N} \rightarrow 0$ is exact. On the other hand, identifying V_1 with its image in V_2 we have

$$V_1(N) = V_1 \cap V_2(N).$$

It follows that $V_{1N} \rightarrow V_{2N}$ is injective.

8.7 If $\varphi \in V$ and $n' \in N(F)$ then there is a compact open subgroup $K_N \leq N(F)$ such that $n' \in K_N$. Choosing K_N in this manner we see that

$$\int_{K_N} \pi(n)(\pi(n')\varphi - \varphi)dn = 0.$$

Conversely, suppose that $\varphi \in V$ is such that

$$\int_{K_N} \pi(n)\varphi dn = 0$$

for a compact open subgroup $K_N \leq N(F)$. Choose a compact open subgroup $K'_N \leq K_N$ such that $\varphi \in V^{K'_N}$. Then

$$0 = \int_{K_N} \pi(n)\varphi dn = \frac{1}{[K_N : K'_N]} \sum_{n \in K_N/K'_N} \pi(n)\varphi$$

which implies

$$\varphi = \frac{1}{[K_N : K'_N]} \sum_{n \in K_N/K'_N} (\pi(n)\varphi - \varphi).$$

8.9 For each i let $\mathfrak{m}_i \leq C_c^\infty(G(F) // K)$ be the annihilator of V_i . It is a maximal ideal, and $\mathfrak{m}_i = \mathfrak{m}_j$ if and only if $i = j$. Choose $f_1, \dots, f_n \in C_c^\infty(G(F) // K)$ such that $f_i \in \mathfrak{m}_i$ and $\sum_{i=1}^n f_i = \mathbb{1}_K$, where we have normalized the Haar measure so that K has measure 1. Then $\mathbb{1}_K - f_1$ is the identity on V_1 and zero on V_i for $i \neq 1$.

8.11 Show that if $Q(\sigma^a, \lambda_a)$ and $Q(\sigma^{a'}, \lambda_{a'})$ are linked then one of their central characters is not unitary.

Chapter 9

9.4 See [DE09, Proposition 5.3.2].

Chapter 10

10.5 Use Proposition 8.3.1.

10.7 See [Cas, §8].

Chapter 11

11.4 For μ dominant, relate the value of the Whittaker function at $\mu(\varpi)$ to the action of $\mathbb{1}_{\mathrm{GL}_2(\mathcal{O}_F)\mu(\varpi)\mathrm{GL}_2(\mathcal{O}_F)}$ on $J(\lambda)$. Then use that the eigenvalues of such an operator can be computed in terms of λ by Proposition 7.6.11.

11.5 See [JS81b, §2.5].

11.7 See [Bum97, Theorem 3.6.1].

Chapter 12

12.1 Use Lemma 5.1.3 to prove that the kernel of the homomorphism is open.

12.3 See [Bor79, §2.5].

12.8 Assume first that the π_i are all square integrable. Then none of them are linked in the sense of §8.4. Hence we conclude by Theorem 8.4.4. The general case can be reduced to this case by Theorem 8.4.5 and the transitivity of induction.

Chapter 14

14.7 Use Theorem 11.5.2 and the local version of (14.13).

Chapter 15

15.4 Let

$$K(N) := \{g \in \mathrm{GL}_n(\widehat{\mathbb{Z}}) : g \equiv I_n \pmod{N\widehat{\mathbb{Z}}}\}.$$

Then $\mathrm{GL}_n(\mathbb{Q}) \cap K(N)$ is the group in the proof of Lemma 15.2.3 and hence is neat for sufficiently divisible N . The set of eigenvalues of matrices in $\mathrm{GL}_n(\mathbb{Q}) \cap g^{-1}K(N)g$ is independent of $g \in \mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}}^{\infty})$.

Chapter 17

17.2 Consider the natural action of $\mathbb{G}_m$ on $X = \mathbb{G}_a$.

17.13 For $1 \leq i \leq 2$ let S_i be a finite set of places of F including S_0 and all infinite places. Moreover let $\mathcal{I}_i$ and $\mathcal{H}_i$ be smooth models (in the sense of §2.4) of I and H over $\mathcal{O}_F^{S_i}$. By replacing $\mathcal{I}_i$ with $\mathcal{I}_{i\mathcal{O}_F^{S_1 \cup S_2}}$ and $\mathcal{H}_i$ with $\mathcal{H}_{i\mathcal{O}_F^{S_1 \cup S_2}}$ we may assume $S = S_1 = S_2$. The definition of a model yields identifications

$$\mathcal{I}_{1F} = \mathcal{I}_{2F} \quad \text{and} \quad \mathcal{H}_{1F} = \mathcal{H}_{2F}.$$

After replacing S by a larger finite set of places of F if necessary the right identification extends to an isomorphism $\mathcal{H}_1 \xrightarrow{\sim} \mathcal{H}_2$ mapping $\mathcal{I}_1$ to $\mathcal{I}_2$.

Chapter 18

18.1 Let $H \leq \mathbb{R}^n$ be a closed real Lie subgroup with finitely many connected components. Then by the correspondence between connected Lie groups and Lie algebras we deduce that the neutral component $H^0 \leq H$ (in the real topology) is isomorphic to $\mathbb{R}^k$ for some $k \leq n$. Thus the quotient

$$H/H^0 \leq \mathbb{R}^n/\mathbb{R}^k$$

is a finite subgroup of $\mathbb{R}^{n-k}$. Such a subgroup is trivial.

18.2 Any connected subgroup of a torus over a characteristic zero field is again a torus [Mil17, Corollary 17.25]. Thus the assertion follows from Theorem 1.7.4.

18.6 Let $G = \mathbb{G}_m \times \mathrm{SL}_2$ and let $H < G$ be the subgroup whose points in an F -algebra R are given by

$$H(R) := \left\{ \left(t, \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix} \right) : t \in R^\times \right\}.$$

18.7 See [Kot86a].

References

- [AC89] J. Arthur and L. Clozel. *Simple algebras, base change, and the advanced theory of the trace formula*, volume 120 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989. [351](#), [353](#)
- [AG94] A. Ash and D. Ginzburg. p -adic L -functions for $GL(2n)$. *Invent. Math.*, 116(1-3):27–73, 1994. [384](#)
- [AGR93] A. Ash, D. Ginzburg, and S. Rallis. Vanishing periods of cusp forms over modular symbols. *Math. Ann.*, 296(4):709–723, 1993. [375](#), [392](#), [393](#)
- [Alt20] S. A. Altuğ. Beyond endoscopy via the trace formula—III The standard representation. *J. Inst. Math. Jussieu*, 19(4):1349–1387, 2020. [368](#)
- [Art78] J. Arthur. A trace formula for reductive groups. I. Terms associated to classes in $G(\mathbf{Q})$. *Duke Math. J.*, 45(4):911–952, 1978. [276](#), [429](#), [501](#), [502](#)
- [Art80] J. Arthur. A trace formula for reductive groups. II. Applications of a truncation operator. *Compositio Math.*, 40(1):87–121, 1980. [276](#)
- [Art89] J. Arthur. Unipotent automorphic representations: conjectures. *Astérisque*, (171-172):13–71, 1989. Orbites unipotentes et représentations, II. [333](#), [339](#)
- [Art90] J. Arthur. Unipotent automorphic representations: global motivation. In *Automorphic forms, Shimura varieties, and L -functions, Vol. I (Ann Arbor, MI, 1988)*, volume 10 of *Perspect. Math.*, pages 1–75. Academic Press, Boston, MA, 1990. [333](#), [339](#)
- [Art02] J. Arthur. A note on the automorphic Langlands group. *Canad. Math. Bull.*, 45(4):466–482, 2002. Dedicated to Robert V. Moody. [336](#)
- [Art05] J. Arthur. An introduction to the trace formula. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of

- Clay Math. Proc.*, pages 1–263. Amer. Math. Soc., Providence, RI, 2005. [269](#), [275](#), [276](#), [282](#), [429](#), [465](#)
- [Art13] J. Arthur. *The endoscopic classification of representations*, volume 61 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2013. Orthogonal and symplectic groups. [290](#), [331](#), [338](#), [339](#), [358](#), [362](#), [363](#), [364](#), [366](#), [387](#)
- [AS06] M. Asgari and F. Shahidi. Generic transfer for general spin groups. *Duke Math. J.*, 132(1):137–190, 2006. [362](#)
- [AS14] M. Asgari and F. Shahidi. Image of functoriality for general spin groups. *Manuscripta Math.*, 144(3-4):609–638, 2014. [362](#)
- [AT09] E. Artin and J. T. Tate. *Class field theory*. AMS Chelsea Publishing, Providence, RI, 2009. Reprinted with corrections from the 1967 original. [318](#)
- [Atl] The Atlas of Lie Groups and Representations. <http://liegroups.org/>. [125](#)
- [Ato17] H. Atobe. On the uniqueness of generic representations in an L -packet. *Int. Math. Res. Not. IMRN*, (23):7051–7068, 2017. [332](#)
- [AV16] J. Adams and D. A. Vogan. Contragredient representations and characterizing the local Langlands correspondence. *Amer. J. Math.*, 138(3):657–682, 2016. [340](#)
- [Bad08] A. I. Badulescu. Global Jacquet-Langlands correspondence, multiplicity one and classification of automorphic representations. *Invent. Math.*, 172(2):383–438, 2008. With an appendix by Neven Grbac. [290](#), [515](#)
- [BB20] V. Blomer and J. Buttcane. On the subconvexity problem for L -functions on $GL(3)$. *Ann. Sci. Éc. Norm. Supér. (4)*, 53(6):1441–1500, 2020. [492](#)
- [BCR98] J. Bochnak, M. Coste, and M.-F. Roy. *Real algebraic geometry*, volume 36 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1998. Translated from the 1987 French original, Revised by the authors. [105](#)
- [BDKV84] J. Bernstein, P. Deligne, D. Kazhdan, and M. F. Vigneras. *Représentations des groupes réductifs sur un corps local*. Travaux en Cours. Hermann, Paris, 1984. [515](#)
- [Ber] J. Bernstein. Representations of p -adic groups. http://www.math.tau.ac.il/~bernstei/Unpublished_texts/unpublished_texts/Bernstein93new-harv.lect.from-chic.pdf. [214](#), [215](#), [220](#)
- [Ber74] J. Bernstein. All reductive p -adic groups are tame. *Funkcional. Anal. i Prilozhen.*, 8(2):3–6, 1974. [141](#)
- [Ber84] J. Bernstein. P -invariant distributions on $GL(N)$ and the classification of unitary representations of $GL(N)$ (non-Archimedean case). In *Lie group representations, II (College Park, Md.,*

- 1982/1983), volume 1041 of *Lecture Notes in Math.*, pages 50–102. Springer, Berlin, 1984. [503](#)
- [Ber20] J. Bernstein. Hidden sign in Langlands’ correspondence. *arXiv e-prints*, page [arXiv:2004.10487](#), April 2020. [344](#)
- [BFOR17] K. Bringmann, A. Folsom, K. Ono, and L. Rolén. *Harmonic Maass forms and mock modular forms: theory and applications*, volume 64 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2017. [174](#)
- [BG14] K. Buzzard and T. Gee. The conjectural connections between automorphic representations and Galois representations. In *Automorphic forms and Galois representations. Vol. 1*, volume 414 of *London Math. Soc. Lecture Note Ser.*, pages 135–187. Cambridge Univ. Press, Cambridge, 2014. [341](#), [343](#), [344](#)
- [BH06] C. J. Bushnell and G. Henniart. *The local Langlands conjecture for $GL(2)$* , volume 335 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006. [215](#)
- [BHR94] D. Blasius, M. Harris, and D. Ramakrishnan. Coherent cohomology, limits of discrete series, and Galois conjugation. *Duke Math. J.*, 73(3):647–685, 1994. [406](#)
- [BJ79] A. Borel and H. Jacquet. Automorphic forms and automorphic representations. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 189–207. Amer. Math. Soc., Providence, R.I., 1979. With a supplement “On the notion of an automorphic representation” by R. P. Langlands. [161](#), [286](#)
- [BK00] A. Braverman and D. Kazhdan. γ -functions of representations and lifting. *Geom. Funct. Anal.*, (Special Volume, Part I):237–278, 2000. With an appendix by V. Vologodsky, GAFA 2000 (Tel Aviv, 1999). [491](#), [518](#)
- [BK02] A. Braverman and D. Kazhdan. Normalized intertwining operators and nilpotent elements in the Langlands dual group. volume 2, pages 533–553. 2002. Dedicated to Yuri I. Manin on the occasion of his 65th birthday. [491](#)
- [BK14] J. Bernstein and B. Krötz. Smooth Fréchet globalizations of Harish-Chandra modules. *Israel J. Math.*, 199(1):45–111, 2014. [120](#)
- [BL19] J. Bernstein and E. M. Lapid. On the meromorphic continuation of Eisenstein series. *J. Amer. Math. Soc.*, 37(1):187–234, 2024. [269](#)
- [BLR90] S. Bosch, W. Lütkebohmert, and M. Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*

- (3) [*Results in Mathematics and Related Areas (3)*]. Springer-Verlag, Berlin, 1990. [12](#), [459](#)
- [BLS96] A. Borel, J.-P. Labesse, and J. Schwermer. On the cuspidal cohomology of S -arithmetic subgroups of reductive groups over number fields. *Compositio Math.*, 102(1):1–40, 1996. [405](#)
- [BMS67] H. Bass, J. Milnor, and J.-P. Serre. Solution of the congruence subgroup problem for SL_n ($n \geq 3$) and Sp_{2n} ($n \geq 2$). *Inst. Hautes Études Sci. Publ. Math.*, (33):59–137, 1967. [397](#)
- [BNS16] A. Bouthier, B. C. Ngô, and Y. Sakellaridis. On the formal arc space of a reductive monoid. *Amer. J. Math.*, 138(1):81–108, 2016. [491](#)
- [Bor79] A. Borel. Automorphic L -functions. In *Automorphic forms, representations and L -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 27–61. Amer. Math. Soc., Providence, R.I., 1979. [193](#), [197](#), [202](#), [320](#), [324](#), [567](#)
- [Bor91] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991. [2](#), [24](#), [33](#), [35](#), [199](#), [260](#), [376](#), [550](#)
- [Bor97] A. Borel. *Automorphic forms on $SL_2(\mathbf{R})$* , volume 130 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1997. [241](#), [246](#)
- [Bor98a] A. Borel. *Semisimple groups and Riemannian symmetric spaces*, volume 16 of *Texts and Readings in Mathematics*. Hindustan Book Agency, New Delhi, 1998. [114](#)
- [Bor98b] M. V. Borovoi. Abelian Galois cohomology of reductive groups. *Mem. Amer. Math. Soc.*, 132(626):viii+50, 1998. [447](#)
- [Bor07] A. Borel. Automorphic forms on reductive groups. In *Automorphic forms and applications*, volume 12 of *IAS/Park City Math. Ser.*, pages 7–39. Amer. Math. Soc., Providence, RI, 2007. [162](#)
- [Bor09] M. V. Borovoi. The defect of weak approximation for homogeneous spaces. II. *Dal’nevost. Mat. Zh.*, 9(1-2):15–23, 2009. [66](#)
- [Bou02] N. Bourbaki. *Lie groups and Lie algebras. Chapters 4–6*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2002. Translated from the 1968 French original by Andrew Pressley. [376](#), [536](#)
- [BP16a] R. Beuzart-Plessis. La conjecture locale de Gross-Prasad pour les représentations tempérées des groupes unitaires. *Mém. Soc. Math. Fr. (N.S.)*, (149):vii+191, 2016. [391](#)
- [BP16b] R. Beuzart-Plessis. A short proof of the existence of supercuspidal representations for all reductive p -adic groups. *Pacific J. Math.*, 282(1):27–34, 2016. [128](#)
- [BP21] R. Beuzart-Plessis. Comparison of local relative characters and the Ichino-Ikeda conjecture for unitary groups. *J. Inst. Math. Jussieu*, 20(6):1803–1854, 2021. [378](#), [391](#)

- [BPCZ22] R. Beuzart-Plessis, P-H. Chaudouard, and M. Zydor. The global Gan-Gross-Prasad conjecture for unitary groups: the endoscopic case. *Publ. Math. Inst. Hautes Études Sci.*, 135:183–336, 2022. [391](#)
- [BPLZZ21] R. Beuzart-Plessis, Y. Liu, W. Zhang, and X. Zhu. Isolation of cuspidal spectrum, with application to the Gan-Gross-Prasad conjecture. *Ann. of Math. (2)*, 194(2):519–584, 2021. [391](#), [429](#), [483](#)
- [BR94a] L. Barthel and D. Ramakrishnan. A nonvanishing result for twists of L -functions of $GL(n)$. *Duke Math. J.*, 74(3):681–700, 1994. [305](#)
- [BR94b] D. Blasius and J. D. Rogawski. Zeta functions of Shimura varieties. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 525–571. Amer. Math. Soc., Providence, RI, 1994. [415](#)
- [BR10] A. I. Badulescu and D. Renard. Unitary dual of $GL(n)$ at Archimedean places and global Jacquet-Langlands correspondence. *Compos. Math.*, 146(5):1115–1164, 2010. [515](#)
- [Bri91] C. Brislawn. Traceable integral kernels on countably generated measure spaces. *Pacific J. Math.*, 150(2):229–240, 1991. [422](#), [426](#), [488](#)
- [Bri22] M. Brion. Homomorphisms of algebraic groups: representability and rigidity. *Michigan Math. J.*, 72:51–76, 2022. [411](#)
- [Bru78] R. W. Bruggeman. Fourier coefficients of cusp forms. *Invent. Math.*, 45(1):1–18, 1978. [492](#)
- [BS98] R. Berndt and R. Schmidt. *Elements of the representation theory of the Jacobi group*. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 1998. [2011 reprint of the 1998 original] [MR1634977]. [x](#), [91](#), [115](#), [249](#)
- [BT65] A. Borel and J. Tits. Groupes réductifs. *Inst. Hautes Études Sci. Publ. Math.*, (27):55–150, 1965. [277](#)
- [Bum97] D. Bump. *Automorphic forms and representations*, volume 55 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. [104](#), [119](#), [126](#), [132](#), [178](#), [563](#), [564](#), [567](#)
- [Bum05] D. Bump. The Rankin-Selberg method: an introduction and survey. In *Automorphic representations, L -functions and applications: progress and prospects*, volume 11 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 41–73. de Gruyter, Berlin, 2005. [294](#), [361](#)
- [Bum13] D. Bump. *Lie groups*, volume 225 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013. [307](#)
- [BW00] A. Borel and N. R. Wallach. *Continuous cohomology, discrete subgroups, and representations of reductive groups*, volume 67 of *Mathematical Surveys and Monographs*. American Mathematical

- Society, Providence, RI, second edition, 2000. [223](#), [402](#), [403](#), [406](#), [408](#), [409](#), [410](#)
- [BZ76] J. Bernstein and A. V. Zelevinsky. Representations of the group $GL(n, F)$ where F is a non-Archimedean local field. *Russ. Math. Surv.*, 31(3):1–68, 1976. [224](#)
- [BZ77] J. Bernstein and A. V. Zelevinsky. Induced representations of reductive p -adic groups. I. *Ann. Sci. École Norm. Sup. (4)*, 10(4):441–472, 1977. [223](#), [224](#)
- [Car79] P. Cartier. Representations of p -adic groups: a survey. In *Automorphic forms, representations and L -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977)*, Part 1, Proc. Sympos. Pure Math., XXXIII, pages 111–155. Amer. Math. Soc., Providence, R.I., 1979. [135](#), [199](#), [207](#), [215](#), [564](#)
- [Car05] H. Carayol. Cohomologie automorphe et compactifications partielles de certaines variétés de Griffiths-Schmid. *Compos. Math.*, 141(5):1081–1102, 2005. [406](#)
- [Car15] H. Carayol. Cohomologie automorphe et sous-variétés des variétés de Griffiths-Schmid. *Astérisque*, (369):203–222, 2015. [406](#)
- [Cas] W. Casselman. Introduction to the theory of admissible representations of p -adic reductive groups. <https://www.math.ubc.ca/~cass/research/pdf/p-adic-book.pdf>. [207](#), [215](#), [220](#), [566](#)
- [Cas67] J. W. S. Cassels. Global fields. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 42–84. Thompson, Washington, D.C., 1967. [51](#)
- [Cas73] W. Casselman. An assortment of results on representations of $GL_2(k)$. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 1–54. Lecture Notes in Math., Vol. 349, 1973. [184](#)
- [Cas89] W. Casselman. Canonical extensions of Harish-Chandra modules to representations of G . *Canad. J. Math.*, 41(3):385–438, 1989. [120](#)
- [CF86] J. W. S. Cassels and A. Fröhlich, editors. *Algebraic number theory*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London, 1986. Reprint of the 1967 original. [45](#), [46](#), [563](#)
- [CFGK19] Y. Cai, S. Friedberg, D. Ginzburg, and E. Kaplan. Doubling constructions and tensor product L -functions: the linear case. *Invent. Math.*, 217(3):985–1068, 2019. [340](#), [362](#)
- [CFK18] Y. Cai, S. Friedberg, and E. Kaplan. Doubling constructions: local and global theory, with an application to global functoriality for non-generic cuspidal representations. *arXiv e-prints*, page [arXiv:1802.02637](#), Feb 2018. [362](#)
- [CGP10] B. Conrad, O. Gabber, and G. Prasad. *Pseudo-reductive groups*, volume 17 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2010. [15](#), [16](#), [22](#), [260](#), [261](#)

- [Cha19] P.-H. Chaudouard. On relative trace formulae: the case of Jacquet-Rallis. *Acta Math. Vietnam.*, 44(2):391–430, 2019. [519](#)
- [Che58] C. Chevalley. Séminaire C. Chevalley, 1956–1958. Classification des groupes de Lie algébriques. pages ii+166 + ii+122 pp. (mimerographed), 1958. [25](#)
- [CHH88] M. Cowling, U. Haagerup, and R. Howe. Almost L^2 matrix coefficients. *J. Reine Angew. Math.*, 387:97–110, 1988. [128](#)
- [CHLN11] L. Clozel, M. Harris, J.-P. Labesse, and B. C. Ngô, editors. *On the stabilization of the trace formula*, volume 1 of *Stabilization of the Trace Formula, Shimura Varieties, and Arithmetic Applications*. International Press, Somerville, MA, 2011. [415](#), [517](#)
- [CKM04] J. W. Cogdell, H. H. Kim, and M. R. Murty. *Lectures on automorphic L -functions*, volume 20 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 2004. [340](#), [355](#)
- [CKPSS04] J. W. Cogdell, H. H. Kim, I. I. Piatetski-Shapiro, and F. Shahidi. Functoriality for the classical groups. *Publ. Math. Inst. Hautes Études Sci.*, (99):163–233, 2004. [313](#), [340](#), [358](#)
- [CL10] P.-H. Chaudouard and G. Laumon. Le lemme fondamental pondéré. I. Constructions géométriques. *Compos. Math.*, 146(6):1416–1506, 2010. [363](#)
- [CL12] P.-H. Chaudouard and G. Laumon. Le lemme fondamental pondéré. II. Énoncés cohomologiques. *Ann. of Math. (2)*, 176(3):1647–1781, 2012. [363](#)
- [Clo90a] L. Clozel. The fundamental lemma for stable base change. *Duke Math. J.*, 61(1):255–302, 1990. [511](#), [517](#)
- [Clo90b] L. Clozel. Motifs et formes automorphes: applications du principe de fonctorialité. In *Automorphic forms, Shimura varieties, and L -functions, Vol. I (Ann Arbor, MI, 1988)*, volume 10 of *Perspect. Math.*, pages 77–159. Academic Press, Boston, MA, 1990. [341](#), [342](#)
- [Clo07] L. Clozel. Spectral theory of automorphic forms. In *Automorphic forms and applications*, volume 12 of *IAS/Park City Math. Ser.*, pages 43–93. Amer. Math. Soc., Providence, RI, 2007. [97](#), [211](#), [339](#)
- [Clo16] L. Clozel. Errata: Motifs et formes automorphes: applications du principe de fonctorialité [MR1044819]. In *Autour des motifs—École d’été Franco-Asiatique de Géométrie Algébrique et de Théorie des Nombres Asian-French Summer School on Algebraic Geometry and Number Theory. Vol. III*, volume 49 of *Panor. Synthèses*, pages 61–63. Soc. Math. France, Paris, 2016. [342](#)
- [CMSP17] J. Carlson, S. Müller-Stach, and C. Peters. *Period mappings and period domains*, volume 168 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. Second edition of [MR2012297]. [406](#)

- [Cog07] J. W. Cogdell. L -functions and converse theorems for GL_n . In *Automorphic forms and applications*, volume 12 of *IAS/Park City Math. Ser.*, pages 97–177. Amer. Math. Soc., Providence, RI, 2007. [294](#), [299](#), [309](#)
- [Con02] B. Conrad. A modern proof of Chevalley’s theorem on algebraic groups. *J. Ramanujan Math. Soc.*, 17(1):1–18, 2002. [98](#)
- [Con12a] B. Conrad. Finiteness theorems for algebraic groups over function fields. *Compos. Math.*, 148(2):555–639, 2012. [67](#), [69](#), [375](#)
- [Con12b] B. Conrad. Weil and Grothendieck approaches to adelic points. *Enseign. Math. (2)*, 58(1-2):61–97, 2012. [51](#), [54](#), [63](#), [64](#), [455](#)
- [Con14] B. Conrad. Reductive group schemes. In *Autour des schémas en groupes. Vol. I*, volume 42/43 of *Panor. Synthèses*, pages 93–444. Soc. Math. France, Paris, 2014. [11](#), [20](#), [22](#), [23](#), [59](#), [61](#), [63](#), [188](#), [189](#), [192](#), [542](#), [543](#), [546](#), [549](#)
- [CPS99] J. W. Cogdell and I. I. Piatetski-Shapiro. Converse theorems for GL_n . II. *J. Reine Angew. Math.*, 507:165–188, 1999. [313](#)
- [CPSS11] J. W. Cogdell, I. I. Piatetski-Shapiro, and F. Shahidi. Functoriality for the quasisplit classical groups. In *On certain L -functions*, volume 13 of *Clay Math. Proc.*, pages 117–140. Amer. Math. Soc., Providence, RI, 2011. [357](#), [358](#), [359](#)
- [CS80] W. Casselman and J. A. Shalika. The unramified principal series of p -adic groups. II. The Whittaker function. *Compositio Math.*, 41(2):207–231, 1980. [300](#), [301](#)
- [CST17] J. W. Cogdell, F. Shahidi, and T.-L. Tsai. Local Langlands correspondence for GL_n and the exterior and symmetric square ε -factors. *Duke Math. J.*, 166(11):2053–2132, 2017. [361](#)
- [DE09] A. Deitmar and S. Echterhoff. *Principles of harmonic analysis*. Universitext. Springer, New York, 2009. [113](#), [240](#), [561](#), [566](#)
- [Del71] P. Deligne. Travaux de Shimura. In *Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389*, pages 123–165. Lecture Notes in Math., Vol. 244. Springer, Berlin, 1971. [410](#)
- [Del73a] P. Deligne. Formes modulaires et représentations de $GL(2)$. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 55–105. Lecture Notes in Math., Vol. 349. Springer, Berlin, 1973. [179](#)
- [Del73b] P. Deligne. Les constantes des équations fonctionnelles des fonctions L . pages 501–597. Lecture Notes in Math., Vol. 349, 1973. [323](#)
- [Del75] P. Deligne. Correction to: “Les constantes des équations fonctionnelles des fonctions L ” (*modular functions of one variable, ii* (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), pp. 501–597, Lecture Notes in Math., Vol. 349, Springer, Berlin, 1973). pages p. 149. Lecture Notes in Math., Vol. 476, 1975. [323](#)

- [Del20] P. Delorme. On the spectral Theorem of Langlands. *Adv. Math.*, 426(109107): 36, 2023. [269](#)
- [DG74] C. Demazure and P. Grothendieck. *Schémas en groupes*, volume 151–153 of *Lecture Notes in Mathematics*. Springer-Verlag, New York, 1974. [2](#), [3](#), [16](#), [25](#), [37](#), [411](#), [533](#), [549](#)
- [DG80] M. Demazure and P. Gabriel. *Introduction to algebraic geometry and algebraic groups*, volume 39 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam-New York, 1980. Translated from the French by J. Bell. [39](#)
- [Dix77] J. Dixmier. *C^* -algebras*. North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977. Translated from the French by Francis Jellet, North-Holland Mathematical Library, Vol. 15. [128](#)
- [DKV79a] J. J. Duistermaat, J. A. C. Kolk, and V. S. Varadarajan. Spectra of compact locally symmetric manifolds of negative curvature. *Invent. Math.*, 52(1):27–93, 1979. [509](#)
- [DKV79b] J. J. Duistermaat, J. A. C. Kolk, and V. S. Varadarajan. Erratum: “Spectra of compact locally symmetric manifolds of negative curvature”. *Invent. Math.*, 54(1):101, 1979. [509](#)
- [DKV84] P. Deligne, D. Kazhdan, and M.-F. Vignéras. Représentations des algèbres centrales simples p -adiques. In *Representations of reductive groups over a local field*, Travaux en Cours, pages 33–117. Hermann, Paris, 1984. [485](#)
- [DM78] J. Dixmier and P. Malliavin. Factorisations de fonctions et de vecteurs indéfiniment différentiables. *Bull. Sci. Math. (2)*, 102(4):307–330, 1978. [109](#)
- [DMOS82] P. Deligne, J. S. Milne, A. Ogus, and K.-Y. Shih. *Hodge cycles, motives, and Shimura varieties*, volume 900 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1982. [416](#)
- [Do15] V. C. Do. Le lemme fondamental métaplectique de Jacquet et Mao en égales caractéristiques. *Bull. Soc. Math. France*, 143(1):125–196, 2015. [384](#)
- [Do20] V. C. Do. Transfer to characteristic zero of the Jacquet-Mao’s metaplectic fundamental lemma. *Int. Math. Res. Not. IMRN*, (21):7644–7661, 2020. [384](#)
- [Dri80] V. G. Drinfel’d. Langlands’ conjecture for $GL(2)$ over functional fields. In *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, pages 565–574. Acad. Sci. Fennica, Helsinki, 1980. [368](#)
- [Dri87a] V. G. Drinfel’d. Cohomology of compactified moduli varieties of F -sheaves of rank 2. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 162(Avtomorf. Funkts. i Teor. Chisel. III):107–158, 189, 1987. [368](#)
- [Dri87b] V. G. Drinfel’d. Moduli varieties of F -sheaves. *Funktsional. Anal. i Prilozhen.*, 21(2):23–41, 1987. [368](#)

- [Dri88] V. G. Drinfel'd. Proof of the Petersson conjecture for $GL(2)$ over a global field of characteristic p . *Funktsional. Anal. i Prilozhen.*, 22(1):34–54, 96, 1988. [368](#)
- [DS05] F. Diamond and J. Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005. [174](#), [179](#)
- [Dyn52] E. B. Dynkin. Maximal subgroups of the classical groups. *Trudy Moskov. Mat. Obšč.*, 1:39–166, 1952. [31](#)
- [Edi92] B. Edixhoven. Néron models and tame ramification. *Compositio Math.*, 81(3):291–306, 1992. [527](#)
- [EH00] D. Eisenbud and J. Harris. *The geometry of schemes*, volume 197 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. [3](#), [104](#)
- [EvdGM08] B. Edixhoven, G. van der Geer, and B. Moonen. Modular forms. In *Modular forms on Schiermonnikoog*, pages 1–11. Cambridge Univ. Press, Cambridge, 2008. [174](#)
- [FGKV98] E. Frenkel, D. Gaitsgory, D. Kazhdan, and K. Vilonen. Geometric realization of Whittaker functions and the Langlands conjecture. *J. Amer. Math. Soc.*, 11(2):451–484, 1998. [300](#)
- [FJ93] S. Friedberg and H. Jacquet. Linear periods. *J. Reine Angew. Math.*, 443:91–139, 1993. [383](#)
- [Fli88] Y. Z. Flicker. Twisted tensors and Euler products. *Bull. Soc. Math. France*, 116(3):295–313, 1988. [382](#)
- [FLO12] B. Feigon, E. M. Lapid, and O. Offen. On representations distinguished by unitary groups. *Publications mathématiques de l’IHÉS*, 115(1):185–323, 2012. [381](#)
- [Fol95] G. B. Folland. *A course in abstract harmonic analysis*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995. [77](#), [80](#), [81](#), [82](#), [83](#), [92](#), [95](#), [96](#), [97](#), [170](#), [300](#), [564](#)
- [Fol99] G. B. Folland. *Real analysis*. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, second edition, 1999. Modern techniques and their applications, A Wiley-Interscience Publication. [80](#), [83](#)
- [Fri87] S. Friedberg. Poincaré series for $GL(n)$: Fourier expansion, Kloosterman sums, and algebro-geometric estimates. *Math. Z.*, 196(2):165–188, 1987. [244](#), [495](#), [497](#)
- [FS21] L. Fargues and P. Scholze. Geometrization of the local Langlands correspondence. *arXiv e-prints*, page [arXiv:2102.13459](#), February 2021. [332](#), [369](#)
- [FZ95] Y. Z. Flicker and D. Zinoviev. On poles of twisted tensor L -functions. *Proc. Japan Acad. Ser. A Math. Sci.*, 71(6):114–116, 1995. [382](#)
- [Gai03] D. Gaitsgory. Informal introduction to geometric Langlands. In *An introduction to the Langlands program (Jerusalem, 2001)*, pages 269–281. Birkhäuser Boston, Boston, MA, 2003. [67](#)

- [Gar18] P. Garrett. *Modern analysis of automorphic forms by example. Vol. 2*, volume 174 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2018. [84](#), [107](#)
- [Get12] J. R. Getz. An approach to nonsolvable base change and descent. *J. Ramanujan Math. Soc.*, 27(2):143–211, 2012. [352](#), [354](#), [518](#)
- [Get18] J. R. Getz. Nonabelian Fourier transforms for spherical representations. *Pacific J. Math.*, 294(2):351–373, 2018. [368](#)
- [Get20] J. R. Getz. A summation formula for the Rankin-Selberg monoid and a nonabelian trace formula. *Amer. J. Math.*, 142(5):1371–1407, 2020. [352](#), [368](#), [491](#), [518](#)
- [GG12] J. R. Getz and M. Goresky. *Hilbert modular forms with coefficients in intersection homology and quadratic base change*, volume 298 of *Progress in Mathematics*. Birkhäuser/Springer Basel AG, Basel, 2012. [174](#), [176](#), [408](#)
- [GGP12a] W. T. Gan, B. H. Gross, and D. Prasad. Restrictions of representations of classical groups: examples. *Astérisque*, (346):111–170, 2012. Sur les conjectures de Gross et Prasad. I. [387](#)
- [GGP12b] W. T. Gan, B. H. Gross, and D. Prasad. Symplectic local root numbers, central critical L values, and restriction problems in the representation theory of classical groups. *Astérisque*, (346):1–109, 2012. Sur les conjectures de Gross et Prasad. I. [387](#), [389](#), [390](#), [392](#)
- [GH14] J. R. Getz and H. Hahn. Algebraic cycles and Tate classes on Hilbert modular varieties. *Int. J. Number Theory*, 10(1):161–176, 2014. [383](#), [416](#)
- [GH15] J. R. Getz and H. Hahn. A general simple relative trace formula. *Pacific J. Math.*, 277(1):99–118, 2015. [x](#), [455](#), [508](#)
- [GH20] J. R. Getz and C.-H. Hsu. The Fourier transform for triples of quadratic spaces. *arXiv e-prints*, page [arXiv:2009.11490](#), September 2020. [491](#)
- [GHL21] J. R. Getz, C.-H. Hsu, and S. Leslie. Harmonic analysis on certain spherical varieties. *J. Eur. Math. Soc.*, 2023. [491](#)
- [GJ72] R. Godement and H. Jacquet. *Zeta functions of simple algebras*. Lecture Notes in Mathematics, Vol. 260. Springer-Verlag, Berlin-New York, 1972. [294](#)
- [GJ78] S. Gelbart and H. Jacquet. A relation between automorphic representations of $GL(2)$ and $GL(3)$. *Ann. Sci. École Norm. Sup. (4)*, 11(4):471–542, 1978. [355](#)
- [GJR04] D. Ginzburg, D. Jiang, and S. Rallis. On the nonvanishing of the central value of the Rankin-Selberg L -functions. *J. Amer. Math. Soc.*, 17(3):679–722, 2004. [391](#)
- [GJR05] D. Ginzburg, D. Jiang, and S. Rallis. On the nonvanishing of the central value of the Rankin-Selberg L -functions. II. In *Automorphic representations, L -functions and applications: progress*

- and prospects*, volume 11 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 157–191. de Gruyter, Berlin, 2005. [391](#)
- [GJR09] D. Ginzburg, D. Jiang, and S. Rallis. Models for certain residual representations of unitary groups. In *Automorphic forms and L-functions I. Global aspects*, volume 488 of *Contemp. Math.*, pages 125–146. Amer. Math. Soc., Providence, RI, 2009. [391](#)
- [GK73] I. M. Gel’fand and D. Kazhdan. On the representation of the group $GL(n, K)$ where K is a local field. *Funct. Anal. Appl.*, 6:315–317, 1973. [296](#)
- [GK82] S. Gelbart and A. W. Knap. L -indistinguishability and R groups for the special linear group. *Adv. in Math.*, 43(2):101–121, 1982. [482](#)
- [GK15] J. R. Getz and J. Klassen. Isolating Rankin-Selberg lifts. *Proc. Amer. Math. Soc.*, 143(8):3319–3329, 2015. [368](#)
- [GK19] W. Goldring and J.-S. Koskivirta. Strata Hasse invariants, Hecke algebras and Galois representations. *Invent. Math.*, 217(3):887–984, 2019. [406](#)
- [GL19] J. R. Getz and B. Liu. A summation formula for triples of quadratic spaces. *Adv. Math.*, 347:150–191, 2019. [491](#)
- [GN10] D. Gaitsgory and D. Nadler. Spherical varieties and Langlands duality. *Mosc. Math. J.*, 10(1):65–137, 271, 2010. [379](#)
- [God66] R. Godement. The spectral decomposition of cusp-forms. In *Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965)*, pages 225–234. Amer. Math. Soc., Providence, R.I., 1966. [168](#), [241](#), [248](#)
- [God15] R. Godement. *Analysis. IV*. Universitext. Springer, Cham, 2015. Integration and spectral theory, harmonic analysis, the garden of modular delights, Translated from the French edition by Urmie Ray. [106](#), [107](#)
- [GP92] B. H. Gross and D. Prasad. On the decomposition of a representation of SO_n when restricted to SO_{n-1} . *Canad. J. Math.*, 44(5):974–1002, 1992. [387](#)
- [GR10] B. H. Gross and M. Reeder. Arithmetic invariants of discrete Langlands parameters. *Duke Math. J.*, 154(3):431–508, 2010. [320](#)
- [Gro97] F. D. Grosshans. *Algebraic homogeneous spaces and invariant theory*, volume 1673 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1997. [378](#)
- [Gro98] B. H. Gross. On the Satake isomorphism. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 223–237. Cambridge Univ. Press, Cambridge, 1998. [186](#)
- [GRS01] D. Ginzburg, S. Rallis, and D. Soudry. Generic automorphic forms on $SO(2n + 1)$: functorial lift to $GL(2n)$, endoscopy, and base change. *Internat. Math. Res. Notices*, (14):729–764, 2001. [359](#), [362](#)

- [GS83] D. Goldfeld and P. Sarnak. Sums of Kloosterman sums. *Invent. Math.*, 71(2):243–250, 1983. [492](#)
- [GS01] S. Gelbart and F. Shahidi. Boundedness of automorphic L -functions in vertical strips. *J. Amer. Math. Soc.*, 14(1):79–107, 2001. [276](#), [361](#)
- [GT03a] M. Goresky and Y.-S. Tai. Anti-holomorphic multiplication and a real algebraic modular variety. *J. Differential Geom.*, 65(3):513–560, 2003. [414](#)
- [GT03b] M. Goresky and Y.-S. Tai. The moduli space of real abelian varieties with level structure. *Compositio Math.*, 139(1):1–27, 2003. [414](#)
- [Gu21] M. P. Gu. Automorphic-twisted summation formulae for pairs of quadratic spaces. *arXiv e-prints*, page [arXiv:2102.06113](#), February 2021. [491](#)
- [GW09] R. Goodman and N. R. Wallach. *Symmetry, representations, and invariants*, volume 255 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2009. [226](#), [381](#)
- [GW10] U. Görtz and T. Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Vieweg + Teubner, Wiesbaden, 2010. Schemes with examples and exercises. [3](#), [6](#), [41](#)
- [GW14] J. R. Getz and E. Wambach. Twisted relative trace formulae with a view towards unitary groups. *Amer. J. Math.*, 136(1):1–58, 2014. [385](#), [386](#), [387](#), [416](#), [522](#)
- [GZ86] B. H. Gross and D. B. Zagier. Heegner points and derivatives of L -series. *Invent. Math.*, 84(2):225–320, 1986. [392](#)
- [Hab78] W. J. Haboush. Homogeneous vector bundles and reductive subgroups of reductive algebraic groups. *Amer. J. Math.*, 100(6):1123–1137, 1978. [438](#)
- [Hah09] H. Hahn. A simple twisted relative trace formula. *Int. Math. Res. Not. IMRN*, (21):3957–3978, 2009. [x](#), [488](#)
- [Hah16a] H. Hahn. On classical groups detected by the triple tensor product and the Littlewood-Richardson semigroup. *Res. Number Theory*, 2:Paper No. 19, 12, 2016. [368](#)
- [Hah16b] H. Hahn. On tensor third L -functions of automorphic representations of $GL_n(\mathbb{A}_F)$. *Proc. Amer. Math. Soc.*, 144(12):5061–5069, 2016. [368](#)
- [Hah21] H. Hahn. Poles of triple product L -functions involving monomial representations. *Int. J. Number Theory*, 17(2):479–486, 2021. [368](#)
- [Hal94] T. C. Hales. Hyperelliptic curves and harmonic analysis (why harmonic analysis on reductive p -adic groups is not elementary). In *Representation theory and analysis on homogeneous spaces (New Brunswick, NJ, 1993)*, volume 177 of *Contemp. Math.*, pages 137–169. Amer. Math. Soc., Providence, RI, 1994. [486](#)

- [Hal95] T. C. Hales. On the fundamental lemma for standard endoscopy: reduction to unit elements. *Canad. J. Math.*, 47(5):974–994, 1995. [511](#)
- [Har74] G. Harder. Chevalley groups over function fields and automorphic forms. *Ann. of Math. (2)*, 100:249–306, 1974. [259](#)
- [Har77] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52. [3](#)
- [Har90] M. Harris. Automorphic forms of $\bar{\partial}$ -cohomology type as coherent cohomology classes. *J. Differential Geom.*, 32(1):1–63, 1990. [406](#)
- [Har95] J. Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. A first course, Corrected reprint of the 1992 original. [6](#)
- [Har04] D. Harari. Weak approximation on algebraic varieties. In *Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002)*, volume 226 of *Progr. Math.*, pages 43–60. Birkhäuser Boston, Boston, MA, 2004. [65](#)
- [Har11] G. Harder. *Lectures on algebraic geometry I*. Aspects of Mathematics, E35. Vieweg + Teubner Verlag, Wiesbaden, revised edition, 2011. Sheaves, cohomology of sheaves, and applications to Riemann surfaces. [401](#), [407](#)
- [HC53] Harish-Chandra. Representations of a semisimple Lie group on a Banach space. I. *Trans. Amer. Math. Soc.*, 75:185–243, 1953. [140](#)
- [HC66] Harish-Chandra. Discrete series for semisimple Lie groups. II. Explicit determination of the characters. *Acta Math.*, 116:1–111, 1966. [160](#)
- [HC68] Harish-Chandra. *Automorphic forms on semisimple Lie groups*. Notes by J. G. M. Mars. Lecture Notes in Mathematics, No. 62. Springer-Verlag, Berlin, 1968. [160](#), [161](#)
- [HC70a] Harish-Chandra. Eisenstein series over finite fields. In *Functional analysis and related fields (Proc. Conf. M. Stone, Univ. Chicago, Chicago, Ill., 1968)*, pages 76–88. Springer, New York, 1970. [214](#), [269](#)
- [HC70b] Harish-Chandra. *Harmonic analysis on reductive p -adic groups*. Lecture Notes in Mathematics, Vol. 162. Springer-Verlag, Berlin-New York, 1970. Notes by G. van Dijk. [141](#)
- [HC75] Harish-Chandra. Harmonic analysis on real reductive groups. I. The theory of the constant term. *J. Functional Analysis*, 19:104–204, 1975. [547](#)
- [HC99] Harish-Chandra. *Admissible invariant distributions on reductive p -adic groups*, volume 16 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999. Preface and notes by Stephen DeBacker and Paul J. Sally, Jr. [226](#)

- [Hec37a] E. Hecke. Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung. I. *Math. Ann.*, 114(1):1–28, 1937. [179](#)
- [Hec37b] E. Hecke. Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung. II. *Math. Ann.*, 114(1):316–351, 1937. [179](#)
- [Hel00] A. G. Helminck. On the classification of k -involutions. *Adv. Math.*, 153(1):1–117, 2000. [381](#)
- [Hel01] S. Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original. [381](#)
- [Hen00] G. Henniart. Une preuve simple des conjectures de Langlands pour $GL(n)$ sur un corps p -adique. *Invent. Math.*, 139(2):439–455, 2000. [327](#), [414](#)
- [Her11a] P. E. Herman. Beyond endoscopy for the Rankin-Selberg L -function. *J. Number Theory*, 131(9):1691–1722, 2011. [368](#), [492](#)
- [Her11b] F. Herzig. A Satake isomorphism in characteristic p . *Compos. Math.*, 147(1):263–283, 2011. [550](#), [565](#)
- [Her12] P. E. Herman. The functional equation and beyond endoscopy. *Pacific J. Math.*, 260(2):497–513, 2012. [492](#)
- [Her16] P. E. Herman. Quadratic base change and the analytic continuation of the Asai L -function: a new trace formula approach. *Amer. J. Math.*, 138(6):1669–1729, 2016. [368](#), [492](#)
- [HL04] M. Harris and J.-P. Labesse. Conditional base change for unitary groups. *Asian J. Math.*, 8(4):653–683, 2004. [430](#)
- [HLR86] G. Harder, R. P. Langlands, and M. Rapoport. Algebraische Zyklen auf Hilbert-Blumenthal-Flächen. *J. Reine Angew. Math.*, 366:53–120, 1986. [371](#), [383](#), [396](#)
- [HLTT16] M. Harris, K.-W. Lan, R. Taylor, and J. Thorne. On the rigid cohomology of certain Shimura varieties. *Res. Math. Sci.*, 3:Paper No. 37, 308, 2016. [344](#)
- [HS01] A. G. Helminck and G. W. Schwarz. Orbits and invariants associated with a pair of commuting involutions. *Duke Math. J.*, 106(2):237–279, 2001. [443](#)
- [Hsu21] C.-H. Hsu. Asymptotics of Schwartz functions: nonarchimedean. *arXiv e-prints*, page [arXiv:2112.02403](#), December 2021. [491](#)
- [HT01] M. Harris and R. Taylor. *The geometry and cohomology of some simple Shimura varieties*, volume 151 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2001. With an appendix by Vladimir G. Berkovich. [317](#), [327](#), [351](#), [414](#)
- [Hub91] A. Huber. On the Parshin-Beilinson adèles for schemes. *Abh. Math. Sem. Univ. Hamburg*, 61:249–273, 1991. [51](#)

- [Hum75] J. E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York-Heidelberg, 1975. Graduate Texts in Mathematics, No. 21. [2](#)
- [Hum78] J. E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. Second printing, revised. [26](#)
- [Igu00] J.-I. Igusa. *An introduction to the theory of local zeta functions*, volume 14 of *AMS/IP Studies in Advanced Mathematics*. American Mathematical Society, Providence, RI; International Press, Cambridge, MA, 2000. [235](#)
- [II10] A. Ichino and T. Ikeda. On the periods of automorphic forms on special orthogonal groups and the Gross-Prasad conjecture. *Geom. Funct. Anal.*, 19(5):1378–1425, 2010. [380](#)
- [IK04] H. Iwaniec and E. Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004. [492](#)
- [Ive86] B. Iversen. *Cohomology of sheaves*. Universitext. Springer-Verlag, Berlin, 1986. [401](#)
- [Iwa02] H. Iwaniec. *Spectral methods of automorphic forms*, volume 53 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI; Revista Matemática Iberoamericana, Madrid, second edition, 2002. [492](#)
- [Jac64] N. Jacobson. *Structure of rings*, volume 37 of *American Mathematical Society Colloquium Publications, Revised edition*. American Mathematical Society, Providence, R.I., 1964. [122](#)
- [Jac71] H. Jacquet. Représentations des groupes linéaires p -adiques. In *Theory of group representations and Fourier analysis (Centro Internaz. Mat. Estivo (C.I.M.E.), II Ciclo, Montecatini Terme, 1970)*, pages 119–220. Edizioni Cremonese, Rome, 1971. [219](#)
- [Jac84] H. Jacquet. On the residual spectrum of $GL(n)$. In *Lie group representations, II (College Park, Md., 1982/1983)*, volume 1041 of *Lecture Notes in Math.*, pages 185–208. Springer, Berlin, 1984. [289](#)
- [Jac86] H. Jacquet. Sur un résultat de Waldspurger. *Ann. Sci. École Norm. Sup. (4)*, 19(2):185–229, 1986. [384](#), [474](#), [484](#)
- [Jac97] H. Jacquet. Automorphic spectrum of symmetric spaces. In *Representation theory and automorphic forms (Edinburgh, 1996)*, volume 61 of *Proc. Sympos. Pure Math.*, pages 443–455. Amer. Math. Soc., Providence, RI, 1997. [423](#), [490](#)
- [Jac04] H. Jacquet. Kloosterman identities over a quadratic extension. *Ann. of Math. (2)*, 160(2):755–779, 2004. [381](#), [519](#)
- [Jac05a] H. Jacquet. A guide to the relative trace formula. In *Automorphic representations, L-functions and applications: progress and*

- prospects*, volume 11 of *Ohio State Univ. Math. Res. Inst. Publ.*, pages 257–272. de Gruyter, Berlin, 2005. [276](#), [380](#), [423](#), [474](#)
- [Jac05b] H. Jacquet. Kloosterman identities over a quadratic extension. II. *Ann. Sci. École Norm. Sup. (4)*, 38(4):609–669, 2005. [381](#), [491](#), [519](#)
- [Jac09] H. Jacquet. Archimedean Rankin-Selberg integrals. In *Automorphic forms and L-functions II. Local aspects*, volume 489 of *Contemp. Math.*, pages 57–172. Amer. Math. Soc., Providence, RI, 2009. [286](#), [303](#), [306](#), [309](#)
- [Jac10] H. Jacquet. Distinction by the quasi-split unitary group. *Israel J. Math.*, 178:269–324, 2010. [381](#), [491](#)
- [Jan14] J. C. Jantzen. Tempered representations for classical p -adic groups. *Manuscripta Math.*, 145(3-4):319–387, 2014. [223](#)
- [Ji98] L. Ji. The trace class conjecture for arithmetic groups. *J. Differential Geom.*, 48(1):165–203, 1998. [242](#)
- [JL70] H. Jacquet and R. P. Langlands. *Automorphic forms on GL(2)*. Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin-New York, 1970. [140](#), [507](#), [514](#), [515](#)
- [JL13] D. Jiang and B. Liu. On Fourier coefficients of automorphic forms of $GL(n)$. *Int. Math. Res. Not. IMRN*, (17):4029–4071, 2013. [502](#)
- [JLR93] H. Jacquet, K.-F. Lai, and S. Rallis. A trace formula for symmetric spaces. *Duke Math. J.*, 70(2):305–372, 1993. [276](#), [379](#), [380](#)
- [JLR99] H. Jacquet, E. M. Lapid, and J. Rogawski. Periods of automorphic forms. *J. Amer. Math. Soc.*, 12(1):173–240, 1999. [374](#)
- [JLZ20] D. Jiang, Z. Luo, and L. Zhang. Harmonic Analysis and Gamma Functions on Symplectic Groups. *arXiv e-prints*, page [arXiv:2006.08126](#), June 2020. [491](#)
- [JPSS79a] H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika. Automorphic forms on $GL(3)$. I. *Ann. of Math. (2)*, 109(1):169–212, 1979. [310](#)
- [JPSS79b] H. Jacquet, I. I. Piatetski-Shapiro, and J. A. Shalika. Automorphic forms on $GL(3)$. II. *Ann. of Math. (2)*, 109(2):213–258, 1979. [310](#)
- [JPSS83] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika. Rankin-Selberg convolutions. *Amer. J. Math.*, 105(2):367–464, 1983. [303](#), [308](#), [312](#)
- [JR92] H. Jacquet and S. Rallis. Symplectic periods. *J. Reine Angew. Math.*, 423:175–197, 1992. [385](#)
- [JR11] H. Jacquet and S. Rallis. On the Gross-Prasad conjecture for unitary groups. In *On certain L-functions*, volume 13 of *Clay Math. Proc.*, pages 205–264. Amer. Math. Soc., Providence, RI, 2011. [391](#), [519](#)

- [JS81a] H. Jacquet and J. A. Shalika. On Euler products and the classification of automorphic forms. II. *Amer. J. Math.*, 103(4):777–815, 1981. [210](#), [287](#), [311](#)
- [JS81b] H. Jacquet and J. A. Shalika. On Euler products and the classification of automorphic representations. I. *Amer. J. Math.*, 103(3):499–558, 1981. [210](#), [309](#), [567](#)
- [JS83] H. Jacquet and J. A. Shalika. The Whittaker models of induced representations. *Pacific J. Math.*, 109(1):107–120, 1983. [223](#)
- [JY96] H. Jacquet and Y. Ye. Distinguished representations and quadratic base change for $GL(3)$. *Trans. Amer. Math. Soc.*, 348(3):913–939, 1996. [381](#), [519](#)
- [Kal13] T. Kaletha. Genericity and contragredience in the local Langlands correspondence. *Algebra Number Theory*, 7(10):2447–2474, 2013. [340](#)
- [Kal16] T. Kaletha. The local Langlands conjectures for non-quasi-split groups. In *Families of automorphic forms and the trace formula*, Simons Symp., pages 217–257. Springer, [Cham], 2016. [328](#), [329](#), [330](#), [331](#), [337](#), [338](#)
- [Kal19] T. Kaletha. Supercuspidal L-packets. *arXiv e-prints*, page [arXiv:1912.03274](#), December 2019. [332](#)
- [Kap15] E. Kaplan. Complementary results on the Rankin-Selberg gamma factors of classical groups. *J. Number Theory*, 146:390–447, 2015. [361](#)
- [Kil88] W. Killing. Die Zusammensetzung der stetigen endlichen Transformationsgruppen. *Math. Ann.*, 33(1):1–48, 1888. [31](#)
- [Kil90] W. Killing. Die Zusammensetzung der stetigen endlichen Transformationsgruppen. *Math. Ann.*, 36(2):161–189, 1890. [31](#)
- [Kim03] H. H. Kim. Functoriality for the exterior square of GL_4 and the symmetric fourth of GL_2 . *J. Amer. Math. Soc.*, 16(1):139–183, 2003. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak. [355](#), [356](#)
- [Kis09] M. Kisin. Modularity of 2-adic Barsotti-Tate representations. *Invent. Math.*, 178(3):587–634, 2009. [354](#)
- [KL08] A. Knightly and C. Li. Petersson’s trace formula and the Hecke eigenvalues of Hilbert modular forms. In *Modular forms on Schiermonnikoog*, pages 145–187. Cambridge Univ. Press, Cambridge, 2008. [177](#), [492](#)
- [KL13] A. Knightly and C. Li. Kuznetsov’s trace formula and the Hecke eigenvalues of Maass forms. *Mem. Amer. Math. Soc.*, 224(1055):vi+132, 2013. [492](#)
- [KMRT98] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol. *The book of involutions*, volume 44 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits. [525](#), [530](#)

- [Kna86] A. W. Knaapp. *Representation theory of semisimple groups*, volume 36 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1986. An overview based on examples. [37](#), [104](#), [125](#), [128](#), [129](#), [131](#), [230](#), [515](#)
- [Kna94] A. W. Knaapp. Local Langlands correspondence: the Archimedean case. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 393–410. Amer. Math. Soc., Providence, RI, 1994. [284](#)
- [Kna02] A. W. Knaapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 2002. [83](#), [88](#), [105](#), [113](#), [343](#), [547](#)
- [Kno91] F. Knop. The Luna-Vust theory of spherical embeddings. In *Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989)*, pages 225–249. Manoj Prakashan, Madras, 1991. [491](#)
- [Kob93] N. Koblitz. *Introduction to elliptic curves and modular forms*, volume 97 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1993. [174](#), [179](#)
- [Kon02] T. Konno. Twisted endoscopy and the generic packet conjecture. *Israel J. Math.*, 129:253–289, 2002. [332](#)
- [Kon03] T. Konno. A note on the Langlands classification and irreducibility of induced representations of p -adic groups. *Kyushu J. Math.*, 57(2):383–409, 2003. [223](#)
- [Kot82] R. E. Kottwitz. Rational conjugacy classes in reductive groups. *Duke Math. J.*, 49(4):785–806, 1982. [513](#), [516](#)
- [Kot84] R. E. Kottwitz. Stable trace formula: cuspidal tempered terms. *Duke Math. J.*, 51(3):611–650, 1984. [338](#), [447](#), [448](#), [517](#)
- [Kot86a] R. E. Kottwitz. Base change for unit elements of Hecke algebras. *Compositio Math.*, 60(2):237–250, 1986. [517](#), [568](#)
- [Kot86b] R. E. Kottwitz. Stable trace formula: elliptic singular terms. *Math. Ann.*, 275(3):365–399, 1986. [447](#), [448](#), [517](#)
- [Kot88] R. E. Kottwitz. Tamagawa numbers. *Ann. of Math. (2)*, 127(3):629–646, 1988. [91](#)
- [Kot92] R. E. Kottwitz. On the λ -adic representations associated to some simple Shimura varieties. *Invent. Math.*, 108(3):653–665, 1992. [414](#)
- [Kot05] R. E. Kottwitz. Harmonic analysis on reductive p -adic groups and Lie algebras. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 393–522. Amer. Math. Soc., Providence, RI, 2005. [486](#)
- [KP23] T. Kaletha and G. Prasad. *Bruhat-Tits theory—a new approach*, volume 44 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2023. [547](#), [548](#), [550](#)
- [Kri19] R. Krishna. A new proof of the Waldspurger formula, I. *Algebra Number Theory*, 13(3):577–642, 2019. [392](#)

- [KRY06] S. S. Kudla, M. Rapoport, and T. Yang. *Modular forms and special cycles on Shimura curves*, volume 161 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2006. [416](#)
- [KS99] R. E. Kottwitz and D. Shelstad. Foundations of twisted endoscopy. *Astérisque*, (255):vi+190, 1999. [448](#), [517](#)
- [KS02] H. H. Kim and F. Shahidi. Functorial products for $GL_2 \times GL_3$ and the symmetric cube for GL_2 . *Ann. of Math. (2)*, 155(3):837–893, 2002. With an appendix by Colin J. Bushnell and Guy Henniart. [355](#), [356](#)
- [KS15] T. Kobayashi and B. Speh. Symmetry breaking for representations of rank one orthogonal groups. *Mem. Amer. Math. Soc.*, 238(1126):v+110, 2015. [408](#)
- [KS17] F. Knop and B. Schalke. The dual group of a spherical variety. *Trans. Moscow Math. Soc.*, 78:187–216, 2017. [379](#), [381](#), [384](#)
- [Kud97] S. S. Kudla. Algebraic cycles on Shimura varieties of orthogonal type. *Duke Math. J.*, 86(1):39–78, 1997. [416](#)
- [Kud04] S. S. Kudla. Special cycles and derivatives of Eisenstein series. In *Heegner points and Rankin L-series*, volume 49 of *Math. Sci. Res. Inst. Publ.*, pages 243–270. Cambridge Univ. Press, Cambridge, 2004. [416](#)
- [Kuz80] N. V. Kuznetsov. The Petersson conjecture for cusp forms of weight zero and the Linnik conjecture. Sums of Kloosterman sums. *Mat. Sb. (N.S.)*, 111(153)(3):334–383, 479, 1980. [492](#)
- [KW09a] C. Khare and J.-P. Wintenberger. Serre’s modularity conjecture. I. *Invent. Math.*, 178(3):485–504, 2009. [354](#)
- [KW09b] C. Khare and J.-P. Wintenberger. Serre’s modularity conjecture. II. *Invent. Math.*, 178(3):505–586, 2009. [354](#)
- [Lab90] J.-P. Labesse. Fonctions élémentaires et lemme fondamental pour le changement de base stable. *Duke Math. J.*, 61(2):519–530, 1990. [511](#), [517](#)
- [Lab99] J.-P. Labesse. Cohomologie, stabilisation et changement de base. *Astérisque*, (257):vi+161, 1999. Appendix A by Laurent Clozel and Labesse, and Appendix B by Lawrence Breen. [434](#), [447](#), [448](#), [517](#)
- [Laf02] L. Lafforgue. Chtoucas de Drinfeld et correspondance de Langlands. *Invent. Math.*, 147(1):1–241, 2002. [368](#)
- [Laf14] L. Lafforgue. Noyaux du transfert automorphe de Langlands et formules de Poisson non linéaires. *Jpn. J. Math.*, 9(1):1–68, 2014. [518](#)
- [Laf18] V. Lafforgue. Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale. *J. Amer. Math. Soc.*, 31(3):719–891, 2018. [369](#)
- [Lan] R. P. Langlands. Letter to André Weil. <https://publications.ias.edu/rpl/paper/43>. [187](#), [276](#), [347](#)

- [Lan56] S. Lang. Algebraic groups over finite fields. *Amer. J. Math.*, 78:555–563, 1956. [543](#)
- [Lan70] R. P. Langlands. Problems in the theory of automorphic forms. In *Lectures in modern analysis and applications, III*, pages 18–61. Lecture Notes in Math., Vol. 170. Springer, Berlin, 1970. [187](#), [276](#), [339](#), [347](#)
- [Lan71] R. P. Langlands. *Euler products*. Yale University Press, New Haven, Conn.-London, 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1967, Yale Mathematical Monographs, 1. [276](#), [340](#), [355](#)
- [Lan76] R. P. Langlands. *On the functional equations satisfied by Eisenstein series*. Lecture Notes in Mathematics, Vol. 544. Springer-Verlag, Berlin, 1976. [269](#), [276](#), [282](#), [474](#)
- [Lan79a] R. P. Langlands. Automorphic representations, Shimura varieties, and motives. Ein Märchen. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 205–246. Amer. Math. Soc., Providence, R.I., 1979. [336](#)
- [Lan79b] R. P. Langlands. Stable conjugacy: definitions and lemmas. *Canadian J. Math.*, 31(4):700–725, 1979. [517](#)
- [Lan80] R. P. Langlands. *Base change for $GL(2)$* , volume 96 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1980. [351](#), [354](#), [516](#)
- [Lan83] R. P. Langlands. *Les débuts d’une formule des traces stable*, volume 13 of *Publications Mathématiques de l’Université Paris VII [Mathematical Publications of the University of Paris VII]*. Université de Paris VII, U.E.R. de Mathématiques, Paris, 1983. [448](#), [517](#)
- [Lan89] R. P. Langlands. On the classification of irreducible representations of real algebraic groups. In *Representation theory and harmonic analysis on semisimple Lie groups*, volume 31 of *Math. Surveys Monogr.*, pages 101–170. Amer. Math. Soc., Providence, RI, 1989. [325](#), [341](#)
- [Lan95] S. Lang. *Introduction to modular forms*, volume 222 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1995. With appendixes by D. Zagier and Walter Feit, Corrected reprint of the 1976 original. [174](#), [179](#)
- [Lan04] R. P. Langlands. Beyond endoscopy. In *Contributions to automorphic forms, geometry, and number theory*, pages 611–697. Johns Hopkins Univ. Press, Baltimore, MD, 2004. [352](#), [368](#), [518](#)
- [Lan16] K.-W. Lan. Vanishing theorems for coherent automorphic cohomology. *Res. Math. Sci.*, 3:Paper No. 39, 43, 2016. [406](#)

- [Lap06] E. M. Lapid. On the fine spectral expansion of Jacquet’s relative trace formula. *J. Inst. Math. Jussieu*, 5(2):263–308, 2006. [474](#)
- [Lau96] G. Laumon. *Cohomology of Drinfeld modular varieties. Part I*, volume 41 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1996. Geometry, counting of points and local harmonic analysis. [144](#), [184](#), [230](#), [233](#), [565](#)
- [Lee21] C.-R. Lee. Endoscopic Relative Orbital Integrals on U_3 . *arXiv e-prints*, page [arXiv:2110.11890](#), October 2021. [387](#)
- [Les19] S. Leslie. The endoscopic fundamental lemma for unitary Friedberg-Jacquet periods. *arXiv e-prints*, page [arXiv:1911.07907](#), November 2019. [387](#)
- [Les22] S. Leslie. On the stabilization of relative trace formulae: descent and the fundamental lemma. *Adv. Math.*, 394:Paper No. 108026, 68, 2022. [387](#)
- [Li17] W.-W. Li. Basic functions and unramified local L -factors for split groups. *Sci. China Math.*, 60(5):777–812, 2017. [187](#)
- [LM09] E. M. Lapid and W. Müller. Spectral asymptotics for arithmetic quotients of $SL(n, \mathbb{R})/SO(n)$. *Duke Math. J.*, 149(1):117–155, 2009. [509](#)
- [Lor08] F. Lorenz. *Algebra. Vol. II*. Universitext. Springer, New York, 2008. Fields with structure, algebras and advanced topics, Translated from the German by Silvio Levy, With the collaboration of Levy. [46](#), [86](#)
- [LR79] D. Luna and R. W. Richardson. A generalization of the Chevalley restriction theorem. *Duke Math. J.*, 46(3):487–496, 1979. [442](#)
- [LR98] E. M. Lapid and J. D. Rogawski. On twists of cuspidal representations of $GL(2)$. *Forum Math.*, 10(2):175–197, 1998. [352](#)
- [LR03] E. M. Lapid and J. D. Rogawski. Periods of Eisenstein series: the Galois case. *Duke Math. J.*, 120(1):153–226, 2003. [374](#)
- [LRS93] G. Laumon, M. Rapoport, and U. Stuhler. $\mathcal{D}$ -elliptic sheaves and the Langlands correspondence. *Invent. Math.*, 113(2):217–338, 1993. [327](#)
- [LS87] R. P. Langlands and D. Shelstad. On the definition of transfer factors. *Math. Ann.*, 278(1-4):219–271, 1987. [326](#), [331](#), [448](#), [518](#)
- [LV83] D. Luna and T. Vust. Plongements d’espaces homogènes. *Comment. Math. Helv.*, 58(2):186–245, 1983. [491](#)
- [LV07] E. Lindenstrauss and A. Venkatesh. Existence and Weyl’s law for spherical cusp forms. *Geom. Funct. Anal.*, 17(1):220–251, 2007. [429](#), [483](#), [508](#)
- [LW13] J.-P. Labesse and J.-L. Waldspurger. *La formule des traces tordue d’après le Friday Morning Seminar*, volume 31 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2013. With a foreword by Robert Langlands [dual English/French text]. [488](#)

- [Mac17] M. Maculan. Maximality of hyperspecial compact subgroups avoiding Bruhat-Tits theory. *Ann. Inst. Fourier (Grenoble)*, 67(1):1–21, 2017. [59](#), [549](#)
- [Mai22] A. Maiti. Weyl’s Law for Arbitrary Archimedean Type. *arXiv e-prints*, page [arXiv:2210.05986](#), October 2022. [429](#), [483](#), [508](#)
- [Mao98] Z. Mao. A fundamental lemma for metaplectic correspondence. *J. Reine Angew. Math.*, 496:107–129, 1998. [384](#)
- [Mat60] Y. Matsushima. Espaces homogènes de Stein des groupes de Lie complexes. *Nagoya Math. J.*, 16:205–218, 1960. [438](#)
- [Mat13] N. Matringe. Essential Whittaker functions for $GL(n)$. *Doc. Math.*, 18:1191–1214, 2013. [285](#), [306](#)
- [Men65] J. L. Mennicke. Finite factor groups of the unimodular group. *Ann. of Math. (2)*, 81:31–37, 1965. [397](#)
- [MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994. [202](#), [434](#), [440](#), [441](#), [442](#), [443](#), [445](#)
- [Mil80] J. S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980. [414](#)
- [Mil05] J. S. Milne. Introduction to Shimura varieties. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 265–378. Amer. Math. Soc., Providence, RI, 2005. [12](#), [410](#), [411](#)
- [Mil10] J. Milnor. *Collected papers of John Milnor. V. Algebra*. American Mathematical Society, Providence, RI, 2010. Edited by Hyman Bass and T. Y. Lam. [397](#)
- [Mil17] J. S. Milne. *Algebraic groups*, volume 170 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. The theory of group schemes of finite type over a field. [2](#), [9](#), [10](#), [11](#), [13](#), [14](#), [15](#), [16](#), [18](#), [21](#), [22](#), [23](#), [25](#), [27](#), [29](#), [30](#), [31](#), [34](#), [40](#), [41](#), [189](#), [190](#), [191](#), [199](#), [221](#), [235](#), [249](#), [389](#), [434](#), [436](#), [437](#), [439](#), [441](#), [442](#), [494](#), [527](#), [528](#), [532](#), [533](#), [535](#), [537](#), [546](#), [568](#)
- [Miy06] T. Miyake. *Modular forms*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2006. Translated from the 1976 Japanese original by Yoshitaka Maeda. [174](#), [179](#)
- [MM97] M. R. Murty and V. K. Murty. *Non-vanishing of L-functions and applications*, volume 157 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1997. [352](#)
- [Mœg02] C. Mœglin. Sur la classification des séries discrètes des groupes classiques p -adiques: paramètres de Langlands et exhaustivité. *J. Eur. Math. Soc. (JEMS)*, 4(2):143–200, 2002. [223](#)

- [Mok15] C. P. Mok. Endoscopic classification of representations of quasi-split unitary groups. *Mem. Amer. Math. Soc.*, 235(1108):vi+248, 2015. [192](#), [290](#), [332](#), [338](#), [359](#), [363](#), [368](#), [383](#), [387](#)
- [Moo65] C. C. Moore. Decomposition of unitary representations defined by discrete subgroups of nilpotent groups. *Ann. of Math. (2)*, 82:146–182, 1965. [97](#)
- [Mor] L. J. Mordell. Note on Certain Modular Relations Considered by Messrs. Ramanujan, Darling, and Rogers. *Proc. London Math. Soc.*, S2-20(1):408. [179](#)
- [Mor08] S. Morel. Complexes pondérés sur les compactifications de Baily-Borel: le cas des variétés de Siegel. *J. Amer. Math. Soc.*, 21(1):23–61, 2008. [399](#), [415](#)
- [Mor10] S. Morel. *On the cohomology of certain noncompact Shimura varieties*, volume 173 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2010. With an appendix by Robert Kottwitz. [415](#)
- [Mos55] G. D. Mostow. Self-adjoint groups. *Ann. of Math. (2)*, 62:44–55, 1955. [114](#)
- [Mos56] G. D. Mostow. Fully reducible subgroups of algebraic groups. *Amer. J. Math.*, 78:200–221, 1956. [16](#)
- [MP49] S. Minakshisundaram and Å. Pleijel. Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds. *Canadian J. Math.*, 1:242–256, 1949. [508](#)
- [MR87] V. K. Murty and D. Ramakrishnan. Period relations and the Tate conjecture for Hilbert modular surfaces. *Invent. Math.*, 89(2):319–345, 1987. [416](#)
- [MT02] C. Mœglin and M. Tadić. Construction of discrete series for classical p -adic groups. *J. Amer. Math. Soc.*, 15(3):715–786, 2002. [223](#)
- [Mül89] W. Müller. The trace class conjecture in the theory of automorphic forms. *Ann. of Math. (2)*, 130(3):473–529, 1989. [242](#)
- [Mül98] W. Müller. The trace class conjecture in the theory of automorphic forms. II. *Geom. Funct. Anal.*, 8(2):315–355, 1998. [242](#)
- [Mül00] W. Müller. On the singularities of residual intertwining operators. *Geom. Funct. Anal.*, 10(5):1118–1170, 2000. [276](#)
- [Mül07] W. Müller. Weyl’s law for the cuspidal spectrum of SL_n . *Ann. of Math. (2)*, 165(1):275–333, 2007. [508](#)
- [Mum99] D. Mumford. *The red book of varieties and schemes*, volume 1358 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 1999. Includes the Michigan lectures (1974) on curves and their Jacobians, With contributions by Enrico Arbarello. [3](#), [105](#)
- [Mur] F. Murnaghan. Representations of reductive p -adic groups. <http://www.math.toronto.edu/murnaghan/courses/mat1197/notes.pdf>. [215](#)

- [MV07] I. Mirković and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. of Math. (2)*, 166(1):95–143, 2007. [198](#)
- [MW89] C. Mœglin and J.-L. Waldspurger. Le spectre résiduel de $GL(n)$. *Ann. Sci. École Norm. Sup. (4)*, 22(4):605–674, 1989. [286](#), [289](#), [290](#), [502](#)
- [MW95] C. Mœglin and J.-L. Waldspurger. *Spectral decomposition and Eisenstein series*, volume 113 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1995. Une paraphrase de l'Écriture [A paraphrase of Scripture]. [73](#), [264](#), [269](#), [274](#), [275](#), [276](#), [277](#)
- [MW12] C. Mœglin and J.-L. Waldspurger. La conjecture locale de Gross-Prasad pour les groupes spéciaux orthogonaux: le cas général. *Astérisque*, (347):167–216, 2012. Sur les conjectures de Gross et Prasad. II. [391](#)
- [MW16a] C. Mœglin and J.-L. Waldspurger. *Stabilisation de la formule des traces tordue. Vol. 1*, volume 316 of *Progress in Mathematics*. Birkhäuser/Springer, Cham, 2016. [331](#), [363](#), [517](#)
- [MW16b] C. Mœglin and J.-L. Waldspurger. *Stabilisation de la formule des traces tordue. Vol. 2*, volume 317 of *Progress in Mathematics*. Birkhäuser/Springer, Cham, 2016. [331](#), [363](#), [517](#)
- [Nag59] M. Nagata. On the 14-th problem of Hilbert. *Amer. J. Math.*, 81:766–772, 1959. [441](#)
- [Nek94] J. Nekovář. Beilinson's conjectures. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 537–570. Amer. Math. Soc., Providence, RI, 1994. [415](#)
- [Neu99] J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder. [45](#)
- [Ngô99a] B. C. Ngô. Faisceaux pervers, homomorphisme de changement de base et lemme fondamental de Jacquet et Ye. *Ann. Sci. École Norm. Sup. (4)*, 32(5):619–679, 1999. [381](#), [491](#), [519](#)
- [Ngô99b] B. C. Ngô. Le lemme fondamental de Jacquet et Ye en caractéristique positive. *Duke Math. J.*, 96(3):473–520, 1999. [381](#), [491](#), [519](#)
- [Ngô10a] B. C. Ngô. Endoscopy theory of automorphic forms. In *Proceedings of the International Congress of Mathematicians. Volume I*, pages 210–237. Hindustan Book Agency, New Delhi, 2010. [518](#)
- [Ngô10b] B. C. Ngô. Le lemme fondamental pour les algèbres de Lie. *Publ. Math. Inst. Hautes Études Sci.*, (111):1–169, 2010. [363](#), [439](#)
- [Ngô10c] B. C. Ngô. Survey on the fundamental lemma. In *Current developments in mathematics, 2009*, pages 1–21. Int. Press, Somerville, MA, 2010. [363](#)

- [Ngô14] B. C. Ngô. On a certain sum of automorphic L -functions. In *Automorphic forms and related geometry: assessing the legacy of I. I. Piatetski-Shapiro*, volume 614 of *Contemp. Math.*, pages 337–343. Amer. Math. Soc., Providence, RI, 2014. [491](#), [518](#)
- [Ngô20] B. C. Ngô. Hankel transform, Langlands functoriality and functional equation of automorphic L -functions. *Jpn. J. Math.*, 15(1):121–167, 2020. [491](#)
- [NT22] J. Newton and J. A. Thorne. Symmetric power functoriality for Hilbert modular forms. *arXiv e-prints*, page [arXiv:2212.03595](#), December 2022. [355](#)
- [Oes84] J. Oesterlé. Nombres de Tamagawa et groupes unipotents en caractéristique p . *Invent. Math.*, 78(1):13–88, 1984. [88](#), [91](#)
- [Off06a] O. Offen. On symplectic periods of the discrete spectrum of GL_{2n} . *Israel J. Math.*, 154:253–298, 2006. [374](#)
- [Off06b] O. Offen. Residual spectrum of GL_{2n} distinguished by the symplectic group. *Duke Math. J.*, 134(2):313–357, 2006. [385](#)
- [Pet32] H. Petersson. Über die Entwicklungskoeffizienten der automorphen Formen. *Acta Math.*, 58(1):169–215, 1932. [492](#)
- [Pol15] J. Polák. Exposing relative endoscopy in unitary symmetric spaces. *Res. Math. Sci.*, 2:2:9, 2015. [387](#)
- [Pol16] J. Polák. *Relative endoscopy for unitary symmetric spaces*. PhD thesis, McGill University, 2016. [387](#)
- [Poo17] B. Poonen. *Rational points on varieties*, volume 186 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2017. [x](#), [38](#), [40](#), [61](#), [62](#), [63](#), [434](#), [437](#), [449](#), [452](#), [453](#), [454](#), [455](#), [459](#), [460](#), [515](#), [546](#)
- [PR94] V. Platonov and A. Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen. [65](#), [66](#), [71](#)
- [Pra77] G. Prasad. Strong approximation for semi-simple groups over function fields. *Ann. of Math. (2)*, 105(3):553–572, 1977. [66](#)
- [Pra19] D. Prasad. Generic representations for symmetric spaces. *Adv. Math.*, 348:378–411, 2019. With an appendix by Yiannis Sakellaridis. [393](#)
- [PS16a] V. Pilloni and B. Stroh. Cohomologie cohérente et représentations Galoisiennes. *Ann. Math. Qué.*, 40(1):167–202, 2016. [406](#)
- [PS16b] V. Pilloni and B. Stroh. Surconvergence, ramification et modularité. *Astérisque*, (382):195–266, 2016. [354](#)
- [PWZ21] A. Pollack, C. Wan, and M. Zydor. On the residue method for period integrals. *Duke Mathematical Journal*, 170(7):1457 – 1515, 2021. [387](#)
- [Raj02] C. S. Rajan. On the image and fibres of solvable base change. *Math. Res. Lett.*, 9(4):499–508, 2002. [352](#)

- [Ram94] D. Ramakrishnan. Pure motives and automorphic forms. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 411–446. Amer. Math. Soc., Providence, RI, 1994. [336](#)
- [Ram00a] D. Ramakrishnan. Modularity of the Rankin-Selberg L -series, and multiplicity one for $SL(2)$. *Ann. of Math. (2)*, 152(1):45–111, 2000. [356](#)
- [Ram00b] S. Ramanujan. On certain arithmetical functions [Trans. Cambridge Philos. Soc. **22** (1916), no. 9, 159–184]. In *Collected papers of Srinivasa Ramanujan*, pages 136–162. AMS Chelsea Publ., Providence, RI, 2000. [179](#)
- [Ram04] D. Ramakrishnan. Algebraic cycles on Hilbert modular fourfolds and poles of L -functions. In *Algebraic groups and arithmetic*, pages 221–274. Tata Inst. Fund. Res., Mumbai, 2004. [383](#), [416](#)
- [Ram15] D. Ramakrishnan. A mild Tchebotarev theorem for $GL(n)$. *J. Number Theory*, 146:519–533, 2015. [309](#)
- [Rap14] A. S. Rapinchuk. Strong approximation for algebraic groups. In *Thin groups and superstrong approximation*, volume 61 of *Math. Sci. Res. Inst. Publ.*, pages 269–298. Cambridge Univ. Press, Cambridge, 2014. [66](#)
- [Ren10] D. Renard. *Représentations des groupes réductifs p -adiques*, volume 17 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2010. [215](#), [220](#), [222](#), [278](#)
- [Ric77] R. W. Richardson. Affine coset spaces of reductive algebraic groups. *Bull. London Math. Soc.*, 9(1):38–41, 1977. [438](#)
- [Ric82] R. W. Richardson. Orbits, invariants, and representations associated to involutions of reductive groups. *Invent. Math.*, 66(2):287–312, 1982. [379](#), [443](#), [490](#)
- [Rog83] J. D. Rogawski. Representations of $GL(n)$ and division algebras over a p -adic field. *Duke Math. J.*, 50(1):161–196, 1983. [485](#), [515](#)
- [Rud90] Z. Rudnick. *Poincaré series*. PhD thesis, Yale University, 1990. [368](#)
- [Rud91] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991. [77](#)
- [RV99] D. Ramakrishnan and R. J. Valenza. *Fourier analysis on number fields*, volume 186 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999. [45](#), [68](#), [551](#)
- [Sai75] H. Saito. Automorphic forms and algebraic extensions of number fields. *Proc. Japan Acad.*, 51(4):229–233, 1975. [351](#), [488](#), [516](#)
- [Sak08] Y. Sakellaridis. On the unramified spectrum of spherical varieties over p -adic fields. *Compos. Math.*, 144(4):978–1016, 2008. [379](#)
- [Sak12] Y. Sakellaridis. Spherical varieties and integral representations of L -functions. *Algebra Number Theory*, 6(4):611–667, 2012. [491](#), [518](#)

- [Sak13] Y. Sakellaridis. Spherical functions on spherical varieties. *Amer. J. Math.*, 135(5):1291–1381, 2013. [379](#)
- [Sak18] Y. Sakellaridis. Inverse Satake transforms. In *Geometric aspects of the trace formula*, Simons Symp., pages 321–349. Springer, Cham, 2018. [187](#)
- [Sak22] Y. Sakellaridis. Transfer operators and Hankel transforms between relative trace formulas, II: Rankin-Selberg theory. *Adv. Math.*, 394:Paper No. 108039, 104, 2022. [368](#), [492](#)
- [Sar] P. Sarnak. Comments on Langlands’ lecture ”Endoscopy and Beyond”. <http://publications.ias.edu/sarnak/section/513/guide.html>. [368](#), [492](#)
- [Sar05] P. Sarnak. Notes on the generalized Ramanujan conjectures. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 659–685. Amer. Math. Soc., Providence, RI, 2005. [288](#), [339](#)
- [Sat63] I. Satake. Theory of spherical functions on reductive algebraic groups over p -adic fields. *Inst. Hautes Études Sci. Publ. Math.*, (18):5–69, 1963. [199](#)
- [Sch85] W. Scharlau. *Quadratic and Hermitian forms*, volume 270 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1985. [526](#)
- [Sch88] P. Schneider. Introduction to the Beĭlinson conjectures. In *Beĭlinson’s conjectures on special values of L -functions*, volume 4 of *Perspect. Math.*, pages 1–35. Academic Press, Boston, MA, 1988. [415](#)
- [Sch11] P. Schneider. *p -adic Lie groups*, volume 344 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. [235](#)
- [Sch15] P. Scholze. On torsion in the cohomology of locally symmetric varieties. *Ann. of Math. (2)*, 182(3):945–1066, 2015. [344](#)
- [Sel56] A. Selberg. Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series. *J. Indian Math. Soc. (N.S.)*, 20:47–87, 1956. [276](#), [419](#), [473](#), [508](#)
- [Sel63] A. Selberg. Discontinuous groups and harmonic analysis. In *Proc. Internat. Congr. Mathematicians (Stockholm, 1962)*, pages 177–189. Inst. Mittag-Leffler, Djursholm, 1963. [276](#)
- [Ser72] J.-P. Serre. Propriétés galoisiennes des points d’ordre fini des courbes elliptiques. *Invent. Math.*, 15(4):259–331, 1972. [316](#)
- [Ser73] J.-P. Serre. *A course in arithmetic*. Springer-Verlag, New York-Heidelberg, 1973. Translated from the French, Graduate Texts in Mathematics, No. 7. [174](#), [179](#)

- [Ser79] J.-P. Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg. [46](#)
- [Ser02] J.-P. Serre. *Galois cohomology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2002. Translated from the French by Patrick Ion and revised by the author. [447](#), [449](#), [454](#)
- [Ser06] J.-P. Serre. *Lie algebras and Lie groups*, volume 1500 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition. [59](#), [138](#), [235](#)
- [SGA72] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*. Lecture Notes in Mathematics, Vol. 269. Springer-Verlag, Berlin-New York, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat. [x](#)
- [Sha74] J. A. Shalika. The multiplicity one theorem for GL_n . *Ann. of Math. (2)*, 100:171–193, 1974. [296](#), [303](#)
- [Sha90] F. Shahidi. A proof of Langlands’ conjecture on Plancherel measures; complementary series for p -adic groups. *Ann. of Math. (2)*, 132(2):273–330, 1990. [331](#), [361](#)
- [Sha10] F. Shahidi. *Eisenstein series and automorphic L -functions*, volume 58 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2010. [276](#), [294](#), [340](#), [355](#)
- [Sha11] F. Shahidi. Arthur packets and the Ramanujan conjecture. *Kyoto J. Math.*, 51(1):1–23, 2011. [339](#)
- [She82] D. Shelstad. L -indistinguishability for real groups. *Math. Ann.*, 259(3):385–430, 1982. [326](#), [448](#)
- [Shi76] T. Shintani. On an explicit formula for class-1 “Whittaker functions” on GL_n over P -adic fields. *Proc. Japan Acad.*, 52(4):180–182, 1976. [300](#)
- [Shi79] T. Shintani. On liftings of holomorphic cusp forms. In *Automorphic forms, representations and L -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 97–110. Amer. Math. Soc., Providence, R.I., 1979. [351](#), [488](#), [516](#)
- [Shi94] G. Shimura. *Introduction to the arithmetic theory of automorphic functions*, volume 11 of *Publications of the Mathematical Society of Japan*. Princeton University Press, Princeton, NJ, 1994. Reprint of the 1971 original, Kanô Memorial Lectures, 1. [174](#), [179](#)

- [Shi11] S. W. Shin. Galois representations arising from some compact Shimura varieties. *Ann. of Math. (2)*, 173(3):1645–1741, 2011. [415](#)
- [Sil79] A. J. Silberger. *Introduction to harmonic analysis on reductive p -adic groups*, volume 23 of *Mathematical Notes*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1979. Based on lectures by Harish-Chandra at the Institute for Advanced Study, 1971–1973. [128](#), [129](#)
- [Sli84] M. H. Sliman. Théorie de Mackey pour les groupes adéliques. *Astérisque*, (115):151, 1984. [91](#), [97](#), [249](#)
- [Sol20] M. Solleveld. Conjugacy of Levi subgroups of reductive groups and a generalization to linear algebraic groups. *J. Pure Appl. Algebra*, 224(6):106254, 16, 2020. [273](#)
- [Sou05] D. Soudry. On Langlands functoriality from classical groups to GL_n . *Astérisque*, (298):335–390, 2005. Automorphic forms. I. [362](#)
- [Spe83] B. Speh. Unitary representations of $GL(n, \mathbb{R})$ with nontrivial $(\mathfrak{g}, K)$ -cohomology. *Invent. Math.*, 71(3):443–465, 1983. [290](#)
- [Spr79] T. A. Springer. Reductive groups. In *Automorphic forms, representations and L -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 3–27. Amer. Math. Soc., Providence, R.I., 1979. [389](#)
- [Spr94] T. A. Springer. Reduction theory over global fields. *Proc. Indian Acad. Sci. Math. Sci.*, 104(1):207–216, 1994. K. G. Ramanathan memorial issue. [72](#)
- [Spr09] T. A. Springer. *Linear algebraic groups*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, second edition, 2009. [2](#), [33](#), [35](#), [546](#)
- [Sta16] The Stacks Project Authors. Stacks project. <http://stacks.math.columbia.edu>, 2016. [4](#), [5](#), [6](#), [10](#), [40](#), [41](#), [58](#), [438](#)
- [Ste65] R. Steinberg. Regular elements of semisimple algebraic groups. *Inst. Hautes Études Sci. Publ. Math.*, (25):49–80, 1965. [187](#), [232](#), [439](#), [442](#), [485](#)
- [Ste68] R. Steinberg. *Endomorphisms of linear algebraic groups*. Memoirs of the American Mathematical Society, No. 80. American Mathematical Society, Providence, R.I., 1968. [234](#), [527](#)
- [Ste87] G. Stevens. Poincaré series on $GL(r)$ and Kloostermann sums. *Math. Ann.*, 277(1):25–51, 1987. [244](#), [497](#)
- [Sun17] B. Sun. The nonvanishing hypothesis at infinity for Rankin-Selberg convolutions. *J. Amer. Math. Soc.*, 30(1):1–25, 2017. [408](#)
- [Sun19] B. Sun. Cohomologically induced distinguished representations and cohomological test vectors. *Duke Math. J.*, 168(1):85–126, 2019. [384](#), [408](#)

- [SV17] Y. Sakellaridis and A. Venkatesh. Periods and harmonic analysis on spherical varieties. *Astérisque*, (396):viii+360, 2017. [373](#), [379](#), [380](#), [385](#)
- [SW20] P. Scholze and J. Weinstein. *Berkeley lectures on p -adic geometry*, volume 207 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2020. [369](#)
- [SW22] Y. Sakellaridis and J. Wang. Intersection complexes and unramified L -factors. *J. Amer. Math. Soc.*, 35(3):799–910, 2022. [491](#)
- [Tat67] J. T. Tate. Fourier analysis in number fields, and Hecke’s zeta-functions. In *Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965)*, pages 305–347. Thompson, Washington, D.C., 1967. [294](#), [323](#), [551](#)
- [Tat79] J. T. Tate. Number theoretic background. In *Automorphic forms, representations and L -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 3–26. Amer. Math. Soc., Providence, R.I., 1979. [316](#), [317](#), [318](#), [319](#), [320](#), [321](#), [323](#), [324](#), [344](#)
- [Tat94] J. T. Tate. Conjectures on algebraic cycles in l -adic cohomology. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 71–83. Amer. Math. Soc., Providence, RI, 1994. [415](#)
- [Tit79] J. Tits. Reductive groups over local fields. In *Automorphic forms, representations and L -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 29–69. Amer. Math. Soc., Providence, R.I., 1979. [59](#)
- [Tun81] J. Tunnell. Artin’s conjecture for representations of octahedral type. *Bull. Amer. Math. Soc. (N.S.)*, 5(2):173–175, 1981. [354](#)
- [Č22] K. Česnavičius. Problems about torsors over regular rings. *Acta Math. Vietnam.*, 47(1):39–107, 2022. With an appendix by Yifei Zhao. [454](#)
- [Ven04] A. Venkatesh. “Beyond endoscopy” and special forms on $GL(2)$. *J. Reine Angew. Math.*, 577:23–80, 2004. [368](#), [492](#)
- [Vog93] D. A. Vogan. The local Langlands conjecture. In *Representation theory of groups and algebras*, volume 145 of *Contemp. Math.*, pages 305–379. Amer. Math. Soc., Providence, RI, 1993. [343](#)
- [Vus74] T. Vust. Opération de groupes réductifs dans un type de cônes presque homogènes. *Bull. Soc. Math. France*, 102:317–333, 1974. [381](#)
- [VZ84] D. A. Vogan and G. J. Zuckerman. Unitary representations with nonzero cohomology. *Compositio Math.*, 53(1):51–90, 1984. [410](#)
- [Wal88] N. R. Wallach. *Real reductive groups. I*, volume 132 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988. [104](#), [115](#), [119](#), [122](#), [124](#), [125](#), [325](#)

- [Wal92] N. R. Wallach. *Real reductive groups. II*, volume 132 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1992. [104](#), [120](#), [129](#), [297](#)
- [Wal97] J.-L. Waldspurger. Le lemme fondamental implique le transfert. *Compositio Math.*, 105(2):153–236, 1997. [439](#)
- [Wal03] J.-L. Waldspurger. La formule de Plancherel pour les groupes p -adiques (d’après Harish-Chandra). *J. Inst. Math. Jussieu*, 2(2):235–333, 2003. [129](#), [224](#), [225](#)
- [Wal06] J.-L. Waldspurger. Endoscopie et changement de caractéristique. *J. Inst. Math. Jussieu*, 5(3):423–525, 2006. [439](#)
- [Wal08] J.-L. Waldspurger. L’endoscopie tordue n’est pas si tordue. *Mem. Amer. Math. Soc.*, 194(908):x+261, 2008. [439](#)
- [Wal09a] J.-L. Waldspurger. à propos du lemme fondamental pondéré tordu. *Math. Ann.*, 343(1):103–174, 2009. [439](#)
- [Wal09b] J.-L. Waldspurger. Endoscopie et changement de caractéristique: intégrales orbitales pondérées. *Ann. Inst. Fourier (Grenoble)*, 59(5):1753–1818, 2009. [439](#)
- [War72] G. Warner. *Harmonic analysis on semi-simple Lie groups. I*. Die Grundlehren der mathematischen Wissenschaften, Band 188. Springer-Verlag, New York-Heidelberg, 1972. [107](#), [301](#)
- [Wei95] A. Weil. *Basic number theory*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the second (1973) edition. [552](#), [554](#)
- [Wey12] H. Weyl. Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung). *Math. Ann.*, 71(4):441–479, 1912. [474](#), [508](#)
- [Wil91] F. L. Williams. Spectral multiplicity of derived functor modules. *Adv. Math.*, 85(2):145–160, 1991. [406](#)
- [Xue17] H. Xue. On the global Gan-Gross-Prasad conjecture for unitary groups: Approximating smooth transfer of Jacquet-Rallis. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 01 2017. [391](#)
- [Ye00] Y. Ye. A Kuznetsov formula for Kloosterman sums on GL_n . *Ramanujan J.*, 4(4):385–395, 2000. [492](#)
- [Yu] J. K. Yu. Smooth models associated to concave functions in Bruhat-Tits theory. In *Autour des schemas en groupes. Vol.III*, volume 47 of *Panor. Synthèses*, pages 227–258. Soc. Math. France, Paris, 2015. [58](#)
- [Yun11] Z. Yun. The fundamental lemma of Jacquet and Rallis. *Duke Math. J.*, 156(2):167–227, 2011. With an appendix by Julia Gordon. [391](#), [520](#)
- [Yun13] Z. Yun. Orbital integrals and Dedekind zeta functions. In *The legacy of Srinivasa Ramanujan*, volume 20 of *Ramanujan Math.*

- Soc. Lect. Notes Ser.*, pages 399–420. Ramanujan Math. Soc., Mysore, 2013. [486](#)
- [YZ17] Z. Yun and W. Zhang. Shtukas and the Taylor expansion of L -functions. *Ann. of Math. (2)*, 186(3):767–911, 2017. [392](#), [417](#)
- [YZ19] Z. Yun and W. Zhang. Shtukas and the Taylor expansion of L -functions (II). *Ann. of Math. (2)*, 189(2):393–526, 2019. [392](#), [417](#)
- [Zel80] A. V. Zelevinsky. Induced representations of reductive p -adic groups. II. On irreducible representations of $GL(n)$. *Ann. Sci. École Norm. Sup. (4)*, 13(2):165–210, 1980. [223](#), [224](#), [285](#), [328](#)
- [Zha12a] W. Zhang. Gross-Zagier formula and arithmetic fundamental lemma. In *Fifth International Congress of Chinese Mathematicians. Part 1, 2*, volume 2 of *AMS/IP Stud. Adv. Math.*, 51, pt. 1, pages 447–459. Amer. Math. Soc., Providence, RI, 2012. [417](#)
- [Zha12b] W. Zhang. On arithmetic fundamental lemmas. *Invent. Math.*, 188(1):197–252, 2012. [392](#), [417](#)
- [Zha14] W. Zhang. Fourier transform and the global Gan-Gross-Prasad conjecture for unitary groups. *Ann. of Math. (2)*, 180(3):971–1049, 2014. [391](#), [520](#)
- [Zha21] W. Zhang. Weil representation and Arithmetic Fundamental Lemma. *Ann. of Math. (2)*, 193(3):863–978, 2021. [417](#)
- [Zhu07] K. H. Zhu. *Operator theory in function spaces*, volume 138 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2007. [246](#)
- [Zhu17] X. Zhu. Affine Grassmannians and the geometric Satake in mixed characteristic. *Ann. of Math. (2)*, 185(2):403–492, 2017. [198](#)
- [Zhu20] X. Zhu. A note on Integral Satake isomorphisms. *arXiv e-prints*, page [arXiv:2005.13056](#), May 2020. [344](#)
- [Zyd19] M. Zydor. Periods of automorphic forms over reductive subgroups. *arXiv e-prints*, page [arXiv:1903.01697](#), Mar 2019. [374](#), [474](#)
- [Zyd20] M. Zydor. Les formules des traces relatives de Jacquet-Rallis grossières. *J. Reine Angew. Math.*, 762:195–259, 2020. [474](#)

Index

- (H, χ) -distinguished representation
 - cuspidal automorphic, 374
 - smooth, 372
 - (H, χ) -functional, 372
 - $(\mathfrak{g}, K)$ -module, 115
 - admissible, 116
 - $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -module, 163
 - 1-cocycle, 447
 - cohomologous, 447
 - A -packet, 338
 - local, 366
 - A -parameter, 365
 - C^* -algebra, 96
 - D_v -compatible, 515
 - F -supercuspidal function, 431
 - G -intertwining map, 78
 - H -distinguished representation
 - admissible, 372
 - cuspidal automorphic, 374
 - H -torsor, 451
 - trivial, 451
 - K -finite function, 163
 - K -type σ , 114
 - L -group, 191
 - parabolic subgroup of, 194
 - Levi subgroup of, 195
 - relevant, 195
 - relevant Levi subgroup of, 195
 - standard, 194
 - L -indistinguishable, 339
 - weakly globally, 208
 - L -map, 192
 - L -packet
 - automorphic global, 335
 - global, 334
 - local, 329
 - unramified, 330
 - Vogan, 380, 390
 - weak automorphic, 208
 - L -parameter, 324
 - adelic, 334
 - almost tempered, 333
 - equivalent, 324
 - equivalent adelic, 336
 - tempered, 328
 - unramified, 329
 - R -points, 3
 - R -valued points, 3
 - $Z(\mathfrak{g})$ -finite, 159
 - χ -distinguished representation
 - admissible, 372
 - cuspidal automorphic, 374
 - $\mathfrak{w}$ -generic, 331
 - Fr-semisimple, 197
 - ψ -Whittaker function
 - global, 297
 - local, 296
 - θ -centralizer, 487
 - θ -conjugation, 486
 - v -adic metric, 46
- A**
- absolute value, 46
 - Archimedean, 46
 - equivalent, 46
 - non-Archimedean, 46
 - normalized, 46
 - trivial, 46
 - action
 - natural, 80

regular, 80
 adèle ring, 48
 adelic quotient, 69
 admissible
 $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_F^\infty)$ -module, 163
 admissible $C_c^\infty(G)$ -module, 140
 affine algebraic group, 13
 affine group scheme, 7
 abelian, 8
 additive, 8
 constant, 23
 linear, 9
 monomorphism of, 9
 multiplicative, 8
 quotient, 10
 quotient map of, 9
 trivial, 23
 affine scheme, 3
 étale, 4
 connected, 13
 finite, 4
 flat, 58
 integral, 4
 irreducible, 4
 of finite presentation, 4
 of finite type, 4
 reduced, 4
 smooth, 4
 affine subgroup scheme, 9
 approximate identity, 86
 approximation
 strong, 50, 64
 weak, 64
 Arthur-SL₂, 365
 Artin conjecture, 352
 strong, 352
 Artin reciprocity map, 317
 automorphic form
 classical, 159
 cuspidal, 167, 264
 over a function field, 166
 over a number field, 163
 automorphic induction, 350
 automorphic kernel, 429
 automorphic representation, 169
 C -algebraic, 343
 L -algebraic, 343
 discrete, 91
 over a function field, 167
 over a number field, 165
 automorphy factor, 176

B

base change

of a scheme, 12
 of an admissible representation, 350
 Borel subgroup scheme, 542
 Bruhat cell, 494

C

Casselman-Shalika formula, 299, 300
 Cauchy identity, 307
 center, 11
 central isogeny
 equivalent, 30
 of affine algebraic groups, 29
 of root data, 29
 centralizer, 11
 character, 82
 generic, 294
 Hecke, 184
 infinitesimal, 120, 125
 of an admissible representation,
 129, 225
 of an algebraic group, 20
 Chevalley restriction theorem, 442
 class, 440
 association, 273
 geometric, 444
 Langlands, 198
 map, 448
 relatively semisimple, 440
 classification
 Bernstein-Zelevinsky, 222, 223
 endoscopic, 363
 Langlands, 131
 closed immersion, 5
 closed subscheme, 5
 cocharacter, 20
 cocycle condition, 447
 coefficient of π , 227
 cohomology
 $(\mathfrak{g}, K_\infty)$, 403
 $(\mathfrak{g}, \mathfrak{k})$, 403
 cuspidal, 405
 Eisenstein, 405
 coinvariant, 218
 commensurable, 70
 complex dual, 31
 conductor of π , 306
 constant term, 168
 of f along B , 199
 of f along P , 230
 continuous spectrum, 282
 contracted product, 452
 converse theorem, 312
 coordinate ring, 3
 coroot, 28

cuspid form, 167, 174
 cuspidal automorphic representation,
 168, 169
 discrete, 169
 cuspidal support, 222

D

datum
 cuspidal, 221
 Shimura, 410
 Whittaker, 331
 Dirac sequence, 246
 discrete series representation, 127
 discrete spectrum, 242
 distinction, 371
 distribution, 225
 Dixmier-Malliavin lemma, 110
 dual root system, 28
 dynamic, 545, 546

E

Eisenstein series, 275
 element
 θ -elliptic, 487
 θ -semisimple, 487
 absolutely semisimple, 232
 Casimir, 133
 elliptic, 485
 gcf, 466
 invariant, 440
 neat, 397
 neutral, 447
 nilpotent, 14
 relatively elliptic, 465
 relatively regular, 438
 relatively semisimple, 438
 relatively unimodular, 450
 relevant, 450, 463
 semisimple, 14
 unipotent, 14
 equalizer, 39
 exact diagram, 39
 exponent of the conductor, 306

F

factorizable module, 151
 fiber
 generic, 58
 product, 7
 special, 58
 field
 function, 45
 global, 45
 local, 46

 number, 45
 field of Hilbert spaces, 92
 direct integral, 93
 measurable, 93
 field of operators, 93
 direct integral, 94
 measurable, 93
 finite adeles, 49
 finite length, 206
 finiteness of class numbers, 67
 first Galois cohomology, 447
 Flath's theorem, 152
 form
 equivalent, 388
 equivalent pure inner, 389, 390
 inner, 389, 449
 of an algebraic group, 448
 outer, 532
 pure inner, 389, 390
 formally smooth, 4
 Fourier transform
 adelic, 554
 local, 553
 fppf, 451
 fppf cover, 38
 fppf sheaf, 39
 Fréchet space, 105
 Frobenius
 arithmetic, 317
 geometric, 317
 fundamental lemma, 511
 Hecke algebra, 511
 Jacquet-Mao, 384
 Jacquet-Ye, 381

G

Gårding subspace, 109
 Gan-Gross-Prasad conjecture, 387
 Gelfand-Pettis integral, 84
 geometric
 fiber, 542
 point, 542
 geometric correspondence, 509
 geometric Langlands program, 198
 geometric Satake correspondence, 198
 global A -packet
 discrete, 365
 global A -parameter
 discrete generic, 365
 global L -packet
 weak, 208
 good position, 198, 229, 541
 group
 adjoint, 29

- almost simple, 66
 - anisotropic, 33
 - definite unitary, 43
 - equal rank, 325
 - inertia, 321
 - isotropic, 33
 - Langlands, 336
 - Langlands dual, 191
 - quasi-split classical, 357
 - reductive, 15
 - root, 34
 - semisimple, 15
 - simply connected, 29
 - solvable, 14
 - split reductive group, 23
 - td-, 135
 - td-type, 135
 - twisted endoscopic, 517
 - unimodular, 81
 - unipotent, 14
 - unramified reductive, 59, 146, 184
 - Weil, 317
 - Weil-Deligne, 320
 - Weyl, 23
- H**
- Haar measure
 - left, 81
 - right, 81
 - Hamel basis, 154
 - Harish-Chandra homomorphism, 124
 - Harr measure, 81
 - Hasse principle, 338
 - Hecke algebra, 137
 - of K -type Ξ , 121
 - spherical, 146
 - unramified, 146, 184
 - Hecke eigenform, 174
 - Hecke eigenvalue, 174
 - Hecke operator, 174
 - Hilbert space, 78
 - separable, 78
- I**
- Ichino-Ikeda conjecture, 380
 - including data, 277
 - infinitesimally equivalent, 119
 - involution
 - of orthogonal type, 530
 - of symplectic type, 530
 - of unitary type, 530
 - isometry group, 527
 - isomorphism
 - of affine group schemes, 7
- Iwasawa decomposition, 71, 129, 198
- J**
- Jacobi identity, 17
 - Jacobson density theorem, 122
 - Jacquet functor, 218
 - Jacquet module, 219
 - Jordan decomposition, 14
- K**
- Künneth formula, 409
 - kernel function, 244, 420
 - automorphic, 421
 - Kloosterman integral, 496
 - Kloosterman sum, 492
 - Kuznetsov formula, 492
- L**
- Langlands L -function, 340
 - Langlands correspondence, 315
 - Archimedean, 325
 - local, 327, 329
 - Langlands data, 131, 223
 - Langlands decomposition, 130
 - Langlands functoriality conjecture, 210, 338
 - Langlands quotient, 131, 222
 - Langlands reciprocity, 315, 337
 - Langlands-Shahidi method, 355
 - Levi decomposition, 16
 - Lie algebra, 16
 - Lie bracket, 17
 - lift
 - of the correspondence, 401
 - weak, 349
 - limit of discrete series, 127
 - local system, 399
 - locally symmetric space, 396
- M**
- Maass forms, 178
 - matching, 509, 511, 515, 516
 - matrix coefficient, 112, 214
 - Matsushima's formula, 405
 - maximal split torus, 22
 - measurable field of representations, 94
 - direct integral, 94
 - unitary, 94
 - model, 58
 - moderate growth, 159
 - adelic function of, 161
 - Fréchet representation of, 119
 - modular form, 174
 - morphism

- dominant, 6
 - fppf, 38
 - of affine group schemes, 7
 - of affine scheme, 3
 - of Lie algebras, 17
 - of presheaf, 38
 - of Shimura data, 412
 - of Shimura variety, 412
 - structure, 7
 - surjective, 6
 - multiplicity one, 303
 - strong, 309
- N**
- neutral component, 14
 - nondegenerate module, 139
 - norm, 516
 - Hilbert-Schmidt, 240
 - trace, 240
 - normalizer, 11
- O**
- operator
 - compact, 246
 - diagonal, 94
 - Hilbert-Schmidt, 240
 - intertwining, 271
 - trace class, 240
 - orbital integral, 233, 485
 - global relative, 463
 - local relative, 450
 - twisted, 487
 - orthogonal type, 364
- P**
- parabolic descent, 203
 - of orbital integrals, 232
 - of representations, 229
 - parabolic induction, 214, 215
 - parabolic subgroup scheme, 542
 - parameter
 - C -algebraic, 342
 - L -algebraic, 342
 - period integral, 243, 374
 - Peter-Weyl Theorem, 113
 - pinning, 189
 - place, 46
 - Archimedean, 46
 - finite, 47
 - infinite, 47
 - non-Archimedean, 46
 - Poincaré duality, 409
 - Poincaré series, 244
 - Poisson summation, 554
 - pre-Hilbert space, 79
 - presheaf, 38
 - primitive ideal, 96
 - product formula, 48
- Q**
- quasi-character, 82
 - central, 133
 - modular, 82, 203, 213
 - unramified, 203
 - quasi-complete space, 84
 - quasi-split, 33
 - quotient
 - categorical, 440
 - GIT, 440
- R**
- radical, 15
 - unipotent, 14
 - Ramanujan conjecture, 288
 - Ramanujan's Δ -function, 178
 - rank
 - of a torus, 19
 - of an affine algebraic group, 22
 - Rankin-Selberg L -function, 293
 - global, 308
 - local, 303
 - unramified, 307
 - Rankin-Selberg integral
 - global, 310
 - local, 304
 - rapidly decreasing function, 263
 - reduced norm, 526
 - reduced subscheme, 437
 - reduced trace, 526
 - reduction theory, 72
 - reductive group scheme, 542
 - reflex field, 412
 - regular element of $\mathfrak{a}_{T\mathbb{C}}^*$, 206
 - relative Langlands program, 380
 - relative to the action of H , 438
 - representable
 - fppf sheaf, 40
 - functor, 3
 - representation
 - adjoint, 19
 - admissible, 115, 140
 - admissible Hilbert, 141
 - Asai, 360, 382
 - cohomological, 405
 - conjugate dual, 362
 - conjugate self-dual, 362
 - contragredient, 144
 - dihedral, 354

- discrete series, 128
 - equivalent, 78
 - essentially unitary, 286
 - faithful, 9
 - Frobenius semisimple, 320
 - generic, 224, 295
 - Hermitian contragredient, 144
 - Hilbert space, 141
 - icosahedral, 354
 - induced, 130, 213, 269
 - isobaric, 210, 288
 - nearly equivalent, 389
 - nondegenerate, 296
 - octahedral, 354
 - odd, 354
 - of a locally compact group, 78
 - of an affine group scheme, 9
 - pre-unitarizable, 79
 - pre-unitary, 79
 - principal series, 127, 203
 - quasi-cuspidal, 214
 - quotient, 78
 - regular, 81, 239, 419
 - smooth, 119, 139
 - Speh, 290
 - spherical, 148
 - square integrable, 128
 - essentially, 128
 - Steinberg, 291
 - subquotient, 78
 - supercuspidal, 128, 214
 - tempered, 128
 - essentially, 128
 - tetrahedral, 354
 - unitarily equivalent, 78
 - unitarizable, 78
 - unitary, 78
 - unramified, 148, 184
 - unramified principal series, 206
 - residual spectrum, 282
 - restricted direct product, 56
 - restricted product topology, 49
 - restricted tensor product
 - of algebras, 150
 - of vector spaces, 149
 - restriction map, 39
 - ring of integers, 47
 - root, 25
 - group, 27
 - positive, 26
 - defined by P , 37
 - simple, 26
 - spaces, 25
 - root datum, 27, 28
 - dual, 31
 - base, 188
 - reduced, 29
 - root system, 26
 - base, 26
 - rank, 26
 - reduced, 26
 - Weyl group, 27
- S**
- Sakellaridis-Venkatesh conjecture, 380, 387
 - Satake isomorphism, 186, 197
 - Satake parameter, 198
 - section, 38, 93
 - measurable, 93
 - segment, 224
 - linked, 224
 - self-dual Haar measure
 - adelic, 554
 - local, 553
 - semisimplification, 206
 - sheaf of sections, 400
 - sheafification, 40
 - Shimura manifold, 396
 - Siegel set, 71
 - Siegel upper-half space, 268
 - slowly increasing function, 159
 - adelic, 161
 - smooth dual, 143
 - smooth endomorphisms, 227
 - smooth function, 136
 - smooth linear functional, 143
 - space of K -finite vectors, 115
 - Speh representation
 - type, 364
 - stability of γ -factor, 359
 - subgroup
 - $\ker \phi$, 10
 - arithmetic, 70, 397
 - associated parabolic, 273
 - Borel, 32
 - congruence, 70, 397
 - derived, 14
 - hyperspecial, 59
 - Levi, 16
 - minimal parabolic, 33, 34
 - neat arithmetic, 397
 - neat compact open, 398
 - normal, 10
 - opposite parabolic, 35
 - parabolic, 32
 - proper parabolic, 32
 - spherical, 379

- standard parabolic, 34, 222, 270
- symmetric, 381
- subquotient theorem, 131
- subrepresentation, 78
- subspace
 - σ -isotypic, 114
 - cuspidal, 168, 241
- supercuspidal function, 430
- symplectic type, 364

T

- Tamagawa measure, 91
- Tamagawa number, 91
- topology
 - Fell, 95, 97
 - Hausdorff, 52
 - hull-kernel, 96
 - Jacobson, 96
 - Zariski, 5
- torus, 19
 - anisotropic, 21
 - Deligne, 12, 410
 - induced, 66
 - maximal, 22
 - quasi-trivial, 66
 - split, 19, 21
- trace, 240
 - of $\pi(f)$, 225
 - relative, 426
 - twisted, 487
- trace formula
 - geometric side, 479
 - relative, 371, 474
 - simple, 485
 - simple general relative, 479, 483
 - simple relative, 474
 - simple twisted, 486, 488
 - spectral side, 479
- transfer, 511
 - Jacquet-Langlands, 514
 - weak, 349
- transfer factor, 511
- twisted endoscopy, 517
- type I group, 95, 97

- type of a Levi subgroup, 36
- type of a parabolic subgroup, 36

U

- unfolding, 244
- uniform moderate growth, 159
 - adelic function of, 161
- uniformizer, 48
- unitary dual, 95
- unramified character, 299
- unramified outside of S
 - admissible representation, 209
 - group, 209

V

- valuation, 45
- variety
 - Hilbert modular, 413
 - Kottwitz, 413
 - Shimura, 395, 412
 - Siegel modular, 413
 - spherical, 379
 - symmetric, 381
- vector
 - K -finite, 115
 - V -cohomological, 404
 - cohomological, 404, 405
 - smooth, 107, 171
- vector field, 93

W

- weight, 24
 - spaces, 24
- Weil restriction of scalars, 12
- Weyl chamber, 37
 - positive, 37
- Weyl law, 508
 - relative, 509
- Whittaker expansion, 298
- Whittaker functional, 295
- Whittaker model
 - global, 298
 - local, 296